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New Perspectives on Fractional Milne-Type Inequalities: Insights from Twice-Differentiable Functions

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Abstract

This paper delves into an inquiry that centers on the exploration of fractional adaptations of Milne-type inequalities by employing the framework of twice-differentiable convex mappings. Leveraging the fundamental tenets of convexity, Hölder's inequality, and the power-mean inequality, a series of novel inequalities are deduced. These newly acquired inequalities are fortified through insightful illustrative examples, bolstered by rigorous proofs. Furthermore, to lend visual validation, graphical representations are meticulously crafted for the showcased examples.

1. Introduction

Convex functions and the inequalities that describe their properties are fundamental concepts in mathematical optimization and economic theory, among other fields. These functions are characterized by the shape of their graph, which curves such that any line segment between two points on the graph does not fall below it. This curvature leads to interesting and useful properties, particularly in the realm of optimization where they guarantee local minima are also global minima. Inequalities related to convex functions, such as Jensen's inequality, play a crucial role in various analytical and theoretical proofs. They are also instrumental in establishing conditions for optimality and convergence in more complex scenarios. Understanding these functions and their associated inequalities provides a solid foundation for delving into more advanced topics in mathematics and economics. Now, let's define the basic notion of a convex function to further grasp the essence of these intriguing concepts.

Definition 1.1 ([1]). *Let I be convex set on $\mathbb{R}$. The function $f: I \to \mathbb{R}$ is called convex on I, if it satisfies the following inequality:*

$$f(tx+(1-t)\,y) \leq t f(x)+(1-t)f(y) \tag{1.1}$$

for all $(x,y) \in I$ and $t \in [0,1]$. The mapping f is a concave on I if the inequality (1.1) holds in reversed direction for all $t \in [0,1]$ and $x,y \in I$.

Within the domain of mathematical analysis, inequalities serve as pivotal tools for examining the intricate nuances of numerical relationships. These inequalities provide a framework for exploring the dynamic interplay between quantities, shedding light on the disparities that permeate mathematical landscapes. As researchers endeavor to unveil the mysteries of mathematical systems, inequalities offer a lens through which the fundamental variations between values can be rigorously scrutinized. Their presence underscores the recognition that the mathematical continuum is far from a monolithic entity; rather, it is a rich tapestry of gradations and magnitudes. By meticulously navigating the terrain of inequalities, scholars gain insights into the profound structure and underlying principles governing mathematical phenomena.
In recent times, the attention of researchers has turned significantly toward diverse classes of integral inequalities, including types like Trapezoid, Midpoint, and Simpson inequalities. Numerous scholars have made substantial contributions to extending and generalizing these fundamental inequalities. For instance, noteworthy progress has been achieved by Dragomir and Agarwal in investigating error estimates for the trapezoidal formula, as highlighted in [2]. The variations of the trapezoid formula's boundedness were explored by Dragomir in [3]. Additionally, Sarikaya and Aktan, in their work [4], delved into novel inequalities of both the Simpson and Trapezoid types, focusing particularly on functions characterized by a convex absolute value of the second derivative. Fractional trapezoid-type inequalities found their

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exploration in [5, 6]. Kırmacı, in [7] introduced midpoint-type inequalities tailored for differentiable convex functions, while Sarıkaya and colleagues derived an array of fresh inequalities suitable for twice differentiable functions as expounded in [8]. The fractional counterparts of these findings are also comprehensively discussed in [9, 10]. Moreover, a series of mathematical luminaries have established results applicable to twice differentiable convex functions, exemplified by works such as [11–13].

Explorations within the realm of numerical integration and the establishment of error bounds have assumed a pivotal position within the tapestry of mathematical literature. Furthermore, scholars have meticulously examined the error bounds of functions that exhibit diverse levels of differentiability from once to multiple times. The spectrum of mathematical inequalities, including those of Simpson, Newton, and Milne types, emerges with distinct purposes and applications. These inequalities find their roles reverberating across various domains of mathematics and numerical analysis, facilitating meticulous scrutiny and enhancement of the efficiency and accuracy of computational techniques. Remarkably, this journey doesn't halt here a multitude of researchers have ventured into uncharted territories, harnessing the potency of fractional calculus to derive novel bounds, expanding the frontiers of understanding and application in this intricate mathematical terrain.

