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On Level Hypersurfaces of the Vertical Lift of a Submersion

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Abstract. Suppose that (M, G) be a Riemannian manifold and $f : M \to \mathbb{R}$ be a submersion. Then the vertical lift of f, $f^v : TM \to \mathbb{R}$ defined by $f^v = f \circ \pi$ is also a submersion. This interesting case, differently from [10], leads us to investigation of the level hypersurfaces of f^v in tangent bundle TM. In this paper we obtained some differential geometric relations between level hypersurfaces of f and f^v. In addition, we noticed that, unlike [13], a level hypersurface of f^v is always lightlike, i.e., it doesn't depend on any additional condition.

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1. Introduction

We denote by $\mathfrak{I}_0^0(M)$ the algebra of smooth functions on M. We consider $f \in \mathfrak{I}_0^0(M)$, the vertical lift of f to tangent bundle TM is defined by $f^v = f \circ \pi$. From definition of f^v we say that f^v is induced by f. In this case some geometrical relations can be found between the level hypersurfaces of f and f^v. A similar study was conducted by M. Yıldırım [13] in 2009 and some important relations are obtained.

We need some tools to do these investigations. These tools are vertical and complete lifts of differentiable elements defined on M. The notion of vertical and complete lift was introduced by K. Yano and S. Kobayashi in [12]. By using these lifts, in [10], M. Tani introduced the notion of prolongations of hypersurfaces to tangent bundle.

In [10], Tani showed that there exist some geometrical relations between the geometry of S in M and TS in TM for a given hypersurface S. We should emphasize here that in Tani's study [10], complete lift metric on TM was taken into consideration. In [11], it is stated that this metric is a semi-Riemannian metric with $n-index$. In this case, the geometry of the level hypersurfaces of f^v is examined within the (TM, G^c) semi-Riemann structure. In this study, it has been seen that all level surfaces of f^v are lightlike hypersurfaces.

Lightlike hypersurfaces of semi-Riemannian manifolds have been studied by Many authors [2, 6–8] and others.

In this paper, we discuss the relationships between the geometry of level surfaces of a real-valued function and its vertical lift. The importance of this paper is that, differently from [10], we find a class of hypersurfaces in tangent bundle TM such that these are derived from hypersurfaces in M. Because, in [10] obtained submanifold in TM such that it is tangent to original submanifold in M, but it isn't so in this work.

In last section, we establish lightlike structure on a level hypersurface of vertical lift of f and see that fundamental notions of degenerate submanifold geometry were obtained by a natural way. That is, we needn't to any strong condition. This case shows that the problem, studied here, is completely suitable and interesting.

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In section 2, we shall give an introductory information. In section 3, we shall show that the vertical lift of a submersion is also a submersion and its any level set is a hypersurface (denoted by $\bar{S}$) in tangent bundle. In section 4, we obtain Gauss and Weingarten formulas for $\bar{S}$. In addition, it is obtained that $\bar{S}$ is a semi-Riemannian hypersurface with index $n-1$ with respect to G^c (G is a Riemannian metric on M). In section 5, we give a lightlike (null) structure on $\bar{S}$. In addition, considering the lightlike structure on $\bar{S}$ we obtain some geometrical relations between the level hypersurfaces of f and $\bar{S}$ as well.

2. Notations and Preliminaries

Let M be an n- dimensional differentiable manifold. We denote by TM its tangent bundle with the projection $\pi_M : TM \longrightarrow M$ and by $T_p(M)$ its tangent space at a point p of M. $\Im_s^r(M)$ is the space of tensor fields of class C^∞ and of type (r,s). An element of $\Im_0^0(M)$ is a C^∞ function defined on M. V be a coordinate neighborhood in M and (x^i) , $1 \leq i \leq n$, are certain local coordinates defined in V. We introduce a system of coordinates (x^i, y^i) in $\pi_M^{-1}(V)$ such that (y^i) are cartesian coordinates in each tangent space $T_p(M)$, p being an arbitrary point of V, with respect to the natural frame $(\frac{\partial}{\partial x^i})$ of local coordinates (x^i). We call (x^i, y^i) the coordinates induced in $\pi_M^{-1}(V)$ from (x^i). We suppose that all the used maps belong to the class C^∞ and we shall adopt the Einstein summation convention through this paper.

Now, we must recall the definition of vertical and complete lifts of differentiable elements defined on M. Let f, X, w, G, F and $\hat{\nabla}$ be a function, a vector field, a 1-form, a tensor field of type $(0,2)$, $(1,1)$- tensor and a linear connection, respectively. We denote by f^v, X^v, w^v , G^v and F^v the vertical lifts and by f^c, X^c, w^c, G^c , F^c and $\hat{\nabla}^c$ the complete lifts, respectively. For a function f on M, we have

$$\begin{aligned} f^v &= f \circ \pi_M, \\ f^c &= y^i \frac{\partial f}{\partial x^i}, \end{aligned}$$

with respect to induced coordinates. Moreover, these lifts have those properties:

$$\left.\begin{aligned} (fX)^v &= f^v X^v, & F^c X^c &= (FX)^c, \\ (fX)^c &= f^v X^c + f^c X^v, & F^c X^v &= (FX)^v, \\ X^v f^v &= 0, & F^v X^c &= (FX)^v, \\ X^c f^c &= (Xf)^c, & F^v X^v &= 0, \\ [X,Y]^c &= [X^c, Y^c], & G^c(X^v, Y^v) &= 0, \\ [X^v, Y^v] &= 0, & G^c(X^c, Y^c) &= (G(X,Y))^c, \\ w^c(X^c) &= (w(X))^c, & \hat{\nabla}^c_{X^c} Y^c &= (\hat{\nabla}_X Y)^c, \\ w^v(X^v) &= 0, & \hat{\nabla}^c_{X^v} Y^v &= 0, \end{aligned}\right\} \tag{2.1}$$

$$\left.\begin{aligned} X^v f^c &= X^c f^v = (Xf)^v, \\ w^v(X^c) &= w^c X^v = (w(X))^v, \\ [X,Y]^v &= [X^v, Y^c] = [X^c, Y^v], \\ G^c(X^v, Y^c) &= G^c(X^c, Y^v) = (G(X,Y))^v, \\ \hat{\nabla}^c_{X^v} Y^c &= \hat{\nabla}^c_{X^c} Y^v = (\hat{\nabla}_X Y)^v \end{aligned}\right\} \tag{2.2}$$

(cf. [11]). Hence, it is easily seen that if G is a Riemannian metric on M, then G^c is a semi Riemannian metric on TM and index of G is equal to dimension of M. Thus, if (M,G) is a Riemannian manifold then (TM, G^c) is a semi Riemannian manifold with index n. Let $\hat{\nabla}$ be a metrical connection on M with respect to G. In this case, by considering equalities in (2.1) we can say that $\hat{\nabla}^c$ is a metrical connection on TM with respect to G^c. Through this paper, as a semi-Riemannian structure on TM we shall consider $(TM, G^c, \hat{\nabla}^c)$.

