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RESEARCH ARTICLE

A New Representation for Slant Curves in Sasakian 3-Manifolds

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this paper, we define a new kind of curve called N-slant curve whose principal normal vector field makes a constant angle with the Reeb vector field ξ in Sasakian 3-manifolds. Then, we give some characterizations of N-slant curves in Sasakian 3-manifolds and we obtain some properties of the curves in $\mathbb{R}^3(-3)$. Moreover, we investigate the conditions of C-parallel and C-proper mean curvature vector fields along N-slant curves in Sasakian 3-manifolds. Finally, we study N-slant curves of type $AW(k)$ where k=1,2 or 3.

Keywords: *Slant helix, Mean curvature vector field, Curves of AW(k)-type, Sasakian manifold.*
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1. Introduction

Characterizations of the special curves such as geodesic, circles, circular helices, general helices, slant helices, etc. have been studied for a long time. *A curve of constant slope* or *helix* is defined by the property that its tangent vector field makes a constant angle θ with a fixed line l which is axis of the curve in space. A necessary and sufficient condition that a curve be of constant slope or a general helix is that the ratio of curvature to torsion be constant. This classical result was stated by Lancret in 1802 (see [18]) and firstly proved by B. de Saint Venant in 1845 (see [28]). The Lancret theorem was revised and solved by Barros [6] in 3-dimensional real space forms by using Killing vector fields along curves. Izumiya and Takeuchi [16] have introduced the concept of *slant helices* and *conical geodesic curves* in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ is defined by the property that its principal normal vector field makes a constant angle with a fixed line u. Moreover, they gave a classification of special developable surfaces under the condition of the existence of such a special curve as a geodesic.

After that, the notion of slant helix in Euclidean 3-space can be generalized to higher dimensions [1], [2], [29]. Kula and Yaylı [17] studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix. Then Okuyucu *et al.* [21] gave a generalization of slant helices using the property that its normal vector field makes a constant angle with a left invariant vector field in a three dimensional compact Lie group G.

As a generalization of Legendre curves, Cho *et al.* [11] have introduced the notion of a slant curve in Sasakian 3-manifolds. In their study, a curve in a contact manifold is said to be slant curve if its tangent vector field makes a constant angle with the Reeb vector field ξ. In particular, if the contact angle is equal to $\frac{\pi}{2}$, then the curve is called a Legendre curve. Moreover, in another study, Cho *et al.* [12] showed that biharmonic curves in 3-dimensional Sasakian space forms are slant helices. Also, Cho *et al.* [13] construct the slant curves using the Pseudo-Hermitian connection. Yıldırım [30] obtained curvatures of non-geodesic Frenet curves on 3-dimensional normal almost contact manifolds and provided the results of their characterization.

The mean curvature vector field $\mathbb{H}$ of the curve α is defined by $\mathbb{H} = \nabla_{\alpha'}\alpha' = \varkappa N$ in 3-dimensional contact Riemannian manifolds. Lee *et al.* [20] introduced the notions of C-parallel and C-proper mean curvature

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vector fields along slant curves of Sasakian 3-manifolds in the tangent and normal bundles. They considered $\nabla_{\alpha'}\mathbb{H} = \lambda\xi$ and $\Delta_{\alpha'}\mathbb{H} = \lambda\xi$ $(\lambda \in \mathbb{R})$ corresponding to $\nabla_{\alpha'}\mathbb{H} = \lambda\mathbb{H}$ and $\Delta_{\alpha'}\mathbb{H} = \lambda\mathbb{H}$, respectively.

The mean curvature vector field $\mathbb{H}$ is said to be C-parallel and C-proper mean curvature vector field in the tangent bundle (or the normal bundle) if $\nabla_{\alpha'}\mathbb{H} = \lambda\xi$ (or $\nabla^{\perp}_{\alpha'}\mathbb{H} = \lambda\xi$) and $\Delta_{\alpha'}\mathbb{H} = \lambda\xi$ (or $\Delta^{\perp}_{\alpha'}\mathbb{H} = \lambda\xi$) where ∇ and $\nabla^{\perp}$ denote the operator of covariant differentiation in the tangent bundle and the normal bundle of Sasakian 3-manifolds, respectively.

After these studies, Özgür and Güvenç [25] studied non-geodesic slant curves in pseudo-Hermitian proper and pseudo-Hermitian harmonic mean curvature vector fields for the Tanaka-Webster connection in the tangent and normal bundles, respectively. Also, they obtained some differential equations of slant curves for Tanaka-Webster connection [26].

In [3], Arslan and West defined the notion of AW(k)-type submanifolds. Then, curvature conditions and characterizations related to these curves were given in n-dimensional Euclidean space $\mathbb{E}^n$ [4, 23]. Furthermore, Yoon [31] studied curves of AW(k)-type in the Lie group G with a bi-invariant metric. Also, the characterization of the general helices in terms of AW(k)-type curve in the Lie group G was given by the same author. Moreover, The geometry of AW(k)-type submanifolds in different ambient spaces was intensively studied by many authors [14, 15, 19, 24].

In the present paper, we consider a new kind of curve called N-slant curve whose principal normal vector field makes a constant angle with the Reeb vector field ξ. Moreover, we introduce the notion of C-parallel and C-proper mean curvature vector fields along N-slant curves in Sasakian 3-manifolds. Finally, we investigate N-slant curves of type $AW(k)$ in Sasakian 3-manifolds.

2. Contact Manifolds and Frenet Curves

Let M be a $(2n+1)$-dimensional differentiable manifold which has a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. In this case, M is said to be a contact manifold and η is called a contact form [7]. M admits a global non-vanishing vector field ξ and a field φ of endomorphisms of tangent spaces.

