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Research Article

A Generalization of Curve Mates: Normal Mate of a Curve

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Abstract

This a paper, a new curve pair is defined that generalizes some pairs of curves well known as Mannheim and Bertrand curve pairs. A normal curve pair is defined in such a way that a vector $\vec{w}$ obtained by overlapping the normal planes of the P and P^* curves makes the Γ angle as the binormals of these curves. The relationship between torsions and curvatures of curve pairs was analyzed. Moreover, a unit quaternion q was defined corresponding to the rotation matrix between the Frenet vectors of the curves. In the conclusion, it is expressed which famous curve pairs will be obtained in which particular case.

Keywords: Normal mate, Curve mates, Mannheim mate, Quaternion.

Eğri Çiftlerinin Genelleştirilmesi: Bir Eğrinin Normal Eğri Çifti

Öz

Bu çalışmada Bertrand ve Mannheim gibi çok bilinen eğri çiftlerini genelleştiren yeni bir eğri çifti tanımlanmıştır. Normal eğri çifti, P ve P^* eğrilerinin normal düzlemlerinin kesişimleri ile elde edilen bir $\vec{w}$ vektörünün bu eğrilerin binormalleri ile Γ açısı yapacak şekilde tanımlanır. Eğri çiftlerinin eğrilikleri ve burulmaları arasındaki ilişki analiz edilmiştir. Ayrıca, eğrilerin Frenet vektörleri arasındaki dönme matrisine karşılık gelen birim q kuaterniyonu tanımlanmıştır. Sonuç olarak, hangi özel durumda hangi ünlü eğri çiftinin elde edileceğini ifade edilmiştir.

Anahtar Kelimeler: Normal eğri çifti, Eğri çiftleri, Mannheim eğri çifti, Kuaterniyon.

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1. Introduction

Establishing a connection between two points corresponding to two curves and defining a new curve pair from these curves has been a subject that has attracted the attention of many researchers in classical differential geometry. These curve pairs have applications in fields like robotics, computer-aided geometric design and planning of paths. Some of the famous curve pairs are like Parallel, Evolute, Involute, Bertrand, Natural and Mannheim mates. Some properties and basic definitions of these curve pairs can be found in various papers [1,3,5,9,10,11,17,21,23,24,28,33,36]. These curves have been studied by many authors and some have been generalized to larger dimensions [8,14,16,25,32]. Moreover, Curve pairs in Lorentzian space have also been studied by many researchers [2,13,15,18,22,31,34,35,37]. The aim of the authors in this study is to express these curves in three-dimensional Euclidean space and make a new generalization and to investigate whether they give one of them by examining the special cases in the general definition. For this, A new pair of curves called normal mate is defined. If a vector obtained by overlapping normal planes makes the same angle as the binormals of the curves, the pair of curves formed by these curves is called normal mates. In this paper, we'll only analyze the normal mate. Moreover, we'll also expressed firstly the states between the Frenet vectors of curves of P and $P*$. Then the states between their curvatures and torsions of curves of P and $P*$. In the result section, it will be given which specific result represents which well-known curve.

If the tangent of a regular α curve with curvature different from zero at each point is the normal of an I curve, the involute curve of the α curve is expressed as the I curve. Evolute of a curve is defined as the place of the centers of curvatures of the curve. The evolute of an involute is the original curve. The definitions of involute and evolute were introduced by the Dutch mathematician Christian Huygens.

If the points on the curves that correspond to each other are also in the same direction as the normal of one curve and the normal of another these two curves are called the Bertrand curve pair. Moreover, a Bertrand curve is defined as a space curve B whose normal vector is identical to the normal vector of another curve B^*, known as its Bertrand mate. If λ is nonzero constant B^* and B are Bertrand mates as

$$B^*(s) = B(s) + \lambda\mathbf{N}(s) \text{ and } \mathbf{N}^*(s) = \pm\mathbf{N}(s)$$

for all $s \in I$. Bertrand curves stated by Bertrand in 1850 [1,5,17,21,24].

If the points on the curves that correspond to each other are also in the same direction as the normal of one curve and the binormal of another these two curves are called the Mannheim curve pair. Moreover, a Mannheim curve is defined as a space curve M whose normal vector is identical to the binormal vector of another curve M^*, known as its Mannheim mate. If λ is nonzero constant M^* and M are Mannheim mates as

$$M^*(s) = M(s) + \lambda\mathbf{N}(s) \text{ and } \mathbf{B}^*(s) = \pm\mathbf{N}(s)$$

for all $s \in I$. It was stated by A. Mannheim in 1878 [17,23,28].