Let $f : M \to \mathbb{R}$ be a submersion. In this case for each $t \in range f$, $f^{-1}(t) = S$ is a level hypersurfaces in M , i.e. S_t is $(n-1)-$ dimensional submanifold of M [4]. We know that a vector field on M is tangent to S if and only if $X(f) = 0$. According to this

$$\Im_0^1(S) = \{X \in \Im_0^1(M) : X(f) = 0\}.$$

Let us consider a vector field on M, say X. If for each $p \in Dom(X) \cap S$ $X_p \in T_p S$, then we say that X is a tangent vector field to S. We denote by $\Im_0^1(S)^T$ the module of vector fields on M being tangent to S.

If (M,G) is a Riemannian manifold, then we write $\Im_0^1(S)^\perp = Span\{\text{grad} f\}$, where grad$f$ is gradient vector field of f. We also state that $X \in \Im_0^1(S)^T$ if and only if $G(X, \text{grad} f) = 0$.

Let us consider locally orthonormal basis of $\Im_0^1(M)$,

$$\Delta = \{X_1, ..., X_{n-1}, \xi\} \tag{2.3}$$

in a neighbourhood U of a point p in S , such that for each $q \in U$ and $i = 1, 2, ..., n-1$, $X_i(q)$ is an element of T_qS and $\xi = \frac{\text{grad} f}{|\text{grad} f|}$ is a unit normal field of the hypersurface S. We call the set Δ a local basis of M adapted to S. We get the components of $\hat{\nabla}$ with respect to this adapted basis in following equalities.

$$\left.\begin{array}{lcl} \hat{\nabla}_{X_i} X_j & = & \Gamma_{ij}^k X_k + G(HX, Y)\xi, \\ \hat{\nabla}_{X_i}\xi & = & -HX_i = -h_{ij}X_j, \\ \hat{\nabla}_{\xi} X_i & = & \omega_{ij}X_j + \sigma_i\xi, \\ \hat{\nabla}_{\xi}\xi & = & -\sigma_i X_k, \end{array}\right\} \tag{2.4}$$

where, $\Gamma_{ij}^k, \omega_{ij}, \sigma_i \in \Im_0^0(M)$ and $H = [h_{ij}]$ is shape operator of S.

We denote by $\Im_0^1(TS)^\intercal$ the vector fields on TM being tangent to the TS, from [10] and [11],

$$\Im_0^1(TS)^\intercal = Span\{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v, \xi^v\}, \tag{2.5}$$

and

$$\Im_0^1(TM)\mid_{TS} = \Im_0^1(TS)^\intercal \oplus \Im_0^1(TS)^\perp. \tag{2.6}$$

From (2.1), (2.2), (2.5) and (2.6) as a local basis for $\Im_0^1(TM)$ along TS, we get

$$\Psi = \{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v, \xi^v, \xi^c\}.$$

Lemma 2.1. *If the basis Δ has same orientation with the natural basis $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$, then Ψ has also same orientation with the induced basis $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$.*

In semi- Riemannian geometry, this basis Ψ is known as a quasi orthonormal basis of $\Im_0^1(M)$.

3. Level Hypersurfaces of f^v

In this section, we will interest a special level hypersurface of f^v. If f is an element of $\Im_0^0(M)$ and $Dom(f) = U$ is an open subset of M, then the vertical lift of f is defined on TU.

If $f : M \to \mathbb{R}$ is a submersion, then f^v is also. Indeed, let $f : M \to \mathbb{R}$ is a submersion, then f has rank one for each p in U . This means that, for at least i, $(1 \leq i \leq n)$, $\frac{\partial f}{\partial x^i}\mid_p \neq 0$, $p \in U$. Furthermore, we can write the jacobien matrix of f^v as follows,

$$J(f^v)\mid_{v_p} = \begin{bmatrix} \frac{\partial f}{\partial x^i}\mid_p & 0 \end{bmatrix}_{1\times 2n}$$

for a point $v_p \in TU$. It follows that f^v has rank one.

From definition of f^v, it is easily seen that

$$\begin{aligned} \bar{S} &= (f^v)^{-1}(t) \\ &= S \times \mathbb{R}^n, \\ &= TM\mid_S \\ &= \bigcup_{p\in S} T_pM. \end{aligned}$$

Let (V, φ) be a coordinate neighbourhood in M. Then, $(\hat{V} = \pi^{-1}(V), d\varphi)$ is a coordinate neighbourhood in TM. Let us construct the differentiable structure of $\bar{S}$:

$$\begin{aligned} \bar{S} \cap \hat{V} &= \bar{V} \\ &= \left\{(p, v) \in \hat{V} : p \in S, v_p \in T_pM\right\} \end{aligned}$$

Thus, a local coordinate system on $\bar{V}$ is written as to be $\bar{\varphi} = (u^a, y^i)$, $(1 \leq a \leq n-1)$ and we take $\{\bar{V}_\alpha, \bar{\varphi}_\alpha\}_{\alpha\in I}$ as a differentiable structure on $\bar{S}$. In addition we can also say that $(\bar{S}, \bar{\pi}, M, \mathbb{R}^n)$ has a vector bundle structure with rank n and by this structure it is a vector subbundle of TM, where $\bar{\pi}$ is restriction of π_M to $\bar{S}$.

Let $\bar{\imath} : \bar{S} \to TM$ be natural injection in terms of local coordinates (x^i, y^i), $\bar{\imath}$ has following local expressions

$$x^i = x^i(u^a), \qquad y^i = y^i.$$

Definition 3.1 ([1]). Let $(M, G = (g_{ij}))$ be a semi- Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function. The following vector field is called gradient of f,

$$\mathrm{grad} f \mid_p = g^{ij}(p) \frac{\partial f}{\partial x^j}(p) \frac{\partial}{\partial x^i} \mid_p,$$

where $p \in dom(f)$, $\{x^1, x^2, ..., x^n\}$ is a localy coordinate system on M around p and the matrix $[g^{ij}]$ is invers of $[g_{ij}]$,