If φ, ξ, η satisfy

$$\eta(\xi) = 1 \ , \ \varphi^2(X) = -X + \eta(X)\xi \ , \ \varphi\xi = 0 \ \text{ and } \eta \circ \varphi = 0,$$

then M is called an almost contact manifold with an almost contact structure (φ, ξ, η).

M becomes an almost contact metric manifold with an almost contact structure (φ, ξ, η, g) if

$$\begin{gathered} g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \\ g(X, \varphi Y) = -g(\varphi X, Y), \\ \eta(X) = g(X, \xi), \end{gathered} \tag{2.1}$$

where $X, Y \in \chi(M)$ and g is a Riemannian tensor of M [8].

Next, we define a 2-form Ω on M by

$$\Omega(X, Y) = g(\varphi X, Y),$$

for all $X, Y \in \chi(M)$, called the fundamental 2-form of the almost contact metric structure (φ, ξ, η, g).

If $\Omega = d\eta$, then M is called a contact metric manifold. Here $d\eta$ is defined by

$$d\eta(X, Y) = \frac{1}{2}\Big(X\eta(Y) - Y\eta(X) - \eta([X, Y])\Big) \ \text{ for any } X, Y \in \chi(M).$$

The Reeb vector field ξ is a unique vector field satisfying

$$\eta(\xi) = 1 \text{ and } d\eta(\xi, X) = 0,$$

for all $X \in \chi(M)$.

$(2n+1)$-dimensional almost contact metric manifold M is said to be normal if the normality tensor $S(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)$ vanishes, where N_φ is the Nijenhuis torsion of φ defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \text{ for any } X, Y \in \chi(M).$$

Also, in 3-dimensional almost contact metric manifolds, Olszak [22] showed that

$$(\nabla_X\varphi)Y = g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi,$$

for all $X, Y \in \chi(M)$.

$(2n+1)$-dimensional manifold M is called a Sasakian manifold if it is endowed with a normal contact metric structure (φ, ξ, η, g). We know that an almost contact metric structure on M is a Sasakian structure if and only if

$$(\nabla_X \varphi)\, Y = g(X, Y)\xi - \eta(Y)X, \tag{2.2}$$

for all $X, Y \in \chi(M)$ where ∇ is the Levi-Civita connection on M [7]. From the last equation it follows that

$$\nabla_X \xi = -\varphi X, \tag{2.3}$$

for all $X, Y \in \chi(M)$.

1-dimensional integral submanifold of a contact manifold is called a Legendre curve. We know from [5] that a 3-dimensional contact metric manifold M is a Sasakian manifold if and only if the torsion of its Legendre curves is equal to 1.

Let us briefly recall some notions and results about the structure of the Sasakian space forms $\mathbb{R}^3(-3)$. Consider on $\mathbb{R}^3(-3)$ with elements of the form (x,y,z), its standard contact structure defined by the 1-form $\eta = \frac{1}{2}(dz - ydx)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field φ is defined by the matrix

$$\varphi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{bmatrix}.$$

Then $g = \eta \otimes \eta + \frac{1}{4}(dx^2 + dy^2)$ is an associated Riemannian metric and $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is a Sasakian form with constant ϕ-sectional curvature equal to -3, denoted $\mathbb{R}^3(-3)$ [5]. The vector field

$$\left\{ X = 2\frac{\partial}{\partial y}, \quad \varphi X = 2\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \quad \xi = 2\frac{\partial}{\partial z} \right\} \tag{2.4}$$

form an orthonormal basis in $\mathbb{R}^3(-3)$ and after some straightforward, one obtains

$$\begin{aligned} \nabla_X \varphi X &= \xi = -\nabla_{\varphi X} X, \\ \nabla_\xi X &= -\varphi X = \nabla_X \xi, \\ \nabla_\xi \varphi X &= X = \nabla_{\varphi X} \xi, \\ \nabla_X X &= \nabla_{\varphi X} \varphi X = \nabla_\xi \xi = 0. \end{aligned}$$

Let α be a curve in a Riemannian 3-manifold parametrized by the arc length with Frenet-Serret apparatus $\{T, N, B, \varkappa, \tau\}$. Here T, N, B are orthonormal vector fields and $\varkappa, \tau$ are the curvature and torsion of the curve α, respectively. Then the Frenet-Serret formulas of the curve α satisfy:

$$\nabla_T T = \varkappa N, \quad \nabla_T N = -\varkappa T + \tau B, \quad \nabla_T B = -\tau N, \tag{2.5}$$

where ∇ is the Levi-Civita connection of M.

Geodesics can be regarded as Frenet curves with $\varkappa = 0$. Note that, in general, an ambient space (M^3, g), geodesics may have non-vanishing torsion. In fact, as we shall see later, Legendre geodesics in a Sasakian 3-manifold have constant torsion 1 [5].

The Frenet-Serret formulas of the curve α imply that the mean curvature vector field $\mathbb{H}$ of a Frenet curve α is given by

$$\mathbb{H} = \nabla_T T = \varkappa N.$$

Chen [10] proved the following identity:

$$\Delta \mathbb{H} = -\nabla_T \nabla_T \nabla_T T. \tag{2.6}$$

Moreover, the Laplacian of the mean curvature in the normal bundle (see [27]) is defined by

$$\Delta^\perp \mathbb{H} = -\nabla_T^\perp \nabla_T^\perp \nabla_T^\perp T, \tag{2.7}$$

where $T = \dot{\alpha}$ and $\nabla^\perp$ denotes the normal connection in the normal bundle.