We define curve pairs resulting from a Backlund transformation as pairs of constant torsion curves, which we will refer to as Backlund curves in this study. In classical differential geometry, a Backlund map transforms a surface with constant negative Gauss curvature into another surface with the same constant negative Gauss curvature. Moreover, the Backlund transformation can be restricted to produce a transformation that maps constant torsion curves to other constant torsion curves [6]. Let β and β^* be a smooth curves and Frenet vector fields of the curves β be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. C is constant, τ is torsion and κ is curvature of β. It can be given the relation

$$\beta^*(s) = \beta(s) + ((2C)/(C^2 + \tau^2))((cos\Gamma)\mathbf{T} + (sin\Gamma)\mathbf{N})$$

where, $\beta' = Csin\Gamma - \kappa$. The τ and τ^* torsions of β and β^* respectively constant and are

$$\tau^* = \tau = \frac{sin\Phi}{\lambda}$$

in the above equation λ is the distance between these points and Φ is the angle between binormals of points on the curves that correspond to each other. Studies about Backlund mates can be found in the following articles: [6,26,27].

Our study will involve curve pairs where the vectors connecting corresponding points are always situated in the normal plane. Since we should not confuse these curves with the concepts of normal curves found in the literature, we will refer to these curves as normal pairs.

2. Preliminaries

Now, shortly reminding the Frenet elements of curves. Let's $\delta: I \to E^3$ a regular curve, $v = \|\delta'\| \neq 0$. The tangent, binormal and normal vector fields of δ are expressed as follows

$$\mathbf{T} = \frac{\delta'}{v}, \quad \mathbf{B} = \frac{\delta' \times \delta''}{\|\delta' \times \delta''\|} \text{ and } \mathbf{N} = \mathbf{B} \times \mathbf{T}$$

in order of. Additionally, curvature and torsion of the curve δ are

$$\kappa = \frac{\|\delta' \times \delta''\|}{\|\delta'\|^3} \text{ and } \tau = \frac{det(\delta', \delta'', \delta''')}{\|\delta' \times \delta''\|^2}.$$

Frenet-Serret formula for the curve δ is

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & v\kappa & 0 \\ -v\kappa & 0 & v\kappa \\ 0 & -v\kappa & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

The planes spanned by $\{\mathbf{N},\mathbf{B}\}$ $\{\mathbf{T},\mathbf{B}\}$ and $\{\mathbf{T},\mathbf{N}\}$ are called respectively the normal, rectifying and osculating plane at each point of a curve. The focus of this paper is on curve pairs, where lines connecting corresponding points are found within their normal planes.

Main Theorem and Proof

In this section, we will introduce a new curve pairs that can generalize the famous curve pairs briefly expressed in the previous sections.

3.1. Normal Mates in the Euclidean 3-space

Definition 3.1. Let's map the corresponding points of the curves P and P^* given on the space E^3 and define the curve

$$P^*(s) = P(s) + \lambda(s)\vec{\mathbf{w}}(s)$$

where $\vec{\mathbf{w}}$ is a unit vector and λ is the distance function.

The vector $\vec{\mathbf{w}}$ lies on the line where the normal planes of the curves P and P^* intersect. If the vector $\vec{\mathbf{w}}$ makes an angle Γ with the binormal vector fields at points on the curves P and P^* that correspond to each other, then the pair of curves $\{P,P^*\}$ is termed the Normal pair of curve or Normal mates. In this place, P and P^* are called normal curve and equinormal curve, in order of. Moreover, $P^*(s)$ is expressed as the normal mate of $P(s)$

Theorem 3.2. Let $\{P,P^*\}$ be the Normal mate. $\mathbf{B}(s)$ and $\mathbf{N}(s)$ being the binormal and normal vector fields of the curve $P(s)$, in order of. It is written as follows

$$P^*(s) = P(s) + \lambda(s)((cos\Gamma(s))\mathbf{B}(s) + (sin\Gamma(s))\mathbf{N}(s)) \tag{3.1}$$

where $\lambda(s) \neq 0$ is the distance function between curves P and P^*. Hence, the following equation is obtained

$$0 = \lambda' \tag{3.2}$$

to be $v = \|P'\|$, $v^* = \|P^{*\prime}\|$.