Lemma 3.2. *The gradient vector field of f^v with respect to semi Riemannian metric G^c is the vertical lift of $gradf$, i.e*

$$gradf^v = (gradf)^v.$$

Proof. If G has matrix expression $\left[g_{ij}\right]$ then the matrix expression of G^c is as follows:

$$\begin{bmatrix} (g_{ij})^c & (g_{ij})^v \\ (g_{ij})^v & 0 \end{bmatrix},$$

[11]. We can find inverse of this matrix as in following form,

$$\begin{bmatrix} 0 & (g^{ij})^v \\ (g^{ij})^v & (g^{ij})^c \end{bmatrix}.$$

From definition of gradient vector field, we get the following equality,

$$\begin{aligned} \mathrm{grad} f^v &= 0.\frac{\partial f^v}{\partial x^j}\frac{\partial}{\partial x^i} + (g^{ij})^v \frac{\partial f^v}{\partial x^j}\frac{\partial}{\partial y^i} + (g^{ij})^v \frac{\partial f^v}{\partial y^j}\frac{\partial}{\partial x^i} + (g^{ij})^c \frac{\partial f^v}{\partial y^j}\frac{\partial}{\partial y^i} \\ &= (g^{ij})^v \frac{\partial f^v}{\partial x^j}\frac{\partial}{\partial y^i} \\ &= (\mathrm{grad} f)^v. \end{aligned}$$

The proof is complete. □

Since the vector field $(\mathrm{grad} f)^v$ is orthogonal to the submanifold $\bar{S}$ and thus the vector field $\frac{(\mathrm{grad} f)^v}{|(\mathrm{grad} f)^v|} = \left(\frac{(\mathrm{grad} f)}{|(\mathrm{grad} f)|}\right)^v = \xi^v$ is a unit normal vector field of $\bar{S}$.

Theorem 3.3. *If $X \in \Im^1_0(M)$ is a tangent vector field to S, then the complete and vertical lifts of X are tangent to $\bar{S}$.*

Proof. Since X is tangent to S, for each $p \in Dom(X) \cap S$, $X_p \in T_pS$. On the other hand,

$$\begin{aligned} (df^v)_u(X^v_u) &= X^c_u(f^v) \\ &= (X(f))^v(u) \\ &= (X(f))(p) \\ &= X_p(f) \\ &= 0, \end{aligned}$$

where $u = u_p \in \bar{S}$. In addition, we know from formulas of lifts in (2.1) that

$$\begin{aligned} (df^v)_u(X^v_u) &= X^v_u(f^v) \\ &= (X^v(f^v))(u) \\ &= 0, \end{aligned}$$

see (2.1). Thus, X^c and X^v are tangent vector fields to $\bar{S}$. □

4. Lightlike Geometry of $\bar{S}$

In this section, we investigate the lightlike submanifold structure of $\bar{S}$ in semi-Riemannian manifold (TM, G^c). For this purpose we need to some informations about the lightlike submanifold geometry.

Firstly, we note that the notation and fundamental formulas used in this study are the same as [5], following Chap. 4. Let $\bar{M}$ be a $(m+2)$-dimensional semi-Riemannian manifold with index $q \in \{1, ..., m+1\}$. Let M be a hypersurface of $\bar{M}$. Denote by g the induced tensor field by $\bar{g}$ on M. M is called a lightlike hypersurface if g is of constant rank m. Consider the vector bundles $TM^{\perp}$ and $Rad(TM)$ whose fibres are defined by

$$T_xM^{\perp} = \{Y_x \in T_X M |\, g_x(Y_x, X_x) = 0, \forall X_x \in T_xM\}$$

and

$$Rad(T_xM) = T_xM \cap T_xM^{\perp},$$

for any $x \in M$, respectively. Thus, a hypersurface M of $\bar{M}$ is lightlike if and only if $Rad(T_xM) \neq \{0\}$ for all $x \in M$.

If M is a lightlike hypersurface, then we consider the complementary distribution $S(TM)$ of $TM^{\perp}$in TM which is called a screen distribution. From [2], we know that it is nondegenerate. Thus, we have direct orthogonal sum

$$TM = S(TM) \perp TM^{\perp}. \tag{4.1}$$

Since $S(TM)$ is non-degenerate with respect to $\bar{g}$, we have

$$T\bar{M} = S(TM) \perp S(TM)^{\perp},$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\bar{M}|_M$.

Now, we will give an important theorem about lightlike hypersurfaces which enables us to set fundamental equations of M.

Remark 4.1. From now on we denote by $\Gamma(E)$ the module of cross sections of a vector bundle E.

Theorem 4.2 ([5]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\bar{M}$. Then, there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $U \subset M$, there exist a unique section N of $tr(TM)$ on U satisfying*

$$\bar{g}(N, \xi) = 1$$

and

$$\bar{g}(N, N) = \bar{g}(N, W) = 0, \forall W \in \Gamma(S(TM)|_U).$$

From Theorem 4.2 , we have

$$T\bar{M}|_M = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{4.2}$$

$tr(TM)$ is called the null transversal vector bundle of M with respect to $S(TM)$. Let $\bar{\nabla}$ be Levi-Civita connection on $\bar{M}$. We have

$$\bar{\nabla}_X Y = \overset{*}{\nabla}_X Y + h(X, Y), \quad X, Y \in \Gamma(TM) \tag{4.3}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{4.4}$$

where $\overset{*}{\nabla}_X Y, A_V X \in \mathfrak{I}^1_0(TM)$ and $h(X, Y), \nabla^t_X V \in \Gamma(tr(TM))$. ∇ is a symmetric linear connection on M which is called an induced linear connection, ∇^t is a linear connection on the vector bundle $tr(TM)$, h is a $\Gamma(tr(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V .

Locally, suppose $\{\xi, N\}$ is a pair of sections on $U \subset M$ in Theorem 4.2. Then, define a symmetric $\mathfrak{I}^0_0(U)$ −bilinear form B and a 1-form τ on U by

$$B(X, Y) = \bar{g}(h(X, Y), \xi), \forall X, Y \in (TM|_U)$$

and

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi).$$

Thus, (4.3) and (4.4) locally become

$$\bar{\nabla}_X Y = \overset{*}{\nabla}_X Y + B(X, Y)N \tag{4.5}$$

and

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{4.6}$$

respectively.

Let denote P as the projection of TM on $S(TM)$. We consider decomposition

$$\overset{*}{\nabla}_X PY = \nabla_X PY + C(X, PY)\xi$$

and

$$\overset{*}{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi,$$

where $\nabla_X PY$ and $A^*_\xi X$ belong to $S(TM)$ and C is a 1−form on U. Note that ∇ is not metric connection [3]. We have the following equations,

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0,$$
$$g(A^*_\xi X, PY) = B(X, PY), \quad \bar{g}(A^*_\xi X, N) = 0,$$

for any $X, Y \in \Gamma(TM)$.