Lemma 2.1. *[4] Let α be a curve in a Riemannian 3-manifold M. Then, we have*

$$\nabla_T \nabla_T T = -\varkappa^2 T + \varkappa^{'} N + \varkappa\tau B \tag{2.8}$$

$$\nabla_T \nabla_T \nabla_T T = -3\varkappa\varkappa^{'} T + \left(\varkappa^{''} - \varkappa^{3} - \varkappa\tau^2\right) N + \left(2\varkappa^{'}\tau + \varkappa\tau^{'}\right) B, \tag{2.9}$$

$$\nabla_T^{\perp} \nabla_T^{\perp} T = \varkappa^{'} N + \varkappa\tau B, \tag{2.10}$$

$$\nabla_T^{\perp} \nabla_T^{\perp} \nabla_T^{\perp} T = \left(\varkappa^{''} - \varkappa\tau^2\right) N + \left(2\varkappa^{'}\tau + \varkappa\tau^{'}\right) B. \tag{2.11}$$

.

Definition 2.1. [5] The contact angle between the tangent vector field T of a curve α and the Reeb vector field ξ is the function $\psi : I \to [0, 2\pi)$ given by:

$$\cos\psi\,(s) = g\,(T(s), \xi)$$

Then, the curve α is a slant curve if ψ is a constant function [11]. In particular case of $\psi = \frac{\pi}{2}$ $\left(\text{or } \psi = \frac{3\pi}{2}\right)$, the curve α is called a Legendre curve.

Theorem 2.1. *[20] A non-geodesic curve in a Sasakian 3-manifold M is a slant curve if and only if $\eta^{'}(N) = 0$ and the ratio of $\tau - 1$ to $\varkappa$ is constant.*

3. N-Slant Curves in Sasakian 3-Manifolds

In this section, we give the definition of the N-slant curve and its axis in Sasakian 3-manifold M with a contact metric g. Also, we give some characterizations of such a curve.

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \to M$ be a Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the curve α is called an N-slant curve if its normal vector field N makes a constant angle with the Reeb vector field ξ . That is,

$$\eta\,(N(s)) = g\,(N(s), \xi(s)) = \cos\theta \text{ for all } s \in I, \tag{3.1}$$

where θ is a constant angle between the Reeb vector field ξ and the normal vector field N of the curve α.

Claim 1. Let $\alpha : I \subset \mathbb{R} \to M$ be a Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. If we assume that θ is equal to $\frac{\pi}{2}$, then $\eta'\,(N(s)) = 0$ and the ratio of $\tau - 1$ and $\varkappa$ of the curve α is a constant. Then, it follows from Theorem 2.1 that the curve α is a slant one. Consequently, we see that N-slant curves are a generalization of slant curves in Sasakian 3-manifolds.

After the above Claim, we consider $\theta \neq \frac{\pi}{2}$ for an N-slant curve in Sasakian 3-manifolds throught the paper.

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \to M$ be a non-geodesic Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the harmonic curvature function of the curve α is defined by

$$\mathcal{H} = \frac{\tau - 1}{\varkappa}, \tag{3.2}$$

where $\varkappa$ and τ are the principal curvature and torsion of the curve α, respectively.

From now on, we work with N-slant curves but not slant curves, otherwise $\mathcal{H}$ is constant.

Lemma 3.1. *[9] Let $\alpha : I \subset \mathbb{R} \to M$ be a Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the following equalities*

$$\varphi T = \eta\,(B)\,N - \eta\,(N)\,B,$$
$$\varphi N = \eta\,(T)\,B - \eta\,(B)\,T,$$
$$\varphi B = \eta\,(N)\,T - \eta\,(T)\,N,$$

hold.

Our next result gives a decomposition of the Reeb vector field ξ in the Frenet frame of the curve α.

Proposition 3.1. *Let $\alpha : I \subset \mathbb{R} \to M$ be a Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. If the curve α is an N-slant curve in M, then the unique Reeb vector field ξ of the curve α is*

$$\xi = \left\{ \frac{\varkappa\mathcal{H}\left(1+\mathcal{H}^2\right)}{\mathcal{H}'} T + N + \frac{\varkappa\left(1+\mathcal{H}^2\right)}{\mathcal{H}'} B \right\} \cos\theta, \tag{3.3}$$

where $\mathcal{H} = \dfrac{\tau - 1}{\varkappa}$ is the harmonic curvature function of the curve α and $\theta \neq \dfrac{\pi}{2}$ is a constant angle.

Proof. Let $\alpha : I \subset \mathbb{R} \to M$ be a non-geodesic Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the unique Reeb vector field ξ of the curve α is written by

$$\xi = \lambda_1 T + \lambda_2 N + \lambda_3 B,$$

where $\lambda_1 = \eta(T)$, $\lambda_2 = \eta(N)$ and $\lambda_3 = \eta(B)$.

From Definition 3.1, we know

$$g(N(s), \xi(s)) = \cos\theta \text{ for all } s \in I.$$

By differentiating the last equality, we get

$$g(\nabla_T N, \xi) + g(N, \nabla_T \xi) = 0,$$

from (2.3) and using the Frenet-Serret formulas, we have

$$-\varkappa g(T, \xi) + \tau g(B, \xi) - g(N, \varphi(T)) = 0,$$

then with the help of the Lemma 3.1, we find

$$-\varkappa g(T, \xi) + (\tau - 1) g(B, \xi) = 0.$$

Using the Definition 3.2, we obtain

$$g(T, \xi) = \mathcal{H} g(B, \xi). \tag{3.4}$$

Differentiating the (3.4) with respect to s, we have

$$g(\nabla_T T, \xi) + g(T, \nabla_T \xi) = \mathcal{H}' g(B, \xi) + \mathcal{H}\Big(g(\nabla_T B, \xi) + g(B, \nabla_T \xi) \Big).$$