The Milne-type inequality delves into the realm of mathematical analysis within the context of a function's behavior over a closed interval. This inequality offers insights into the intricate connections between a function's values at the endpoints of the interval, its integral over the interval, and the fourth derivative of the function. It forms a crucial bridge between differentiability and integration, highlighting the delicate interplay between these mathematical concepts. The essence of this inequality lies in its ability to encapsulate the behavior of a function in terms of its derivatives and integrals, offering a powerful tool for understanding and quantifying the relationships within the mathematical landscape.

The Milne-type inequality, an essential mathematical inequality within the realm of integral estimation, draws its nomenclature from the distinguished British mathematician Edward Arthur Milne, who bestowed this inequality upon the mathematical community during the early 20th century. The roots of this inequality trace back to the strategic interplay between integral values and specific points, enabling an upper boundary to be established.

Significant strides in Milne-type inequality research have been witnessed, Budak et al. [14] elegantly derived Milne-type inequalities for differentiable convex functions through the application of Riemann-Liouville fractional integrals. Inspired by their contributions to literature, our current study embarks on a journey of exploration, aiming to unveil novel inequalities by harnessing the potential of Riemann-Liouville integrals to characterize twice-differentiable functions.

Our endeavor begins with a thorough review of the established definitions underpinning the Milne-type inequality and the Riemann-Liouville integral. These definitions, widely recognized and foundational within the literature, lay the groundwork for our research.

Furthermore, delving into the context of Newton-Cotes formulas reveals intriguing parallels between Milne's formula, an open-type variant, and the Simpson's formula, representing the closed-type counterpart. These similarities emerge as a result of both formulas adhering to identical conditions. Consider a function $f:[a,b]\to\mathbb{R}$, which boasts four times continuous differentiability over the open interval (a,b). Here, the expression $\left\|f^{(4)}\right\|_{\infty}=\sup_{\upsilon\in(a,b)}\left|f^{(4)}(\upsilon)\right|<\infty$, signifies the supremum of the absolute values of the fourth derivative, symbolizing the upper echelon of its variations. Under these stipulated conditions, our pursuit culminates in the emergence of the Milne-type inequality:

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(\upsilon)d\upsilon\right|\leq\frac{7(b-a)^4}{23040}\left\|f^{(4)}\right\|_{\infty}$$

This compelling result, as unveiled in [15], substantiates the elegant interplay between mathematical constructs in the exploration of Milne-type inequalities. For more studies on Milne type inequalities, you can refer to [16–18].

Definition 1.2 ([19])**.** *Let us consider a function f belonging to the space $L_1[a,b]$. Within this context, we introduce the Riemann-Liouville fractional integrals, denoted as $\mathscr{J}^{\alpha}_{(a)^+}f$ and $\mathscr{J}^{\alpha}_{(b)^-}f$, where $\alpha>0$, by invoking the following equalities:*

$$\mathscr{J}^{\alpha}_{(a)^+}f(\upsilon)=\frac{1}{\Gamma(\alpha)}\int_a^{\upsilon}(\upsilon-t)^{\alpha-1}f(t)dt,\quad \upsilon>a,$$

which represents the definitive expression of the left-sided Riemann-Liouville fractional integral of function f with order α at the point a, and

$$\mathscr{J}^{\alpha}_{(b)^-}f(\upsilon)=\frac{1}{\Gamma(\alpha)}\int_{\upsilon}^{b}(t-\upsilon)^{\alpha-1}f(t)dt,\quad \upsilon<b,$$

depicting the clear-cut equation that defines the right-sided Riemann-Liouville fractional integral of function f with order α at the point b. It is pertinent to note that $\Gamma(\alpha)$ represents the Gamma function, and $\mathscr{J}^{0}_{(a)^+}f(\upsilon)=\mathscr{J}^{0}_{(b)^-}f(\upsilon)=f(\upsilon)$ in accordance with the defined context.