Now, we will apply the above theory to the hypersurface $\bar{S}$.

Theorem 4.3. *$\bar{S}$ is a lightlike hypersurface of TM.*

Proof. We know that a vector field $\bar{X} \in \Im^1_0(\bar{S})$ if and only if

$$df^v(\bar{X}) = \bar{X}(f^v) = 0.$$

From (2.1) for all $X \in \Im^1_0(\bar{S})$

$$df^v(X^c) = 0,$$
$$df^v(X^v) = 0,$$
$$df^v(\xi^v) = 0.$$

In addition, $G^c(X^c, \xi^v) = G^c(X^v, \xi^v) = G^c(\xi^v, \xi^v) = 0$. This means that the restriction of G^c to $\Im^1_0(\bar{S})$ is 1- degenerate and

$$Rad(T_u\bar{S}) = Sp\{\xi^v_u\}, \forall u \in \bar{S}.$$

□

To describe a screen subspace of $T\bar{S}$, we must write following decomposition from (4.1),

$$T_u\bar{S} = S(T_u\bar{S}) \perp Rad(T_u\bar{S}), \quad u \in \bar{S}.$$

Since $\{X_1, ..., X_{n-1}, \xi\}$ is a frame of M adapted to S, from [11], [10] and Theorem 4.3, the following set

$$\{X^c_1, ..., X^c_{n-1}, X^v_1, ..., X^v_{n-1}, \xi^v\} \tag{4.7}$$

is also basis for $\bar{S}$ adapted to TS.

In this case we get

$$\Im^1_0(\bar{S}) = Span\{X^c_1, ..., X^c_{n-1}, X^v_1, ..., X^v_{n-1}\} \perp Span\{\xi^v\}.$$

On the other hand, from (4.2), we have the following decomposition for $\Im^1_0(TM)$,

$$\begin{aligned}\Im^1_0(TM)_{|\bar{S}} &= (\Gamma(S(T\bar{S})) \perp \Gamma(Rad(T\bar{S}))) \oplus tr(T\bar{S})) \\ &= (Span\{X^c_1, ..., X^c_{n-1}, X^v_1, ..., X^v_{n-1}\} \perp Span\{\xi^v\}) \oplus tr(T\bar{S}).\end{aligned}$$

By using (2.1) and (2.2), we have those equalities,

$$G^c(\xi^c, \xi^c) = 0, \quad G^c(\xi^v, \xi^c) = 1$$

and

$$G^c(\xi^c, \bar{X}) = 0 \quad \forall \bar{X} \in \Gamma(S(T\bar{S}) \mid_{\bar{U}})$$

on a coordinate neighbourhood $\bar{U} \subset \bar{S}$. Thus, from Theorem 4.2, the lightlike transversal bundle of $\bar{S}$ is as follows,

$$tr(T\bar{S} \mid_{\bar{U}}) = \bigcup_{u \in \bar{U}} Span\{\xi^c \mid_u\}$$

with respect to $S(T\bar{S})$. By means of (4.1) and (4.2) for $\hat{X} \in \Im^1_0(TM)$ we can write the following decomposition,

$$\hat{X}\,|_{\bar{U}} = \tilde{X} + \lambda\xi^v + \mu\xi^c\,,$$

where $\tilde{X} \in \Im^1_0(\bar{S})$ tangent to TS and $\lambda, \mu \in \Im^0_0(\bar{S})$ on a neighbourhood $\bar{U}$.

5. The Induced Geometrical Objects

In this section, we investigate the lightlike submanifold geometry of $\bar{S}$. Because of we shall investigate the level sets of f and f^v, first of all we write fundamental equalities of S.

Let (M, G) be Riemannian manifold, S be a hypersurface in M and g be induced metric on S from G, then by definition we have

$$g(X, Y) = G(X, Y) \qquad \text{for } X, Y \in \Im^1_0(S).$$

We know that with this induced metric g, S is a Riemannian submanifold of M. The Gauss and Weingarten formulae of S as in following, respectively,

$$\hat{\nabla}_X Y = \nabla_X Y + g(HX, Y)\xi, \tag{5.1}$$

$$\hat{\nabla}_X \xi = -HX,$$

where $\hat{\nabla}$ and ∇ are Riemannian covariant differentiations determined by G and g, respectively. In addition H and $g(HX, Y)$ are shape operator and second fundamental form of S, respectively.

By using (4.3) and (4.4) we get,

$$\hat{\nabla}^c_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{h}(\bar{X}, \bar{Y}) \tag{5.2}$$

and

$$\hat{\nabla}^c_{\bar{X}} V = -\bar{A}_V \bar{X} + \nabla^t_{\bar{X}} V \tag{5.3}$$

for any $\bar{X}, \bar{Y} \in \Im^1_0(\bar{S})$ and $V \in \Gamma(trT\bar{S})$. Here, $\bar{\nabla}$ and ∇^t are induced connections on $\bar{S}$ and $tr(T\bar{S})$ respectively. $\bar{h}$ and A_V are second fundamental form and shape operator of $\bar{S}$, respectively. The equalities (5.2) and (5.3) are the Gauss and Weingarten formulae, respectively [5].

Define a symetric bilinear form $\bar{B}$ and a 1-form τ on $\bar{U} \subset \bar{S}$ by

$$\begin{aligned} \bar{B}(\bar{X}, \bar{Y}) &= G^c(\bar{h}(\bar{X}, \bar{Y}), \xi^c), \qquad \forall \bar{X}, \bar{Y} \in \Im^1_0(\bar{S}),\\ \tau(\bar{X}) &= G^c(\nabla^t_{\bar{X}} \xi^c, \xi^c), \qquad \forall \bar{X} \in \Im^1_0(\bar{S}). \end{aligned}$$

It follows that

$$\bar{h}(\bar{X}, \bar{Y}) = \bar{B}(\bar{X}, \bar{Y})\xi^c$$

and

$$\nabla^t_{\bar{X}} \xi^c = \tau(\bar{X})\xi^c.$$

Hence, on $\bar{U}$, (4.5) and (4.6) become

$$\hat{\nabla}^c_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{B}(\bar{X}, \bar{Y})\xi^c$$

and

$$\hat{\nabla}^c_{\bar{X}} \xi^c = -A_{\xi^c} \bar{X} + \tau(\bar{X})\xi^c,$$

respectively.