Further on using (2.5) and the Lemma 3.1, we obtain

$$\varkappa g(N, \xi) = \mathcal{H}' g(B, \xi) - \mathcal{H}(\tau - 1) g(N, \xi),$$

which implies that

$$g(B, \xi) = \frac{\varkappa\left(1+\mathcal{H}^2\right)}{\mathcal{H}'} g(N, \xi). \tag{3.5}$$

Plugging (3.5) into (3.4), we immediately get

$$g(T, \xi) = \frac{\varkappa\mathcal{H}}{\mathcal{H}'}\left(1+\mathcal{H}^2\right) g(N, \xi).$$

Consequently, combining the relations (3.1), (3.4) and (3.5) the Reeb vector field ξ of the curve α is given by

$$\xi = \cos\theta \left\{ \frac{\varkappa\mathcal{H}\left(1+\mathcal{H}^2\right)}{\mathcal{H}'} T + N + \frac{\varkappa\left(1+\mathcal{H}^2\right)}{\mathcal{H}'} B \right\}, \tag{3.6}$$

which completes the proof. □

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \to M$ be a non-geodesic Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then α is a N-slant curve if and only if*

$$\frac{\varkappa(1+\mathcal{H}^2)^{\frac{3}{2}}}{\mathcal{H}'} = \tan\theta \text{ is a constant,} \tag{3.7}$$

where $\mathcal{H}$ is the harmonic curvature function of the curve α and $\theta \neq \dfrac{\pi}{2}$ is a constant angle.

Proof. We know that ξ is a unique Reeb vector field of the curve α. By a straightforward computation, using (3.6), we can obtain that

$$\frac{\varkappa(1+\mathcal{H}^2)^{\frac{3}{2}}}{\mathcal{H}^{\prime}} = \tan\theta$$

is a constant function. Conversely, assume that the condition (3.7) is satisfied. Let be the Reeb vector field ξ as follows:

$$\xi = \rho_1 T + \cos\theta N + \rho_3 B. \tag{3.8}$$

Differentiating the (3.8) and using (2.3) with the Lemma 3.1, we obtain

$$\begin{aligned} \rho_1 &= \rho_3 \mathcal{H}, \\ \rho_1' &= \varkappa \cos\theta, \\ \rho_3' &= -\varkappa\mathcal{H}\cos\theta. \end{aligned} \tag{3.9}$$

Then from (3.9), we have

$$\rho_3 = \frac{\varkappa\left(1+\mathcal{H}^2\right)}{\mathcal{H}^{\prime}}\cos\theta. \tag{3.10}$$

Finally, we are able to write the Reeb vector field

$$\xi = \cos\theta\left\{\frac{\varkappa\mathcal{H}\left(1+\mathcal{H}^2\right)}{\mathcal{H}^{\prime}}T + N + \frac{\varkappa\left(1+\mathcal{H}^2\right)}{\mathcal{H}^{\prime}}B\right\}. \tag{3.11}$$

This means that the curve α is an N-slant curve, concluding the proof.

□

Remark 3.1. It is important to notice here that Izumiya obtained an analogue result (Prop.2.1, [16]) for Euclidean spaces.

With the help of Theorem 3.1, we can easily reconstruct the Proposition 3.1 and give the following Corollary.

Corollary 3.1. *Let $\alpha : I \subset \mathbb{R} \to M$ be a non-geodesic Frenet curve parametrized by arc length in a Sasakian 3-manifold M with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. If the curve α is an N-slant curve, then the unique Reeb vector field ξ of the curve α is*

$$\xi = \cos\epsilon(s)\sin\theta T + \cos\theta N + \sin\epsilon(s)\sin\theta B, \tag{3.12}$$

where $\epsilon(s) = \arccos\left(\dfrac{\mathcal{H}}{\sqrt{1+\mathcal{H}^2}}\right)$.

Now, we may give some results for N-slant curves in $\mathbb{R}^3(-3)$.

Proposition 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3(-3)$ is a non-geodesic N-slant curve parametrized by arc length in a Sasakian 3-manifold $\mathbb{R}^3(-3)$ with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then T, N and B of the curve α can be expressed as follows:*

$$\begin{aligned} T &= \Big(\sin\epsilon(s)\sin\varpi(s) - \cos\epsilon(s)\cos\theta\cos\varpi(s)\Big)X \\ &\quad - \Big(\cos\epsilon(s)\cos\theta\sin\varpi(s) + \sin\epsilon(s)\cos\varpi(s)\Big)\varphi X + \cos\epsilon(s)\sin\theta\xi, \\ N &= \sin\theta\cos\varpi(s)\,X + \sin\theta\sin\varpi(s)\,\varphi X + \cos\theta\xi, \\ B &= \Big(-\cos\epsilon(s)\sin\varpi(s) - \sin\epsilon(s)\cos\theta\cos\varpi(s)\Big)X \\ &\quad - \Big(\sin\epsilon(s)\cos\theta\sin\varpi(s) - \cos\epsilon(s)\cos\varpi(s)\Big)\varphi X + \sin\epsilon(s)\sin\theta\xi \end{aligned} \tag{3.13}$$

for some function $\varpi(s)$.