For an in-depth exploration into the intricacies of Riemann-Liouville fractional integrals, we refer the interested reader to [19–21]. Armed with this foundational understanding from the existing literature, we embark on the journey to unveil our novel contributions in the subsequent sections.

2. Main Results

In this study, we will initially derive an equation for twice differentiable functions. By taking the absolute value of this equation and employing convexity, we will establish an inequality. Furthermore, leveraging Hölder's and the power mean inequalities, we will deduce novel inequalities.

Lemma 2.1. *Consider a mapping $f:[a,b]\to\mathbb{R}$ that is twice differentiable on the interval (a,b) and satisfies $f''\in L_1([a,b])$. Under these conditions, the subsequent lemma establishes the following equality:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\\&=\frac{(b-a)^2}{8(\alpha+1)}\int_0^1\left(t^{\alpha+1}-\frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right]dt.\end{aligned}$$

Proof. Applying the method of integration by parts, we are able to derive the subsequent expression:

$$\begin{aligned}
I_1 &= \int_0^1 \left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right) f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt \\
&= \frac{2}{b-a}\left[\left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right) f'\left((1-t)a + t\frac{a+b}{2}\right)\right]\Bigg|_0^1 - \frac{2(\alpha+1)}{b-a}\int_0^1 \left(t^{\alpha} - \frac{4}{3}\right) f'\left((1-t)a + t\frac{a+b}{2}\right) dt \\
&= \frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] - \frac{2(\alpha+1)}{b-a}\left\{\int_0^1 \left(t^{\alpha} - \frac{4}{3}\right) f'\left((1-t)a + t\frac{a+b}{2}\right) dt\right\} \\
&= \frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] - \frac{2(\alpha+1)}{b-a}\left\{-\frac{2}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{8}{3(b-a)} f(a) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)\right\} \\
&= \frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] + \frac{4(\alpha+1)}{3(b-a)^2} f\left(\frac{a+b}{2}\right) - \frac{16(\alpha+1)}{3(b-a)^2} f(a) + \frac{2^{\alpha+2}\Gamma(\alpha+1)(\alpha+1)}{(b-a)^{\alpha+2}} \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a).
\end{aligned}$$

Likewise, we acquire

$$\begin{aligned}
I_2 &= \int_0^1 \left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right) f''\left(\frac{2-t}{2}b + \frac{t}{2}a\right) dt \\
&= -\frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] + \frac{4(\alpha+1)}{3(b-a)^2} f\left(\frac{a+b}{2}\right) - \frac{16(\alpha+1)}{3(b-a)^2} f(b) + \frac{2^{\alpha+2}\Gamma(\alpha+1)(\alpha+1)}{(b-a)^{\alpha+2}} \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b).
\end{aligned}$$

Subsequently, we can observe the following computation:

$$\frac{(b-a)^2}{8(\alpha+1)}[I_1 + I_2] = \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right].$$

Hence, this concludes the proof. □

Theorem 2.2. *Assuming $f:[a,b]\to\mathbb{R}$ is a function with twice differentiable function on the open interval (a,b), and $f'' \in L_1([a,b])$, with $|f''|$ exhibiting convexity across $[a,b]$, the subsequent inequality is valid:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right]\right| \\
&\leq \frac{(b-a)^2\left(2(\alpha+1)(\alpha+2)-3\right)}{24(\alpha+1)(\alpha+2)}\left[|f''(a)| + |f''(b)|\right].
\end{aligned} \tag{2.1}$$

Proof. Upon applying the absolute value to Lemma 2.1 and leveraging the convex property of $|f''|$, we deduce the following result:

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right]\right| \\
&= \frac{(b-a)^2}{8(\alpha+1)}\left|\int_0^1 \left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right) + f''\left(\frac{2-t}{2}b + \frac{t}{2}a\right)\right] dt\right| \\
&\leq \frac{(b-a)^2}{8(\alpha+1)}\int_0^1 \left|t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right|\left[\frac{2-t}{2}|f''(a)| + \frac{t}{2}|f''(b)| + \frac{2-t}{2}|f''(b)| + \frac{t}{2}|f''(a)|\right] dt \\
&= \frac{(b-a)^2}{8(\alpha+1)}\left[|f''(a)| + |f''(b)|\right]\int_0^1 \frac{4}{3}(\alpha+1)t - t^{\alpha+1} dt \\
&= \frac{(b-a)^2\left(2(\alpha+1)(\alpha+2)-3\right)}{24(\alpha+1)(\alpha+2)}\left[|f''(a)| + |f''(b)|\right].
\end{aligned}$$

The necessary inequality (2.1) is established. □

Corollary 2.3. *The choice $\alpha = 1$ in Theorem 2.2 yields the following result:*

$$\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{16}\left[|f''(a)| + |f''(b)|\right].$$

Example 2.4. *Consider the interval $[a,b]=[0,1]$, and let's define the function $f:[0,1]\to\mathbb{R}$ as $f(t)=\frac{t^4}{12}$, so that $f''(t)=t^2$ and $|f''|$ is convex over the interval $[0,1]$. Given these conditions,*

$$\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] = \frac{31}{576}$$

Using the definition of the Riemann-Liouville fractional integral, we attain

$$\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) = \mathscr{J}^{\alpha}_{\left(\frac{1}{2}\right)^-} f(0) = \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}} t^{\alpha-1}\frac{t^4}{12}dt = \frac{1}{12\Gamma(\alpha)(\alpha+4)2^{\alpha+4}}$$

and

$$\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)=\mathscr{J}^{\alpha}_{\left(\frac{1}{2}\right)^{+}}f(1)=\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{\alpha-1}\frac{t^{4}}{12}dt$$
$$=\frac{\alpha^{4}-50\alpha^{3}+83\alpha^{2}+262\alpha+384}{12\alpha\Gamma(\alpha)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)2^{\alpha+4}}.$$

Hence, we possess

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]\\&=2^{\alpha-1}\Gamma(\alpha+1)\left[\frac{1}{12\Gamma(\alpha)(\alpha+4)2^{\alpha+4}}+\frac{\alpha^{4}-50\alpha^{3}+83\alpha^{2}+262\alpha+384}{12\alpha\Gamma(\alpha)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)2^{\alpha+4}}\right]\\&=\frac{2\alpha^{4}-44\alpha^{3}+94\alpha^{2}+268\alpha+384}{384(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.\end{aligned}$$

Consequently, the left-hand side of inequality (2.1) simplified to

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|\\&=\left|\frac{2\alpha^{4}-44\alpha^{3}+94\alpha^{2}+268\alpha+384}{384(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}-\frac{31}{576}\right|=:LHS.\end{aligned}\tag{2.2}$$

In a similar manner, the right-hand side of inequality (2.1) was brought down to

$$\frac{(b-a)^{2}\left(2(\alpha+1)(\alpha+2)-3\right)}{24(\alpha+1)(\alpha+2)}\left[|f''(a)|+|f''(b)|\right]=\frac{2(\alpha+1)(\alpha+2)-3}{24(\alpha+1)(\alpha+2)}=:RHS.$$

The outcomes from Example 2.4 are illustrated in Figure 2.1.

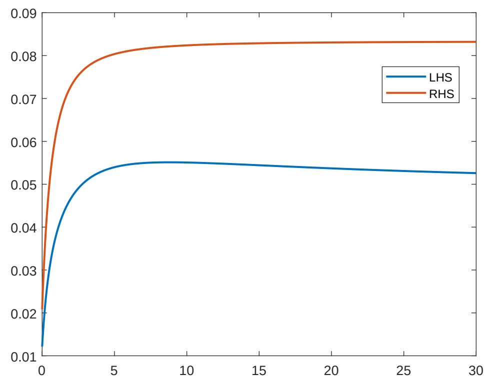

Figure 2.1: Graph of Example 2.4.