On the other hand, if P denotes the projection of $\Im^1_0(\bar{S})$ to $\Im^1_0(TS)$ with respect to the decomposition

$$T_u\bar{S} = S(T_u\bar{S}) \perp Rad(T_u\bar{S})$$

we obtain the local Gauss and Weingarten formulas on $S(T\bar{S})$

$$\bar{\nabla}_{\bar{X}} P\bar{Y} = \tilde{\nabla}_{\bar{X}} P\bar{Y} + \tilde{C}(\bar{X}, P\bar{Y})\xi^v, \tag{5.4}$$

$$\bar{\nabla}_{\bar{X}} \xi^v = -\tilde{A}_{\xi^v} \bar{X} - \tilde{\tau}(\bar{X})\xi^v, \tag{5.5}$$

where $\bar{X} \in \Im^1_0(\bar{S})$, $\bar{Y} \in \Im^1_0(\bar{S})$, $\tilde{C}$, $\tilde{A}_{\xi^c}$ and $\tilde{\nabla}$ are the local second fundamental form, the local shape operator and the linear connection on $S(T\bar{S})$. In [10], we see that the vertical and complete lifts of differentiable elements defined on M

can be described the other differentiable elements defined on TM. For example, let us consider $\hat{X}, \hat{Y} \in \Im_0^1(TM)$, then $\hat{X} = \hat{Y}$ if and only

$$\hat{X}(f^c) = \hat{Y}(f^c)$$

for all $f \in \Im_0^0(M)$. In addition, take two 1- forms $\hat{\omega}$,$\hat{\rho} \in \Im_1^0(TM)$, then $\hat{\omega} = \hat{\rho}$ if and only if

$$\hat{\omega}(X^c) = \hat{\rho}(X^c),$$

for all $X \in \Im_0^1(TM)$. Because of this, instead of taking any vector field, we take the complete and vertical lifts of vector fields tangent and orthogonal to S.

Using theorem 4.3 and the information above, it is sufficient for us to use the vertical and complete lift of the vector fields tangent and normal to S.

Now, we shall write the Gauss and Weingarten formulae of $\bar{S}$ and screen distribution. Let X and Y be vector fields in $\Im_0^1(M)$ tangent to S. By taking into account (2.1), (2.2), (5.1) and (2.4), we have the following aqualities,

$$\left.\begin{array}{rcl}
\hat{\nabla}^c_{X^c} Y^c & = & \left(\hat{\nabla}_X Y\right)^c \\
& = & \nabla^c_{X^c} Y^c + G^c\left(H^c X^c, Y^c\right)\xi^v \\
& & +G^c\left(H^v X^c, Y^c\right)\xi^c, \\
\hat{\nabla}^c_{X^c} Y^v & = & \left(\hat{\nabla}_X Y\right)^v \\
& = & \nabla^c_{X^c} Y^v + G^c\left(H^v X^c, Y^c\right)\xi^v \\
& = & \hat{\nabla}^c_{X^v} Y^c, \\
\hat{\nabla}^c_{\xi^v} Y^c & = & \left(\hat{\nabla}_\xi Y\right)^v \\
& = & \left(\omega_i(Y)X_i + \sigma(Y)\xi\right)^v, \\
& = & (\omega_i(Y))^v X_i^v + \sigma^v(Y^c)\xi^v, \\
\hat{\nabla}^c_{X^c}\xi^v & = & \left(\hat{\nabla}_X \xi\right)^v = H^v X^c, \\
\hat{\nabla}^c_{X^v} Y^v & = & \hat{\nabla}_{\xi^v} Y^v \quad = \hat{\nabla}_{\xi^v}\xi^v = \hat{\nabla}_{X^v}\xi^v = 0,
\end{array}\right\} \tag{5.6}$$

where σ is a 1− form and ω_i ' s are $\Im_0^0(M)$− valued functions such that , for $i, j = 1, 2, ..., n-1$

$$\sigma(X_i) = \sigma_i,$$
$$\omega_i(X_j) = \omega_{ij} = -\omega_{ji},$$

with respect to adapted basis (4.7). On the other hand, from (5.6) Weingarten formulas of $\bar{S}$ are as in follows,

$$\begin{array}{rcccl}
\hat{\nabla}^c_{X^c}\xi^c & = & \left(\hat{\nabla}_X\xi\right)^c & = & H^c X^c, \\
\hat{\nabla}^c_{X^v}\xi^c & = & \left(\hat{\nabla}_X\xi\right)^v & = & H^c X^v, \\
\hat{\nabla}^c_{\xi^v}\xi^c & = & \left(\hat{\nabla}_\xi\xi\right)^v & = & -\sigma_i^v X_i^v
\end{array}$$

where X_i 's are elements of adapted basis given (2.3).

From (5.2), (5.6) and [10] the second fundamental form of $\bar{S}$ is as in following,

$$\begin{array}{rcll}
\bar{B}(X^c, Y^c) & = & G^c\left(H^v X^c, Y^c\right), & \\
\bar{B}(X^c, Y^v) & = & \bar{B}(X^c, \xi^v) & = 0, \\
\bar{B}(X^v, Y^c) & = & \bar{B}(X^v, Y^v) & = 0, \\
\bar{B}(\xi^v, Y^c) & = & \bar{B}(\xi^v, Y^v) & = 0, \\
\bar{B}(\xi^v, \xi^v) & = & \bar{B}(X^v, \xi^v) & = 0.
\end{array}$$

By virtue of (4.7), we have following Theorem.

Theorem 5.1. *S is a totally geodesic hypersurface in M if and only if $\bar{S}$ is a totally geodesic lightlike hypersurface in TM.*

From (5.3), (5.6) and [10], the shape operator of $\bar{S}$ is as in following,

$$\begin{array}{rcl}
A_{\xi^c} X^c & = & -H^c X^c, \\
A_{\xi^c} X^v & = & -H^c X^v, \\
A_{\xi^c}\xi^v & = & -\sigma_i^v X_i^v.
\end{array}$$

The marix representation of the shape operator A_{ξ^c} of $\bar{S}$ with respect to adapted basis can be represented in matrix form as in follows;

$$A_{\xi^c} = \begin{bmatrix} h_{ij} & 0 & 0 \\ h^c_{ij} & h_{ij} & -\sigma^v_i \\ 0 & 0 & 0 \end{bmatrix},$$

where h_{ij} 's are the components of the shape operator H of S according to basis $\{X_1, X_2, ..., X_{n-1}\}$. By considering [9], Def. 3.2, we give following Theorem.

Theorem 5.2. *If S is a minimal hypersurface in M if and only if $\bar{S}$ is also minimal in TM.*

From equalities (5.6) we have,

$$\nabla^t_{X^c}\xi^c = \nabla^t_{X^v}\xi^c = \nabla^t_{\xi^v}\xi^c = 0.$$

Hence, it is clear that $\tau = 0$.