Proof. If we decompose the normal vector field N as

$$N = \mu_1 X + \mu_2 \varphi X + \cos\theta\xi,$$

then, we have $\mu_1^2 + \mu_2^2 = \sin^2\theta$. So, it follows that $\mu_1 = \sin\theta\cos\varpi(s)$ and $\mu_2 = \sin\theta\sin\varpi(s)$. Now using the Lemma 3.1 and formulas (2.1), we obtain the above equalities of T and B. □

Corollary 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3(-3)$ is a non-geodesic$N$-slant curve parametrized by arc length in a Sasakian 3-manifold $\mathbb{R}^3(-3)$ with Frenet curvatures $\varkappa$ and τ. Then the curvatures are represented by the following*

$$\varkappa = \sin\theta\sin\epsilon(s)\Big(\varpi'(s) - 2\sin\theta\cos\epsilon(s)\Big),$$

$$\tau = \sin\theta\cos\epsilon(s)\Big(\varpi'(s) - 2\sin\theta\cos\epsilon(s)\Big) + 1,$$

for some function $\varpi(s)$ where $\epsilon(s) = \arccos\left(\frac{\mathcal{H}}{\sqrt{1+\mathcal{H}^2}}\right)$.

Theorem 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3(-3)$ be a non-geodesic N-slant curve $(\theta \neq 0, \pi)$ in a Sasakian 3-manifold $\mathbb{R}^3(-3)$ with the curvature $\varkappa$. Then its torsion is*

$$\tau = \frac{\varkappa(s)\cot\theta\int\varkappa(s)ds}{\sqrt{1-\left(\cot\theta\int\varkappa(s)ds\right)^2}} + 1. \tag{3.14}$$

Proof. Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold $\mathbb{R}^3(-3)$. Differentiating the formula

$$g(T,\xi) = \eta(T) = \cos\epsilon(s)\sin\theta,$$

along α, it follows that

$$\sin\theta(\cos\epsilon(s))' = \varkappa(s)g(N,\xi) + g(T,-\varphi T) = \varkappa(s)\cos\theta.$$

Hence, we compute

$$\cos\epsilon(s) = \cot\theta\int\varkappa(s)ds. \tag{3.15}$$

As $\mathcal{H} = \frac{\tau(s)-1}{\varkappa(s)} = \cot\epsilon(s)$, we have

$$\tau(s) = \varkappa(s)\cot\epsilon(s) + 1. \tag{3.16}$$

Plugging (3.15) into (3.16), we obtain (3.14). □

Example 3.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3(-3)$ is a non-geodesic N-slant curve parametrized by arc length in a Sasakian 3-manifold $\mathbb{R}^3(-3)$ with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Consider on $\mathbb{R}^3(-3)$ with elements x,y,z with orthonormal basis (2.4). The tangent vector field can be expressed as

$$T = \alpha'(s) = (x', y', z').$$

Here, the tangent vector T of α is also reprensented by the following

$$T = \frac{y'}{2}X + \frac{x'}{2}\varphi X + \left(\frac{z'-x'y}{2}\right)\xi.$$

Using the above formulas for the Levi-Civita connection, we have

$$\nabla_T T = \varkappa N = \left(\frac{y''}{2} + x'\left(\frac{z'-x'y}{2}\right)\right)X + \left(\frac{x''}{2} - y'\left(\frac{z'-x'y}{2}\right)\right)\varphi X + \left(\frac{z''}{2} - \frac{x''y - x'y'}{2}\right)\xi. \tag{3.17}$$

It follows that

$$z'' - \frac{\partial}{\partial s}(x'y) = 2\cos\theta\varkappa(s). \tag{3.18}$$

After integration we obtain,

$$z' - x'y = 2\cos\theta \int \varkappa(s)ds = 2\sin\theta\cos\epsilon(s). \tag{3.19}$$

From the arc-length parametrization condition, we have

$$x'^2 + y'^2 + (z' - x'y)^2 = x'^2 + y'^2 + 4\sin^2\theta\cos^2\epsilon(s) = 4. \tag{3.20}$$

Finally, the tangent vector fields T of the curve α is represented

$$\begin{aligned} x'(s) &= 2\sin\epsilon(s), \\ y'(s) &= 2\cos\theta\cos\epsilon(s), \\ z'(s) &= 2\sin\theta\cos\epsilon(s) + 4\cos\theta\sin\epsilon(s)\int\cos\epsilon(s)ds. \end{aligned}$$

We take for example $\cos\epsilon(s) = \cos(s)$ and $\theta = \frac{\pi}{4}$. Then we may find an explicit parametric equations of $N-$slant curves with $\xi = \frac{\sqrt{2}}{2}\Big(\cos(s)T + N + \sin(s)B\Big)$ which are not slant curves.

$$\begin{aligned} x(s) &= -2\cos(s) + c_1, \\ y(s) &= \sqrt{2}\sin(s) + c_2, \\ z(s) &= \sqrt{2}\sin(s) - \frac{\sqrt{2}}{2}\sin(2s) + \sqrt{2}s + c_3, \end{aligned}$$

where c_1, c_2 and c_3 are constants. Finally, from (3.15) and (3.16) we find $\varkappa = -\sin(s)$ and $\tau = 1 - \cos(s)$.

4. C-Parallel and C-Proper Mean Curvature Vector Fields along N-Slant Curves in Sasakian 3-Manifolds

4.1. *C-Parallel Mean Curvature Vector Field*

For a non-geodesic N-slant curve α in a Sasakian 3-manifold M, using the equations (2.8) and (3.12), we find that the curve α satisfies $\nabla_T \mathbb{H} = \lambda\xi$ if and only if

$$-\varkappa^2 = \lambda\cos\epsilon(s)\sin\theta, \tag{4.1}$$

$$\varkappa' = \lambda\cos\theta, \tag{4.2}$$

$$\varkappa\tau = \lambda\sin\epsilon(s)\sin\theta, \tag{4.3}$$

where λ is a real constant.