Proof If P, P^* is an Normal curve mate, then the vector $\lambda w(s) = P^*(s) - P(s)$ will make an angle $\Gamma(s)$ with binormal vector fields $\mathbf{B}(s)$ and $\mathbf{B}^*(s)$ along the curves P and P^*. Moreover, given that the unit vector $\mathbf{w}(s)$ is in the normal plane. We can express the following equations

$$\vec{w} = (cos\Gamma)\mathbf{B} + (sin\Gamma)\mathbf{N}$$

and

$$P^* = P + \lambda((cos\Gamma)\mathbf{B} + (sin\Gamma)\mathbf{N}).$$

Now let's take the derivative of equation (3.1). Then we find the following equation

$$v^{*}\mathbf{T}^{*}=v\mathbf{T}+\lambda'\big((cos\Gamma)\mathbf{B}+(sin\Gamma)\mathbf{N}\big)$$

$$+\lambda(\Gamma'(-sin\Gamma)\mathbf{B}+(cos\Gamma)(-\tau v\mathbf{N})+\Gamma'(cos\Gamma)\mathbf{N}+(sin\Gamma)(-\kappa v\mathbf{T}+\tau v\mathbf{B}) \quad (3.3)$$

$$v^{*}\mathbf{T}^{*}=(v-\kappa v sin\Gamma)\mathbf{T}$$

$$+(\lambda' sin\Gamma-\lambda v\tau cos\Gamma+\lambda\Gamma' cos\Gamma)\mathbf{N}$$

$$+(\lambda' cos\Gamma-\lambda\Gamma' sin\Gamma+\lambda v\tau sin\Gamma)\mathbf{B}. \quad (3.4)$$

If we take the inner product of both sides of equation (3.4) with the vector $\vec{w}$, we get

$$0=(\lambda' cos\Gamma-\lambda\Gamma' sin\Gamma+\lambda v\tau sin\Gamma)cos\Gamma+(\lambda' sin\Gamma-\lambda v\tau cos\Gamma+\lambda\Gamma' cos\Gamma)sin\Gamma$$

$$0=\lambda'.$$

Consequently, we obtain the equality

$$0=\lambda'.$$

3.2. The Relations Between Frenet Apparatus of Pair of Normal Curves

In this section, we will begin by exploring the relationship both between Frenet frames of normal curve and their mates, as well as the curvature and the torsion of these curves. Moreover, we'll analyze whether distance function between the points on the curves P and P^{*}that correspond to each other is constant or not. Furthermore, we will discuss whether the angle Γ remains constant, considering special cases such as when Γ is constant, $\Gamma=\pi/2$ or $\Gamma=0$. We will investigate under which specific conditions we can obtain curve pairs that are either in Mannheim curves, Bertrand curves or other type.

Theorem 3.3. Let P and P^{*} respectively be a regular curves and normal mates of given by the equation

$$P^{*}=P+\lambda\big((cos\Gamma)\mathbf{B}+(sin\Gamma)\mathbf{N}\big). \quad (4.1)$$

Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T}^{*},\mathbf{N}^{*},\mathbf{B}^{*}\}$ be Frenet vector fields of curves P and P^{*}, in order of. Provided that the angle Φ is between the opposite tangent of the curves, $\mu=1-cos\Phi$. We obtain the matrix equation

$$\begin{bmatrix}\mathbf{T}^{*}\\ \mathbf{N}^{*}\\ \mathbf{B}^{*}\end{bmatrix}=\begin{bmatrix}cos\Phi & -sin\Phi cos\Gamma & sin\Phi sin\Gamma\\ cos\Gamma sin\Phi & 1-\mu cos^{2}\Gamma & \mu cos\Gamma sin\Gamma\\ -sin\Gamma sin\Phi & \mu sin\Gamma cos\Gamma & 1-\mu sin^{2}\Gamma\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} \quad (4.2)$$

Proof Since the vector field

$$\vec{\mathbf{w}}=(cos\Gamma)\mathbf{B}+(sin\Gamma)\mathbf{N}$$

lying on the normal plane, this vector is perpendicular to the tangent vector fields $\mathbf{T}$ and $\mathbf{T}^*$. Hence, $\mathbf{v} = \mathbf{T} \times \mathbf{w}$ and $\mathbf{v}^* = \mathbf{T}^* \times \mathbf{w}$ lies on the normal planes of the curves. Thus, every one of the collections $\{\mathbf{w}, \mathbf{v}^*, \mathbf{T}^*\}$ and $\{\mathbf{w}, \mathbf{v}, \mathbf{T}\}$ forms an orthonormal frame for the curves P and P^*, respectively. The following Matrix equations can be obtained

$$\begin{bmatrix} \vec{\mathbf{w}} \\ \vec{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} cos\Gamma & sin\Gamma \\ -sin\Gamma & cos\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{B}^* \\ \mathbf{N}^* \end{bmatrix} \tag{4.3}$$

and

$$\begin{bmatrix} \vec{\mathbf{w}} \\ \vec{\mathbf{v}}^* \end{bmatrix} = \begin{bmatrix} cos\Gamma & sin\Gamma \\ -sin\Gamma & cos\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{N} \end{bmatrix}. \tag{4.4}$$