From (5.6), the induced connection on $\bar{S}$ is as in follows,

$$\left.\begin{array}{rcl} \bar{\nabla}_{X^c}Y^c &=& \nabla^c_{X^c}Y^c + G^c\left(H^cX^c, Y^c\right)\xi^v, \\ \bar{\nabla}_{X^c}Y^v &=& \bar{\nabla}_{X^v}Y^c, \\ &=& \nabla^c_{X^c}Y^v + G^c\left(H^vX^c, Y^c\right)\xi^v, \\ \bar{\nabla}_{\xi^v}Y^c &=& (\omega_i(Y))^v X^v_i + \sigma^v(Y^c)\xi^v, \\ \bar{\nabla}_{X^c}\xi^v &=& -H^vX^c, \\ \bar{\nabla}_{X^v}Y^v &=& \bar{\nabla}_{X^v}\xi^v = 0, \\ \bar{\nabla}_{\xi^v}\xi^v &=& \bar{\nabla}_{\xi^v}Y^v = 0. \end{array}\right\} \tag{5.7}$$

From Theorem 3.3, the vertical and complete lifts of vector fields tangent to S are also tangent to $\bar{S}$. In addition,

$$G^c(X^c, \xi^v) = G^c(X^v, \xi^v) = 0.$$

It means that $X^v, X^v \in \Gamma(S(T\bar{S}))$ and as a cosequence of this we have

$$PX^c = X^c \text{ and } PX^v = X^v.$$

From these equalities in (5.7) we obtain

$$\begin{array}{l} \tilde{A}_{\xi^v}X^c = -H^vX^c, \\ \tilde{A}_{\xi^v}X^v = \tilde{A}_{\xi^v}\xi^v = 0. \end{array}$$

Hence, by considering (5.5), the shape operator $\tilde{A}_{\xi^v}$ of screen bundle can be represented in matrix form, with respect to adapted basis (4.7), as in the follows.

$$\tilde{A}_{\xi^v} = \begin{bmatrix} h_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By (5.4) and (5.7), we have the second fundamental form of $S(T\bar{S})$ is as in follows,

$$\left.\begin{array}{rcl} \tilde{C}(X^c, Y^c) &=& G^c\left(H^cX^c, Y^c\right), \\ \tilde{C}(X^c, Y^v) &=& \tilde{C}(X^v, Y^c), \\ &=& G^c\left(H^vX^c, Y^c\right), \\ \tilde{C}(X^v, Y^v) &=& 0, \\ \tilde{C}(\xi^v, Y^c) &=& \sigma^v(Y^c), \\ \tilde{C}(\xi^v, Y^v) &=& 0. \end{array}\right\} \tag{5.8}$$

Thus, by considering (5.8) we have,

Theorem 5.3. *The screen distribution $S(T\bar{S})$ is totally geodesic if and only if the followings are satisfied*
i) S is totally geodesic
ii) σ is identically zero on S, i.e. for all $p \in S$, $T_pS = \ker\sigma_p$.

Corollary 5.4. *The induced linear connection on $S(T\bar{S})$,*

$$\begin{array}{rclcl} \tilde{\nabla}_{X^c}Y^c & = & \nabla^c_{X^c}Y^c, & & \\ \tilde{\nabla}_{X^c}Y^v & = & \tilde{\nabla}_{X^v}Y^c & = & \nabla^c_{X^c}Y^v, \\ \tilde{\nabla}_{X^v}Y^v & = & \tilde{\nabla}_{X^v}\xi^v & & \\ & = & \tilde{\nabla}_{\xi^v}\xi^v & = & \tilde{\nabla}_{\xi^v}X^v = 0, \\ \tilde{\nabla}_{\xi^v}X^c & = & (\omega_i(X))^v X^v_i. & & \end{array}$$

Now, we will demonstrate the structure described above with an example.

Example 5.5. Let us consider 3− dimensional Euclidean space $\mathbb{E}^3$ with standard inner product G as a Riemannian metric and a function $f : \mathbb{E}^3 \to \mathbb{R}$. Let f be defined as in following,

$$\begin{array}{rcl} f & : & \mathbb{E}^3 \to \mathbb{R} \\ f(x, y, z) & = & x^2 + y^2 + z^2. \end{array}$$

Suppose that t_0 be a positive real number. We can easily see that t_0 a regular value of f. Then, $f^{-1}(t_0) = S = S^2_{t_0}$ is a hypersurface in $\mathbb{R}^3$, i.e 2- Sphere with t_0 radius.. We get the gradient vector field of f as follows

$$\text{grad} f = x\partial_x + y\partial_y + z\partial_z,$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$.

The normal vector field of S can be obtained as

$$\xi = x\partial_x + y\partial_y + z\partial_z.$$

Now, take two vector fields in $\Im^1_0(\mathbb{E}^3)$ are tangent to S.

$$X = \frac{\sigma}{\alpha}(zx\partial_x + zy - (x^2 + y^2)),$$

$$Y = \frac{1}{\alpha}(-y\partial_x + x\partial_y),$$

where $\sigma = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ and $\alpha = \sqrt{x^2 + y^2}$.

Thus, we obtained a basis for $\Im^1_0(\mathbb{E}^3)$ adapted to S. Indeed,

$$X(f) = \frac{\sigma}{\alpha}(2zx^2 + 2zy^2 - (x^2 + y^2)z) = 0.$$

Similarly,

$$Y(f) = 0.$$

These mean that for every $p \in S$, X_p and Y_p are tangent to S. Moreover, the set $\{X, Y, \xi\}$ is locally basis of $\Im^1_0(\mathbb{E}^3)$ adapted to S.

Now, we obtain local epression of $\hat{\nabla}$ according to basis $\{X, Y, \xi\}$:

$$\left.\begin{array}{rclrcl} \hat{\nabla}_X X & = & -\sigma\xi, & \hat{\nabla}_Y X & = & z\frac{\sigma}{\alpha}Y, \\ \hat{\nabla}_X Y & = & 0, & \hat{\nabla}_Y Y & = & -z\frac{\sigma}{\alpha}X - \sigma\xi, \\ \hat{\nabla}_X \xi & = & \sigma X, & \hat{\nabla}_Y \xi & = & \sigma Y, \\ \hat{\nabla}_\xi X & = & 0, & \hat{\nabla}_\xi Y & = & 0, \\ \hat{\nabla}_\xi \xi & = & 0. & & & \end{array}\right\} \tag{5.9}$$

From (5.9), we have Gauss and Weingarten formulaes of S as in following,

$$\left.\begin{array}{rclrcl} \hat{\nabla}_X X & = & -\sigma\xi, & \hat{\nabla}_Y X & = & z\frac{\sigma}{\alpha}Y, \\ \hat{\nabla}_X Y & = & 0, & \hat{\nabla}_Y Y & = & -z\frac{\sigma}{\alpha}X - \sigma\xi, \end{array}\right\} \tag{5.10}$$

$$\hat{\nabla}_X \xi = \sigma X, \qquad \hat{\nabla}_Y \xi = \sigma Y. \tag{5.11}$$