Theorem 4.1. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then α has a C-parallel mean curvature vector field if and only if the curve α has the curvature κ and the torsion τ related by $\kappa^2 + \tau^2 = \tau$ where $\varkappa = (\lambda\cos\theta)s + s_0$ and $\tau = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \Big((\lambda\cos\theta)s + s_0\Big)^2}$ where s_0 is a non zero constant.*

Proof. Assume that α be a non-geodesic N-slant curve which has a C-parallel mean curvature vector field. From (4.1) and (4.3), we have

$$\tau^2 - \tau + \kappa^2 = 0. \tag{4.4}$$

After integration (4.2), we obtain

$$\varkappa = (\lambda\cos\theta)s + s_0. \tag{4.5}$$

Combining with (4.4) and (4.5), we get $\tau = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \Big((\lambda\cos\theta)s + s_0\Big)^2}$. □

Remark 4.1. If we consider that λ is equal to zero in equality $\nabla_T \mathbb{H} = \lambda\xi$, we have Corollary 4.3 in [20]. Also, Özgür and Tripathi [24] showed that Legendre curves satisfying $\nabla_T \mathbb{H} = 0$ in a Sasakian 3-manifold M are geodesics.

4.2. C-Proper Mean Curvature Vector Field

For a non-geodesic N-slant curve in a Sasakian 3-manifold M, using the equations (2.9) and (3.12), we find that the curve α satisfies $\Delta_T\mathbb{H} = \lambda\xi$ if and only if

$$3\varkappa\varkappa^{'} = \lambda\cos\epsilon(s)\sin\theta, \tag{4.6}$$

$$-\varkappa^{''} + \varkappa^{3} + \varkappa\tau^2 = \lambda\cos\theta, \tag{4.7}$$

$$-2\varkappa^{'}\tau - \varkappa\tau^{'} = \lambda\sin\epsilon(s)\sin\theta, \tag{4.8}$$

where λ is a real constant.

Theorem 4.2. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then α has a C-proper mean curvature vector field if and only if α is a circular helix with $\lambda = \varkappa^3 + \varkappa\tau^2$ where $\varkappa$ and $\tau \neq 1$ are non zero constants ($\theta = 0$) or α has the curvature $\varkappa = \sqrt{\left(\frac{2}{3}\lambda\sin\theta\int\cos\epsilon(s)ds\right)}$ and the torsion τ satisfy the following non-linear relation $3\varkappa^2\varkappa^{'} = (1-\tau)(2\varkappa^{'}\tau + \varkappa\tau^{'})$ ($\theta \neq 0$).*

Proof. Assume that α be a non-geodesic N-slant curve with C-proper mean curvature vector field.

(i) The case of $\theta = 0$: From (4.6) and (4.8), we can easily see that $\varkappa$ and τ are non zero constant. It follows that $\lambda = \varkappa^3 + \varkappa\tau^2$.

(ii) The case of $\theta \neq 0$: It is clearly obtained from the integrating (4.6) that $\varkappa = \sqrt{\left(\frac{2}{3}\lambda\sin\theta\int\cos\epsilon(s)ds\right)}$. Moreover, from (4.6) and (4.8), we have $3\varkappa^2\varkappa^{'} = (1-\tau)(2\varkappa^{'}\tau + \varkappa\tau^{'})$. □

Remark 4.2. Let α be an N-slant curve in a Sasakian 3-manifold M. Then λ is equal to zero in equality $\Delta_T\mathbb{H} = \lambda\xi$ if and only if α is a geodesic [20]. Also, Özgür and Tripathi [24] showed that Legendre curves satisfying $\Delta_T\mathbb{H} = 0$ in a Sasakian 3-manifold M are geodesics.

4.3. C-Parallel Mean Curvature Vector Field in the Normal Bundle

For a non-geodesic N-slant curve α in a Sasakian 3-manifold M, using the equations (2.10) and (3.12), we find that the curve α satisfies $\nabla_T^{\perp}\mathbb{H} = \lambda\xi$ if and only if

$$\lambda\cos\epsilon(s)\sin\theta = 0, \tag{4.9}$$

$$\lambda\cos\theta = \varkappa^{'}, \tag{4.10}$$

$$\lambda\sin\epsilon(s)\sin\theta = \varkappa\tau, \tag{4.11}$$

where λ is a real constant.

Theorem 4.3. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then α has a C-parallel mean curvature vector field in the normal bundle if and only if α has the curvature $\varkappa = \lambda s + c$ and the torsion $\tau = 0$ where c and $\lambda \neq 0$ are real constants.*

Proof. Assume that α is a non-geodesic N-slant curve with C-parallel mean curvature vector field in the normal bundle. Then from (4.9) and (4.11), we obtain that $\tau = 0$. or $\tau = 1$. Let τ be equal to zero, then using (4.11), we see that $\theta = 0$ or $\theta = \pi$ and it has the curvature $\varkappa = \lambda s + c$. Now, considering the curve α is a Legendre curve. From (4.10) and (4.11) we have $\lambda\cos\theta = 0$. Since $\theta \neq \frac{\pi}{2}$, this case is not possible. □

Remark 4.3. Let α be an N-slant curve in a Sasakian 3-manifold M. Then λ is equal to zero in equality $\nabla_T^{\perp}\mathbb{H} = \lambda\xi$ if and only if the curve α becomes a circle as $\varkappa$ is a non zero constant and $\tau = 0$ [20].