Let Φ be the angle between $\mathbf{T}^*$ and $\mathbf{T}$. Since all of the vectors $\mathbf{T}^*$, $\vec{\mathbf{v}}^*$, $\mathbf{T}$, $\vec{\mathbf{v}}$ are perpendicular to the $\vec{\mathbf{w}}$. Provided that the equation (4.3) was used, we get

$$\begin{aligned} \mathbf{T}^* &= (sin\Phi)\vec{\mathbf{v}} + (cos\Phi)\mathbf{T} \\ &= (cos\Phi)\mathbf{T} - (sin\Phi cos\Gamma)\mathbf{N} + (sin\Phi sin\Gamma)\mathbf{B} \end{aligned}$$

and

$$\begin{aligned} \vec{\mathbf{v}}^* &= (cos\Phi)\vec{\mathbf{v}} + (sin\Phi)\mathbf{B} \\ &= (sin\Phi)\mathbf{T} + (cos\Phi cos\Gamma)\mathbf{N} - cos\Phi sin\Gamma\mathbf{B}. \end{aligned}$$

Thus, using the equlaties (4.4) and (4.3) we have

$$\begin{aligned} \begin{bmatrix} \mathbf{B}^* \\ \mathbf{N}^* \end{bmatrix} &= \begin{bmatrix} cos\Gamma & -sin\Gamma \\ sin\Gamma & cos\Gamma \end{bmatrix} \begin{bmatrix} \vec{\mathbf{w}} \\ \vec{\mathbf{v}} \end{bmatrix} \\ &= \begin{bmatrix} -(sin\Gamma sin\Phi)\mathbf{T} + sin\Gamma cos\Gamma(1 - cos\Phi)\mathbf{N} + (1 + sin^2\Gamma(cos\Phi - 1))\mathbf{B} \\ (cos\Gamma sin\Phi)\mathbf{T} + (1 + cos^2\Gamma(cos\Phi - 1))\mathbf{N} + (sin\Gamma cos\Gamma)(1 - cos\Phi)\mathbf{B} \end{bmatrix}. \end{aligned}$$

As a result, (4.2) equality is obtained where $1 - \mu = cos\Phi$.

3.3. Curvature and Torsion of The Normal Curve *P*

Theorem 3.4. Let P and P^* respectively be a reguler curves and normal mates with the relation $P^* = P + \lambda\big((cos\Gamma)\mathbf{B} + (sin\Gamma)\mathbf{N}\big)$ where $\Gamma \neq 0, \Gamma \neq \pi/2$ and the distance function λ. Provided that the angle $\Phi \neq 0$ is between binormal of the corresponding points of P and P^*, in this case the curvature and torsion of the $P(s)$ curve is

$$\begin{aligned} \kappa &= \frac{v - v^* cos\Phi}{\lambda v sin\Gamma} \\ \tau &= \frac{v^* sin\Phi + \Gamma'\lambda}{\lambda v} \end{aligned} \tag{4.5}$$

respectively, where $\mu = 1 - cos\Phi$.

Proof If we compare the equality (3.4) and (4.1)

$$v^*\mathbf{T}^* = (v - \lambda\kappa v sin\Gamma)\mathbf{T}$$

$$+(\lambda' \sin\Gamma - \lambda v\tau\cos\Gamma + \lambda\Gamma'\cos\Gamma)\mathbf{N}$$

$$+(\lambda'\cos\Gamma - \lambda\Gamma'\sin\Gamma + \lambda v\tau\sin\Gamma)\mathbf{B}.$$

Let's take the inner product of both sides of the above equation with **T**, **N** and **B** respectively, then the following equality is obtained

$$\begin{aligned} v^*(cos\Phi) &= (v - \lambda\kappa v sin\Gamma) \\ -v^* sin\Phi cos\Gamma &= (\lambda' sin\Gamma - \lambda v\tau cos\Gamma + \lambda\Gamma' cos\Gamma) \\ v^* sin\Phi sin\Gamma &= (\lambda' cos\Gamma - \lambda\Gamma' sin\Gamma + \lambda v\tau sin\Gamma). \end{aligned} \tag{4.6}$$