Theorem 2.5. *Consider a function $f:[a,b]\to\mathbb{R}$, twice differentiable on the interval (a,b), with $f''\in L_1([a,b])$. Additionally, let $|f''|^q$ exhibit convexity on $[a,b]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. As a result of these conditions, the following inequality is established.*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|\\&\leq\frac{(b-a)^{2}}{8(\alpha+1)}\left(\int_{0}^{1}\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^{p}dt\right)^{\frac{1}{p}}\times\left[\left(\frac{3|f''(a)|^{q}+|f''(b)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^{q}+|f''(a)|^{q}}{4}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.3}$$

Proof. By considering the absolute value in Lemma 2.1, we find that

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|\\&=\frac{(b-a)^{2}}{8(\alpha+1)}\left|\int_{0}^{1}\left(t^{\alpha+1}-\frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right]dt\right|\\&\leq\frac{(b-a)^{2}}{8(\alpha+1)}\left[\int_{0}^{1}\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt+\int_{0}^{1}\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\right].\end{aligned}\tag{2.4}$$

Exploiting the convex nature of $|f''|^q$ and employing the Holder inequality, leads us to the conclusion that

$$\begin{aligned}&\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\\&\leq\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q\right)^{\frac{1}{q}}\\&\leq\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{2-t}{2}|f''(a)|^q+\frac{t}{2}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\\&=\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left[\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right]^{\frac{1}{q}}.\end{aligned}\tag{2.5}$$

Similarly,

$$\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\leq\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left[\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right]^{\frac{1}{q}}.\tag{2.6}$$

Through the incorporation of inequalities (2.5) and (2.6) into (2.4) we arrive at inequality (2.3), thus finalizing the proof. □

Corollary 2.6. *When $\alpha=1$, based on Theorem 2.5, we get*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\leq\frac{(b-a)^2}{16}\left(\int_0^1\left(\frac{8}{3}t-t^2\right)^p\right)^{\frac{1}{p}}\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Example 2.7. *Considering $[a,b]=[0,1]$, let's define the function $f:[0,1]\to\mathbb{R}$ as $f(t)=\frac{t^4}{12}$, satisfying $f''(t)=t^2$, and ensuring $|f''|$ exhibits convexity over $[0,1]$, with $p=q=2$. The left-hand side of the inequality (2.3) resembles the equation presented in (2.2), while the right-hand side of (2.3) simplifies to*

$$\begin{aligned}&\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right)^{\frac{1}{q}}\right]\\&=\frac{1}{8(\alpha+1)}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^2 dt\right)^{\frac{1}{2}}\times\left[\left(\frac{3|f''(0)|^2+|f''(1)|^2}{4}\right)^{\frac{1}{2}}+\left(\frac{3|f''(1)|^2+|f''(0)|^2}{4}\right)^{\frac{1}{2}}\right]\\&=\frac{\sqrt{3}+1}{16(\alpha+1)}\left[\frac{16(\alpha+1)^2(2\alpha+3)(\alpha+3)-72(2\alpha+3)(\alpha+1)+27(\alpha+3)}{27(2\alpha+3)(\alpha+3)}\right]^{\frac{1}{2}}=:RHS.\end{aligned}$$

The findings from Example 2.7 are visually presented in Figure 2.2.

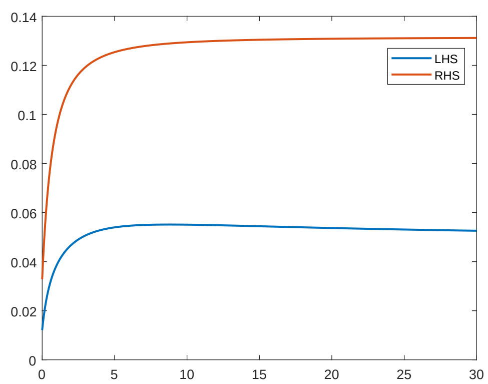

Figure 2.2: Graph of Example 2.7.