From (5.11), it is easily seen that matrix representation ofthe shape operator is as in follows,

$$H = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}.$$

For example, if we take $t_0 = r > 0$, S will be S^2_r and thus we obtain,

$$H = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{bmatrix}.$$

Let us find level hypersurface of the vertical lift of f , f^v

$$\begin{aligned}(f^v)^{-1}(t_0) &= \{(p,u) \in T\mathbb{R}^3 \mid f(p) = t_0,\ u \in \mathbb{R}^3\} \\ &= \bar{S}.\end{aligned}$$

If a locally coordinate system on S is $\{u, v\}$, then the natural inclusion of $\bar{S}$ is given locally in the form

$$\begin{aligned} x &= x \circ \pi = x(u,v), \\ y &= y \circ \pi = y(u,v), \\ z &= z \circ \pi = z(u,v), \\ \bar{x} &= \bar{x}, \\ \bar{y} &= \bar{y}, \\ \bar{z} &= \bar{z}, \end{aligned}$$

where $\{x, y, z, \bar{x}, \bar{y}, \bar{z}\}$ the locally coordinate functions induced by $\{x, y, z\}$ on $T\mathbb{E}^3$, $x \circ \pi$, $y \circ \pi$ and $z \circ \pi$ on $T\mathbb{E}^3$ are identified with x, y and z, respectively.

Being a local basis of $\Im^1_0(T\mathbb{R}^3)$ adapted to $\bar{S}$, we can choose the ordered set $\Phi = \{X^c, Y^c, X^v, Y^v, \xi^v, \xi^c\}$. By considering (5.9), (5.10) and the basis Φ we have following equalities,

$$\left.\begin{array}{lcllcl}
\hat{\nabla}^c_{X^c} X^c & = & -\sigma^c \xi^v - \sigma^v \xi^c, & \hat{\nabla}^c_{X^v} X^c & = & \sigma^v \xi^v, \\
\hat{\nabla}^c_{X^c} X^v & = & -\sigma^v \xi^v, & \hat{\nabla}^c_{Y^v} X^c & = & (z\frac{\sigma}{\alpha})^v Y^v, \\
\hat{\nabla}_{X^c} \xi^v & = & \sigma^v X^v, & \hat{\nabla}^c_{X^v} Y^c & = & \hat{\nabla}^c_{X^v} X^v = 0, \\
\hat{\nabla}^c_{Y^c} X^c & = & (z\frac{\sigma}{\alpha})^c Y^v + (z\frac{\sigma}{\alpha})^v Y^c, & \hat{\nabla}^c_{X^v} Y^v & = & \hat{\nabla}^c_{Y^v} Y^v = 0, \\
\hat{\nabla}^c_{Y^c} X^v & = & (z\frac{\sigma}{\alpha})^v Y^v, & \hat{\nabla}_{X^v} \xi^v & = & \hat{\nabla}^c_{Y^v} X^v = 0, \\
\hat{\nabla}^c_{Y^c} Y^v & = & (-z\frac{\sigma}{\alpha})^v X^v - \sigma^v \xi^v, & \hat{\nabla}_{Y^v} \xi^v & = & \hat{\nabla}_{\xi^v} Y^c = 0, \\
\hat{\nabla}^c_{Y^c} \xi^v & = & \sigma^v Y^v, & \hat{\nabla}^c_{\xi^v} X^c & = & \hat{\nabla}_{\xi^v} X^v = 0, \\
\hat{\nabla}^c_{Y^c} Y^c & = & -(z\frac{\sigma}{\alpha})^v X^c - (z\frac{\sigma}{\alpha})^c X^v & \hat{\nabla}^c_{X^c} Y^v & = & \hat{\nabla}^c_{X^c} Y^c = 0, \\
 & = & -\sigma^c \xi^v - \sigma^v \xi^c, & \hat{\nabla}_{\xi^v} Y^v & = & \hat{\nabla}_{\xi^v} \xi^v = 0, \\
\hat{\nabla}^c_{Y^v} Y^c & = & (-z\frac{\sigma}{\alpha})^v X^v - \sigma^v \xi^v, & & &
\end{array}\right\} \quad (5.12)$$

$$\left.\begin{array}{lcl}
\hat{\nabla}^c_{X^c} \xi^c & = & \sigma^v X^c + \sigma^c X^v, \\
\hat{\nabla}^c_{Y^c} \xi^c & = & \sigma^v Y^c + \sigma^v Y^c, \\
\hat{\nabla}^c_{X^v} \xi^c & = & \sigma^v X^v, \\
\hat{\nabla}^c_{Y^v} \xi^c & = & \sigma^v Y^v, \\
\hat{\nabla}^c_{\xi^v} \xi^c & = & 0.
\end{array}\right\} \quad (5.13)$$

Here, (5.12) and (5.13) are Gauss and Weingarten formulaes of $\bar{S}$, respectively.

By using (5.12) we have the followings,

$$\left.\begin{array}{lcllcl}
\bar{\nabla}_{X^c} X^c & = & -\sigma^c \xi^v, & \bar{\nabla}^c_{X^v} X^c & = & -\sigma^v \xi^v, \\
\bar{\nabla}_{X^c} X^v & = & -\sigma^v \xi^v, & \bar{\nabla}^c_{X^v} Y^c & = & \bar{\nabla}^c_{X^v} X^v = 0, \\
\bar{\nabla}_{X^c} \xi^v & = & \sigma^v X^v, & \bar{\nabla}_{X^v} \xi^v & = & \bar{\nabla}^c_{Y^v} X^v = 0, \\
\bar{\nabla}^c_{Y^c} X^c & = & (z\frac{\sigma}{\alpha})^v Y^c + (z\frac{\sigma}{\alpha})^c Y^v, & \bar{\nabla}_{Y^v} \xi^v & = & \bar{\nabla}_{\xi^v} Y^c = 0, \\
\bar{\nabla}^c_{Y^c} Y^c & = & -(z\frac{\sigma}{\alpha})^v X^c - (z\frac{\sigma}{\alpha})^c X^v & \bar{\nabla}^c_{Y^v} Y^c & = & (-z\frac{\sigma}{\alpha})^v X^v \\
 & = & -\sigma^c \xi^v, & & = & -\sigma^v \xi^v, \\
\bar{\nabla}^c_{Y^c} X^v & = & (z\frac{\sigma}{\alpha})^v Y^v, & \bar{\nabla}^c_{Y^v} X^c & = & (z\frac{\sigma}{\alpha})^v Y^v, \\
\bar{\nabla}^c_{Y^c} Y^v & = & (-z\frac{\sigma}{\alpha})^v X^v - \sigma^v \xi^v, & \bar{\nabla}^c_{\xi^v} X^c & = & \hat{\nabla}_{\xi^v} X^v = 0, \\
\bar{\nabla}^c_{Y^c} \xi^v & = & \sigma^v Y^v, & \bar{\nabla}^c_{X^c} Y^c & = & \bar{\nabla}^c_{X^c} Y^v = 0, \\
\bar{\nabla}^c_{X^v} Y^v & = & \bar{\nabla}^c_{Y^v} Y^v = 0, & \bar{\nabla}_{\xi^v} Y^v & = & \hat{\nabla}_{\xi^v} \xi^v = 0.
\end{array}\right\} \quad (5.14)$$