4.4. C-Proper Mean Curvature Vector Field in the Normal Bundle

For a non-geodesic N-slant curve α in a Sasakian 3-manifold M, using the equations (2.11) and (3.12), we find that the curve α satisfies $\Delta_T^{\perp}\mathbb{H} = \lambda\xi$ if and only if

$$\lambda\cos\epsilon(s)\sin\theta = 0, \tag{4.12}$$

$$\lambda\cos\theta = -\varkappa^{''} + \varkappa\tau^2, \tag{4.13}$$

$$\lambda\sin\epsilon(s)\sin\theta = -2\varkappa^{'}\tau - \varkappa\tau^{'}, \tag{4.14}$$

where λ is a real constant.

Theorem 4.4. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then α has a C-proper mean curvature vector field in the normal bundle if and only if we have following cases*

$i)$ The case of $\lambda = 0$: The curve α has the curvatures

$$\varkappa = \frac{1}{c_0}\left(c_1 - \left(c_0^2 s + c_2\right)^2\right)^{\frac{1}{2}} \text{ and } \tau = \frac{c_0}{\varkappa^2},$$

where $c_0 \neq 0, c_1, c_2$ are real constant.

$ii)$ The case of $\lambda \neq 0$: The curve α is a Legendre helix with curvature $\varkappa = \lambda \cos\theta$ or the curvature of the curve α satisfies the following differential equation $\varkappa^{'} = \pm\left(2\lambda\varkappa + c_0\varkappa^{-2} + d_1\right)^{\frac{1}{2}}$ and its torsion is $\tau = \frac{c_0}{\varkappa^2}$, where c_0, d_1 are real constant.

Proof. Assume that α is a non-geodesic N-slant curve with C-proper mean curvature vector field in the normal bundle.

(i) The case of $\lambda = 0$: From (4.14), we have $2\varkappa^{'}\tau + \varkappa\tau^{'} = 0$. It follows $\tau = \frac{c_0}{\varkappa^2}$. Plugging the last equality into (4.13), we obtain the following differential equation

$$\varkappa^{''} - \frac{c_0}{\varkappa^3} = 0.$$

One immediately gets the solution, namely

$$\varkappa = \frac{1}{c_0}\left(c_1 - \left(c_0^2 s + c_2\right)^2\right)^{\frac{1}{2}}.$$

(ii) For the case of $\lambda \neq 0$, from (4.12) and (4.14), we can easily see that $\left(2\varkappa^{'}\tau + \varkappa\tau^{'}\right)(\tau - 1) = 0$. Considering $\tau = 1$, we get $\varkappa = \lambda\cos\theta$. Then, considering $2\varkappa^{'}\tau + \varkappa\tau^{'} = 0$, we obtain

$$\varkappa^{'} = \pm\left(2\lambda\varkappa + c_0\varkappa^{-2} + d_1\right)^{\frac{1}{2}} \text{ and } \tau = \frac{c_0}{\varkappa^2}.$$

□

5. N-Slant Curves of $AW(k)$-Type in Sasakian 3-Manifolds

In [4], Arslan and Özgur studied curves of $AW(k)$-type. In this section, we investigate N-slant curves of type $AW(k)$ from the viewpoint of Sasakian 3-Manifolds and we find necessary and sufficient conditions for them.

Proposition 5.1. *Let $\alpha : I \subset \mathbb{R} \to M$ be a Frenet curve parametrized by arc length in a Sasakian 3-manifold M with extended Frenet apparatus $\{T, N, B, \varkappa, \tau, \mathcal{H}\}$ then*

$$\begin{aligned}
&\alpha'(s) = T(s),\\
&\alpha^{''}(s) = \nabla_T T(s) = \varkappa(s)N(s),\\
&\alpha^{'''}(s) = \nabla_T\nabla_T T(s) = -\varkappa^2(s)T(s) + \varkappa'(s)N(s) + \Big(\varkappa^2(s)\mathcal{H}(s) + \varkappa(s)\Big)B(s),\\
&\alpha^{iv}(s) = \nabla_T\nabla_T\nabla_T T(s)\\
&\quad = -3\varkappa(s)\varkappa'(s)T(s) + \left\{\varkappa''(s) - \varkappa^3(s) - \varkappa^3(s)\mathcal{H}^2(s) - 2\varkappa^2(s)\mathcal{H}(s) - \varkappa(s)\right\}N(s)\\
&\qquad + \Big(3\varkappa(s)\varkappa'(s)\mathcal{H}(s) + \varkappa^2(s)\mathcal{H}'(s) + 2\varkappa'(s)\Big)B(s).
\end{aligned} \tag{5.1}$$

As a notion, we can easily obtain that

$$\begin{aligned}
&\mathcal{N}_1(s) = \Big(\alpha^{''}(s)\Big)^{\perp} = \varkappa(s)N(s),\\
&\mathcal{N}_2(s) = \Big(\alpha^{'''}(s)\Big)^{\perp} = \varkappa'(s)N(s) + \Big(\varkappa^2(s)\mathcal{H}(s) + \varkappa(s)\Big)B(s),\\
&\mathcal{N}_3(s) = \left(\alpha^{iv}(s)\right)^{\perp} = \Big(\varkappa''(s) - \varkappa^3(s) - \varkappa^3(s)\mathcal{H}^2(s) - 2\varkappa^2(s)\mathcal{H}(s) - \varkappa(s)\Big)N(s)\\
&\qquad + \Big(3\varkappa(s)\varkappa'(s)\mathcal{H}(s) + \varkappa^2(s)\mathcal{H}'(s) + 2\varkappa'(s)\Big)B(s).
\end{aligned} \tag{5.2}$$

Definition 5.1. [4] Let $\alpha : I \subset \mathbb{R} \to M$ be an arclenghted Frenet curve of order 3. Then