Theorem 2.8. *Consider a function $f:[a,b]\to\mathbb{R}$ that is twice differentiable over the interval (a,b), with $f''\in L_1([a,b])$ and $|f''|^q$, where $q\geq 1$, demonstrating convexity across $[a,b]$. As a result of these conditions, the subsequent inequality is satisfied.*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right| \\
&\leq \frac{(b-a)^2}{8(\alpha+1)}\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(a)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(b)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(a)|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.7}$$

Proof. Through the process of taking the absolute value within Lemma 2.1, we arrive at

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right| \\
&=\frac{(b-a)^2}{8(\alpha+1)}\left|\int_0^1\left(t^{\alpha+1}-\frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right]dt\right| \\
&\leq\frac{(b-a)^2}{8(\alpha+1)}\left[\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt+\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\right].
\end{aligned} \tag{2.8}$$

By exploiting the power-mean inequality in conjunction with the convex property of $|f''|^q$, we establish

$$\begin{aligned}
&\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt \\
&\leq\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left(\frac{2-t}{2}|f''(a)|^q+\frac{t}{2}|f''(b)|^q\right)dt\right)^{\frac{1}{q}} \\
&=\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(a)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(b)|^q\right).
\end{aligned} \tag{2.9}$$

Similarly,

$$\begin{aligned}
&\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt \\
&\leq\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(b)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(a)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.10}$$

Upon substituting (2.9) and (2.10) into (2.8), we arrive at the intended inequality denoted as (2.7). □

Corollary 2.9. *If we choose $\alpha=1$ in Theorem 2.8, then we acquire*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq\frac{(b-a)^2}{16}\left[\left(\frac{147|f''(a)|^q+69|f''(b)|^q}{216}\right)^{\frac{1}{q}}+\left(\frac{147|f''(b)|^q+69|f''(a)|^q}{216}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Example 2.10. *Let's take the interval $[a,b]=[0,1]$ and define the function $f:[0,1]\to\mathbb{R}$ as $f:[0,1]\to\mathbb{R}$, $f(t)=\frac{t^4}{12}$, yielding $f''(t)=t^2$. Notably, $|f''|$ exhibits convex behavior across $[0,1]$ and q is assigned the value of 2. Drawing a parallel, the left-hand side of inequality (2.7)* *shares a resemblance with equality (2.2), while the right-hand side of (2.7) simplifies to*

$$
\begin{aligned}
&\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(a)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(b)|^q\right)^{\frac{1}{q}}\right.\\
&+\left.\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(b)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(a)|^q\right)^{\frac{1}{q}}\right]\\
&=\frac{1}{8(\alpha+1)}\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{\frac{1}{2}}\\
&\times\left[\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)^{\frac{1}{2}}+\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)^{\frac{1}{2}}\right]=:RHS.
\end{aligned}
$$

The findings extracted from Example 2.10 have been depicted in Figure 2.3.

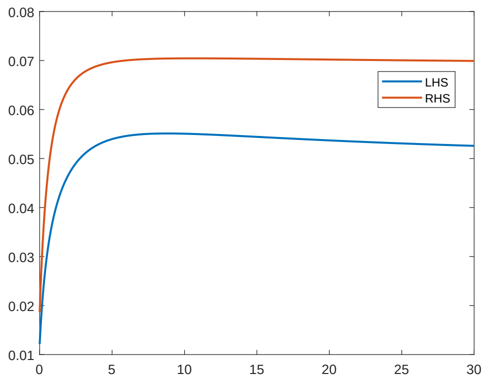

Figure 2.3: Graph of Example 2.10.

3. Conclusion

In this article, Milne-type inequalities have been derived using Fractional Integrals. The obtained inequalities are exemplified, and the accuracy of these examples is validated through graphical representations. Future researchers could explore novel inequalities for different fractional integrals. Furthermore, the current study focused on functions that are twice differentiable. By considering a broader scope of differentiable functions, new inequalities could potentially be discovered. By employing various types of inequalities in Functional Analysis and Numerical Analysis, novel results could be obtained using the methodologies presented in this paper.

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