In terms of $\Gamma \neq 0, \pi/2$,thinking $0 = \lambda'$ and $\mu = 1 - cos\Phi$, we find

$$-v^* sin\Phi cos\Gamma = (\lambda' sin\Gamma - \lambda v\tau cos\Gamma + \lambda\Gamma' cos\Gamma)$$

$$\tau = \frac{v^* sin\Phi + \Gamma'\lambda}{\lambda v}$$

and

$$v^*(cos\Phi) = (v - \lambda\kappa v sin\Gamma)$$

$$\kappa = \frac{v - v^* cos\Phi}{\lambda v sin\Gamma}.$$

Proposition 3.5. Let it be

$$R_\Phi = \begin{bmatrix} cos\Phi & -cos\Gamma sin\Phi & sin\Gamma sin\Phi \\ cos\Gamma sin\Phi & 1 - \mu cos^2\Gamma & \mu cos\Gamma sin\Gamma \\ -sin\Gamma sin\Phi & \mu sin\Gamma cos\Gamma & 1 - \mu sin^2\Gamma \end{bmatrix}$$

is a rotation matrix for any normal pair of the curves. R_Φ rotates a vector through the angle Φ around the axis $\vec{\mathbf{u}} = (0, sin\Gamma, cos\Gamma)$. The unit quaternion q corresponds to R_Φ. Then

$$q = cos\frac{\Phi}{2} + sin\Gamma sin\frac{\Phi}{2}\mathbf{j} + cos\Gamma sin\frac{\Phi}{2}\mathbf{k}.$$

Proof

$$R_{\Phi}=\begin{bmatrix} 1-2(1-a^2)sin^2\frac{\Phi}{2} & ab(1-cos\Phi)-csin\Phi & bsin\Phi+ac(1-cos\Phi) \\ ab(1-cos\Phi)+csin\Phi & 1-2(1-b^2)sin^2\frac{\Phi}{2} & -asin\Phi+bc(1-cos\Phi) \\ -bsin\Phi+ac(1-cos\Phi) & asin\Phi+bc(1-cos\Phi) & 1-2(1-c^2)sin^2\frac{\Phi}{2} \end{bmatrix}$$

is a rotation matrix. R_{Φ} rotates a vector through the angle Φ around the axis $\vec{\mathbf{u}}=(a,b,c)$ and the unit quaternion q corresponds to R_{Φ} is

$q=cos\frac{\Phi}{2}+(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})sin\frac{\Phi}{2}$ [31].

If we have $\vec{\mathbf{u}}=(0,sin\Gamma,cos\Gamma)$ the rotation matrix is

$$R_{\Phi}=\begin{bmatrix} cos\Phi & -cos\Gamma sin\Phi & sin\Gamma sin\Phi \\ cos\Gamma sin\Phi & 1-\mu cos^2\Gamma & \mu sin\Gamma cos\Gamma \\ -sin\Gamma sin\Phi & \mu cos\Gamma sin\Gamma & 1-\mu sin^2\Gamma \end{bmatrix}$$

and the unit quaternion q corresponds to R_{Φ}. Then

$$q=cos\frac{\Phi}{2}+sin\Gamma sin\frac{\Phi}{2}\mathbf{j}+cos\Gamma sin\frac{\Phi}{2}\mathbf{k}.$$

Theorem 3.6. Let P and P^* respectively be a reguler curves and normal mates with the relation $P^*=P+\lambda\big((cos\Gamma)\mathbf{B}+(sin\Gamma)\mathbf{N}\big)$ where $\Gamma\neq 0,\pi/2$ and the distance function λ. Provided that the angle $\Phi\neq 0$ between tangent of the corresponding points of P and P^*, in this case the curvature and torsion of the P^* (s) curve is

$$\begin{aligned} \kappa^* &= \frac{v-v^*cos\Phi}{v^*sin\Gamma}-\frac{\Phi'}{v^*cos\Gamma}, \\ \tau^* &= \frac{\Gamma'+\Phi'tan\Gamma cot\Phi}{v^*}-\frac{v\kappa sin\Gamma}{v^*sin\Phi} \end{aligned} \tag{4.7}$$

respectively, where $\mu=1-cos\Phi$.