(i) The curve α is of type $AW(1)$ if it satisfies

$$\mathcal{N}_3(s) = 0, \tag{5.3}$$

(ii) The curve α is of type $AW(2)$ if it satisfies

$$\|\mathcal{N}_2(s)\|^2 \mathcal{N}_3(s) = \langle \mathcal{N}_3(s), \mathcal{N}_2(s) \rangle \mathcal{N}_2(s), \tag{5.4}$$

(iii) The curve α is of type $AW(3)$ if it satisfies

$$\|\mathcal{N}_1(s)\|^2 \mathcal{N}_3(s) = \langle \mathcal{N}_3(s), \mathcal{N}_1(s) \rangle \mathcal{N}_1(s). \tag{5.5}$$

Proposition 5.2. *[4] Let $\alpha : I \subset \mathbb{R} \to M$ be a Frenet curve parametrized by arc length in a Sasakian 3-manifold M. By using Definition 5.1, we obtain*

(i) The curve α is of type $AW(1)$ if and only if

$$\varkappa''(s) - \varkappa^3(s) - \varkappa(s)\tau^2(s) = 0, \tag{5.6}$$
$$\tau(s) = \frac{c_0}{\varkappa^2(s)}, \; c_0 \in \mathbb{R}$$

(ii) The curve α is of type $AW(2)$ if and only if

$$2(\varkappa'(s))^2\tau(s) + \varkappa(s)\varkappa'(s)\tau'(s) = \varkappa(s)\varkappa''(s)\tau(s) - \varkappa^4(s)\tau(s) - \varkappa^2(s)\tau^3(s), \tag{5.7}$$

(iii) The curve α is of type $AW(3)$ if and only if

$$2\varkappa'(s)\tau(s) + \varkappa(s)\tau'(s) = 0, \tag{5.8}$$

and the solution of this differential equation is $\tau(s) = \dfrac{c_0}{\varkappa^2(s)}$, $c_0 \in \mathbb{R}$.

Theorem 5.1. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then the curve α is of type $AW(1)$ if and only if its curvature $\varkappa$ satisfies the following differential equation*

$$\varkappa'(s)\left(\varkappa^2(s) - 3c_0\right) = d_0\left(\varkappa^2(s) + \frac{\left(c_0 - \varkappa^2(s)\right)^2}{\varkappa^4(s)}\right)^{\frac{3}{2}}, \tag{5.9}$$

where c_0, d_0 are real constants.

Proof. Assume that α is a non-geodesic N-slant curve which is of type $AW(1)$. The equations (3.7) and (5.6) give us

$$\frac{\varkappa(s)(1+\mathcal{H}^2)^{\frac{3}{2}}}{\mathcal{H}'} = \tan\theta = d_0, \tag{5.10}$$
$$\tau = \frac{c_0}{\varkappa^2(s)}. \tag{5.11}$$

Then differentianting last equality, we get

$$\tau'(s) = \frac{-2c_0\varkappa'(s)}{\varkappa^3(s)}. \tag{5.12}$$

Plugging (5.11) and (5.12) into (5.10), we obtain

$$\varkappa'(s)\left(\varkappa^2(s) - 3c_0\right) = d_0\left(\varkappa^2(s) + \left(\tau(s) - 1\right)^2\right)^{\frac{3}{2}}. \tag{5.13}$$

Consequently, if we consider the equations (5.11) in (5.13), we get the (5.9) where $c_0, d_0 \in \mathbb{R}$.

□

Theorem 5.2. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then the curve α is of type $AW(2)$ if and only if its curvature $\varkappa$ satisfies the following differential equation*

$$3\big(\varkappa'(s)\big)^2\tau(s)-\big(\varkappa'(s)\big)^2-d_0\varkappa'(s)\bigg(\varkappa^2(s)+\big(\tau(s)-1\big)^2\bigg)^{\frac{3}{2}}=\varkappa(s)\varkappa''(s)\tau(s)-\varkappa^4(s)\tau(s)-\varkappa^2(s)\tau^3(s),$$

where $d_0\in\mathbb{R}$.

Proof. Assume that α be a non-geodesic N-slant curve which is of type $AW(2)$. We have

$$\tau'(s)=\frac{\varkappa'(s)(\tau(s)-1)-d_0(\varkappa^2(s)+(\tau(s)-1)^2)^{\frac{3}{2}}}{\varkappa(s)}. \tag{5.14}$$

Consequently, using the equations (5.7) and (5.14), we obtain

$$3\big(\varkappa'(s)\big)^2\tau(s)-\big(\varkappa'(s)\big)^2-d_0\varkappa'(s)\bigg(\varkappa^2(s)+\big(\tau(s)-1\big)^2\bigg)^{\frac{3}{2}}=\varkappa(s)\varkappa''(s)\tau(s)-\varkappa^4(s)\tau(s)-\varkappa^2(s)\tau^3(s)$$

. □

Theorem 5.3. *Let α be a non-geodesic N-slant curve in a Sasakian 3-manifold M. Then the curve α is of type $AW(3)$ if and only if its curvature $\varkappa$ satisfies the following differential equation*

$$\frac{\big(\varkappa^2(s)-3c_0\big)\varkappa'(s)}{\varkappa^2(s)}=d_0\left(\varkappa^2(s)+\frac{\big(c_0-\varkappa^2(s)\big)^2}{\varkappa^4(s)}\right)^{\frac{3}{2}}, \tag{5.15}$$

where $c_0,d_0\in\mathbb{R}$

Proof. Assume that α is a non-geodesic N-slant curve is of type $AW(3)$. From (5.8) we know that, $2\varkappa'(s)\tau(s)+\varkappa(s)\tau'(s)=0$ and $\tau=\frac{c_0}{\varkappa^2(s)}$. Consequently, substituting the last equations into (5.14), we obtain (5.15). □

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