Proof Now let's take the derivative of $\mathbf{T}^*$ in equation (4.2). Then we find the following equation

$$\mathbf{T}^*=(cos\Phi)\mathbf{T}-sin\Phi cos\Gamma\mathbf{N}+(sin\Phi sin\Gamma)\mathbf{B}$$

$$\begin{aligned} v^*\kappa^*\mathbf{N}^* &= (-\Phi' sin\Phi+v\kappa sin\Phi cos\Gamma)\mathbf{T} \\ &+(v\kappa cos\Phi-\Phi' cos\Phi cos\Gamma+\Gamma' sin\Phi sin\Gamma-v\tau sin\Phi sin\Gamma)\mathbf{N} \\ &+(-v\tau sin\Phi cos\Gamma+\Phi' cos\Phi sin\Gamma+\Gamma' sin\Phi cos\Gamma)\mathbf{B}. \end{aligned} \tag{4.8}$$

Let’s the inner product of both sides of equation (4.8) with the $\mathbf{T}$, we get

$$v^*\kappa^* cos\Gamma = (-\Phi' + v\kappa cos\Gamma) \tag{4.9}$$

since $< \mathbf{N}^*, \mathbf{T} >= sin\Phi cos\Gamma$. Thus, we get

$$v^*\kappa^* = v\kappa - \frac{\Phi'}{cos\Gamma}$$

or $\Gamma \neq 0$ and $\Gamma \neq \pi/2$. Moreover, in terms of $\Phi \neq 0$, we write $\mu = 1 - cos\Phi$. Hence $\mu' = \Phi' sin\Phi$ and

$$\frac{\mu'}{\mu} = \frac{\Phi' sin\Phi}{1 - cos\Phi} = \frac{\Phi'(1 + cos\Phi}{sin\Phi}.$$

Using these equalities, we obtain

$$v^*\kappa^* = v\kappa - \frac{\Phi'}{cos\Gamma} \tag{4.10}$$

$$\kappa^* = \frac{v\kappa}{v^*} - \frac{\Phi'}{v^* cos\Gamma} \tag{4.11}$$

$$\kappa^* = \frac{v - v^* cos\Phi}{v^*\lambda sin\Gamma} - \frac{\Phi'}{v^* cos\Gamma}\ . \tag{4.12}$$

Let's the inner product of both sides of equation (4.8) with the **B**, we get

$$v^*\kappa^*\mu cos\Gamma sin\Gamma = -v\tau sin\Phi cos\Gamma + \Phi' cos\Phi sin\Gamma + \Gamma' sin\Phi cos\Gamma$$

$$v^*\kappa^* = \frac{-v\tau sin\Phi + \Phi' cos\Phi tan\Gamma + \Gamma' sin\Phi}{\mu sin\Gamma}$$

since $< \mathbf{N}^*, \mathbf{B} >= \mu cos\Gamma sin\Gamma$. Let's write $\tau = \frac{v^* sin\Phi + \Gamma'\lambda}{v\lambda}$ in this equation. Thus, we have

$$v^*\kappa^* = \frac{-v\tau sin\Phi + \Phi' cos\Phi tan\Gamma + \Gamma' sin\Phi}{\mu sin\Gamma} \tag{4.13}$$

$$\kappa^* = \frac{-v\tau sin\Phi + \Phi' cos\Phi tan\Gamma + \Gamma' sin\Phi}{v^*\mu sin\Gamma}\ . \tag{4.14}$$

Additionally, by using equations (4.10) and (4.13) we obtain the following equation

$$\frac{v - v^* cos\Phi}{v^*\lambda sin\Gamma} - \frac{\Phi'}{v^* cos\Gamma} = \frac{-v\tau sin\Phi + \Phi' cos\Phi tan\Gamma + \Gamma' sin\Phi}{v^*\mu sin\Gamma}$$

$$\mu v - \mu v^* cos\Phi - \mu\Phi' tan\Gamma = -\frac{v^* sin\Phi + \Gamma'\lambda}{\lambda} sin\Phi + \Phi' cos\Phi tan\Gamma + \Gamma' sin\Phi.$$

If we substitute the κ curvature expressed in Theorem 4.2 into the equation above, we obtain the following equation

$$\frac{\lambda\Phi'}{sin\Phi} = (v^* - v)cos\Gamma.$$

Thus, considering the equality $0 = \lambda'$ we find

$$\lambda = -\frac{v^* sin\Phi}{v - v^* cos\Phi + \mu\Phi' tan\Gamma} + c \qquad (4.15)$$

with $c \in \mathbb{R}$. The equation (4.15) shows that when Φ is constant, λ will also be constant. If we write equation (4.15) into equation (4.1), we obtain

$$P^* = P + \left(-\frac{v^* sin\Phi}{v - v^* cos\Phi + \mu\Phi' tan\Gamma} + c\right)\left((cos\Gamma)\mathbf{B} + (sin\Gamma)\mathbf{N}\right). \qquad (4.16)$$

If take derivative of $\mathbf{B}^*$ expressed in (4.2). Then, we obtain

$$\mathbf{B}^* = -(sin\Gamma sin\Phi)\mathbf{T} + \mu sin\Gamma cos\Gamma\mathbf{N} + (1 - \mu sin^2\Gamma)\mathbf{B}$$

$$-v^*\tau^*\mathbf{N}^* = -(\Gamma' cos\Gamma sin\Phi + \Phi' sin\Gamma cos\Phi + v\kappa\mu sin\Gamma cos\Gamma)\mathbf{T}$$

$$\left(-v\kappa sin\Gamma sin\Phi + \mu' sin\Gamma cos\Gamma + \mu\Gamma' cos^2\Gamma - \mu\Gamma' sin^2\Gamma - v\tau(1 - \mu sin^2\Gamma)\right)\mathbf{N}$$

$$(v\tau\mu sin\Gamma cos\Gamma - \mu' sin^2\Gamma - 2\mu\Gamma' sin\Gamma cos\Gamma)\mathbf{B}.$$

Using the equality $< \mathbf{N}^*, \mathbf{T} >= cos\Gamma sin\Phi$ we obtain

$$v^*\tau^* cos\Gamma sin\Phi = \Gamma' cos\Gamma sin\Phi + \Phi' sin\Gamma cos\Phi + v\kappa\mu sin\Gamma cos\Gamma$$

$$v^*\tau^* = \Gamma' + \Phi' tan\Gamma cot\Phi + \frac{v\kappa\mu sin\Gamma}{v^* sin\Phi}$$

$$\tau^* = \frac{\Gamma'}{v^*} + \frac{\Phi' tan\Gamma cot\Phi}{v^*} + \frac{v\kappa\mu sin\Gamma}{v^* sin\Phi}$$

for $\Phi \neq 0$.

4. Conclusion

4.1. Both *Φ* and *Γ* are non-constant

Let $\{P, P^*\}$ be an Normal mate of curves. Provided that the angle $\Phi \neq 0$ between binormal of the corresponding points of P and P^*, we have the equation (4.16) for $\Gamma \neq 0, \pi/2$. The curvatures and torsions of P and P^* are

$$\tau = \frac{v^* sin\Phi + \Gamma'\lambda}{v\lambda}, \qquad \tau^* = \frac{\Gamma' + \Phi' tan\Gamma cot\Phi}{v^*} - \frac{v\kappa\mu sin\Gamma}{v^* sin\Phi},$$
$$\kappa = \frac{v - v^* cos\Phi}{\lambda v sin\Gamma}, \qquad \kappa^* = \frac{v\kappa}{v^*} - \frac{\Phi'}{v^* cos\Gamma}.$$

Now, let's analyze special cases.

4.2. Φ is constant and Γ is non-constant with $\Gamma \neq 0, \Gamma \neq \pi/2$.

If Φ is constant, $\lambda = -\frac{v^* sin\Phi}{v - v^* cos\Phi + \mu\Phi' tan\Gamma} + c$ and λ will also be constant. So, if $\Gamma \neq \pi/2$, the equality $0 = \lambda'$. In this case the curvatures and torsions of P and P^* are

$$\tau = \frac{v^* sin\Phi + \Gamma'\lambda}{v\lambda}, \qquad \tau^* = \frac{\Gamma'}{v^*} - \frac{v\kappa sin\Gamma}{v^* sin\Phi},$$
$$\kappa = \frac{v - v^* cos\Phi}{\lambda v sin\Gamma}, \qquad \kappa^* = \frac{v\kappa}{v^*}. \tag{4.17}$$

4.3. Φ is non-constant and Γ are constant

Let $\{P, P^*\}$ be a Normal mate of curves. Provided that the angle $\Phi \neq 0$ between binormal of the corresponding points of P and P^*, we have the equation (4.16) for $\Gamma \neq 0, \pi/2$. The curvatures and torsions of P and P^* are

$$\tau = \frac{v^* sin\Phi}{v\lambda}, \qquad \tau^* = \frac{\Phi' tan\Gamma cot\Phi}{v^*} - \frac{v\kappa sin\Gamma}{v^* sin\Phi},$$
$$\kappa = \frac{v - v^* cos\Phi}{\lambda v sin\Gamma}, \qquad \kappa^* = \frac{v\kappa}{v^*} - \frac{\Phi'}{v^* cos\Gamma}. \tag{4.18}$$

4.4. Both Φ and Γ are constant

Let $\{P, P^*\}$ be an Normal mate of curves. Provided that the angle $\Phi \neq 0, \pi/2$ between binormal of the corresponding points of P and P^*, we have the equation (4.16) for $\Gamma \neq 0, \pi/2$. The curvatures and torsions of P and P^* are

$$\tau = \frac{v^* sin\Phi + \Gamma'\lambda}{v\lambda}, \qquad \tau^* = -\frac{v\kappa sin\Gamma}{v^* sin\Phi},$$
$$\kappa = \frac{v - v^* cos\Phi}{\lambda v sin\Gamma}, \qquad \kappa^* = \frac{v\kappa}{v^*}. \tag{4.19}$$

Now, let's analyze spacial cases.

4.4.1. $\Phi \neq 0, \pi/2$ and $\Gamma = 0$ case

From (4.19) we have

$$\tau = \frac{v^* sin\Phi + \Gamma'\lambda}{v\lambda}, \qquad \tau^* = 0,$$
$$\kappa^* = \frac{v\kappa}{v^*}.$$

4.4.2. $\Phi \neq 0, \pi/2$ and $\Gamma = \pi/2$. (In the case $\{P, P^*\}$ is a Manheim mate)

λ is constant, then

$P^* = P + \lambda N.$

Moreover, with respect to the equality (3.4) and (4.2) we obtain

$v^*\mathbf{T}^* = (v - v\kappa\lambda)\mathbf{T} + v\lambda\tau\mathbf{B}$

$$\begin{bmatrix}\mathbf{T}^*\\ \mathbf{N}^*\\ \mathbf{B}^*\end{bmatrix} = \begin{bmatrix} cos\Phi & 0 & sin\Phi\\ 0 & 1 & 0\\ -sin\Phi & 0 & cos\Phi\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}.$$

Thus, since $P^* = P + \lambda N$, $\lambda \in \mathbb{R}$ $\{P, P^*\}$ is a Mannheim mate. So, using the special case given above, we express

$$\tau = \frac{v^* sin\Phi}{v\lambda}, \qquad \tau^* = -\frac{v\kappa}{v^* sin\Phi},$$
$$\kappa = \frac{v - v^* cos\Phi}{\lambda v}, \qquad \kappa^* = \frac{v\kappa}{v^*}. \tag{4.20}$$

for Mannheim curve P.

Therefore, from equations (4.20) we obtain the following expressions for Mannheim mates

- The product of the twist torsions of Mannheim curves points on the curves P and P^* that correspond to each other must be constant, hence

  $$\tau\tau^* = \frac{\kappa}{\lambda}$$

  from (4.20).

- If a and b are non-zero constants and $a\kappa + b\tau = 1$, this curve is a Mannheim curve. Hence

  $$cot\Phi = \frac{1 - \kappa\lambda}{\tau\lambda} \Leftrightarrow \kappa\lambda + (cot\Phi)\lambda\tau = 1$$

  from (4.20). It means that, $a = \lambda$ and $b = \lambda cot\Phi$.

4.4.3. $\Phi = \pi/2$ and $\Gamma \neq 0, \pi/2$ case

From (4.19) and λ is constant we have

$$\tau = \frac{v^*}{v\lambda}, \qquad \tau^* = -\frac{v}{\lambda v^*},$$
$$\kappa = \frac{1}{\lambda sin\Gamma}, \qquad \kappa^* = \frac{v}{v^*\lambda sin\Gamma}.$$

4.4.4. $\Phi = \pi/2$ and $\Gamma = \pi/2$ case

From (3.1) we have

$$P^* = P + \lambda N.$$

From (4.19) we have

$$\tau = \frac{v^*}{v\lambda}, \qquad \tau^* = -\frac{v}{\lambda v^*},$$
$$\kappa = \frac{1}{\lambda}, \qquad \kappa^* = \frac{v\kappa}{v^*}.$$

$$\tau\tau^* = -\frac{1}{\lambda^2}, \lambda\kappa + \mu\tau = 1.$$

Ethics in publishing

There are no ethical issues regarding the publication of this study.

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