These equalities in (5.14) describe the induced connection $\bar{\nabla}$ on $\bar{S}$. By using (5.12) we have second fundamental form of $\bar{S}$,

$$\begin{array}{lllllll} \bar{h}(X^c,X^c) & = & -\sigma^v\xi^c, & \bar{h}(Y^c,Y^c) & = & -\sigma^v\xi^c, \\ \bar{h}(X^c,Y^c) & = & \bar{h}(X^c,X^v)=0, & \bar{h}(Y^c,X^v) & = & \bar{h}(Y^c,Y^v)=0, \\ \bar{h}(X^c,\xi^v) & = & \bar{h}(\xi^v,X^c)=0, & \bar{h}(\xi^v,Y^c) & = & \bar{h}(\xi^v,Y^v)=0, \\ \bar{h}(\xi^v,\xi^v) & = & \bar{h}(X^c,Y^v)=0, & \bar{h}(\xi^v,X^v) & = & \bar{h}(Y^c,\xi^v)=0, \\ \bar{h}(Y^c,X^v) & = & 0. \end{array}$$

From (5.13) shape operator of $\bar{S}$ can be written as follows,

$$\left.\begin{array}{lllllll} \bar{A}_{\xi^c}(X^c) & = & \sigma^vX^c+\sigma^cX^v, & \bar{A}_{\xi^c}(X^v) & = & \sigma^vX^v, \\ \bar{A}_{\xi^c}(Y^c) & = & \sigma^vY^c+\sigma^cY^v, & \bar{A}_{\xi^c}(Y^v) & = & \sigma^vY^v, \\ \bar{A}_{\xi^c}(\xi^v) & = & 0. \end{array}\right\} \tag{5.15}$$

According to (5.15) the shape operator of $\bar{S}$ in $T\mathbb{R}^3$can be represented as in follows,

$$\bar{A}_{\xi^c}=\begin{bmatrix} \sigma^v & 0 & 0 & 0 & 0 \\ 0 & \sigma^v & 0 & 0 & 0 \\ \sigma^c & 0 & \sigma^v & 0 & 0 \\ 0 & \sigma^c & 0 & \sigma^v & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5}.$$

In addition, according to (5.14)

$$\begin{array}{llllll} \tilde{\nabla}_{X^c}X^c & = & 0, & \tilde{\nabla}^c_{X^v}X^c & = & 0, \\ \tilde{\nabla}^c_{X^c}Y^c & = & 0, & \tilde{\nabla}^c_{X^v}Y^c & = & 0, \\ \tilde{\nabla}_{X^c}X^v & = & 0, & \tilde{\nabla}^c_{X^v}X^v & = & 0, \\ \tilde{\nabla}^c_{X^c}Y^v & = & 0, & \tilde{\nabla}^c_{X^v}Y^v & = & 0, \\ \tilde{\nabla}^c_{Y^c}X^c & = & (z\frac{\sigma}{\alpha})^vY^c+(z\frac{\sigma}{\alpha})^cY^v, & \tilde{\nabla}^c_{Y^v}X^c & = & (z\frac{\sigma}{\alpha})^vY^v, \\ \tilde{\nabla}^c_{Y^c}Y^c & = & -(z\frac{\sigma}{\alpha})^vX^c-(z\frac{\sigma}{\alpha})^cX^v, & \tilde{\nabla}^c_{Y^v}Y^c & = & (-z\frac{\sigma}{\alpha})^vX^v, \\ \tilde{\nabla}^c_{Y^c}X^v & = & (z\frac{\sigma}{\alpha})^vY^v, & \tilde{\nabla}^c_{Y^v}X^v & = & 0, \\ \tilde{\nabla}^c_{Y^c}Y^v & = & (-z\frac{\sigma}{\alpha})^vX^v, & \tilde{\nabla}^c_{Y^v}Y^v & = & 0, \\ \tilde{\nabla}^c_{\xi^v}X^c & = & 0, & \tilde{\nabla}_{\xi^v}Y^c & = & 0, \\ \tilde{\nabla}_{\xi^v}X^v & = & 0, & \tilde{\nabla}_{\xi^v}Y^v & = & 0, \end{array}$$

and

$$\begin{array}{l} \hat{\nabla}^c_{X^c}\xi^v=\sigma^vX^v, \\ \hat{\nabla}^c_{X^v}\xi^v=0, \\ \hat{\nabla}^c_{Y^c}\xi^v=\sigma^vY^v, \\ \hat{\nabla}_{Y^v}\xi^v=0. \end{array}$$

The shape operator of screen bundle $\tilde{A}_{\xi^v}$ is given in following,

$$\begin{array}{l} \tilde{A}_{\xi^v}(X^c)=\sigma^vX^v, \\ \tilde{A}_{\xi^v}(Y^c)=\sigma^vY^c, \\ \tilde{A}_{\xi^v}(X^v)=0, \\ \tilde{A}_{\xi^v}(Y^v)=0. \end{array}$$

Hence, the matrix representation of $\tilde{A}_{\xi^v}$ is as in follows,

$$\tilde{A}_{\xi^v}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^v & 0 & 0 \\ \sigma^v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4},$$

with respect to ordered basis $\{X^c,Y^c,X^v,Y^v,\xi^v\}$. Thus, the second fundamental form of screen bundle is in the following,

$$
\begin{array}{rclrcl}
\tilde{C}(X^c, X^c) & = & -\sigma^c, & \tilde{C}(Y^c, X^v) & = & 0, \\
\tilde{C}(X^c, Y^c) & = & 0, & \tilde{C}(Y^c, Y^v) & = & -\sigma^v, \\
\tilde{C}(X^c, X^v) & = & -\sigma^v, & \tilde{C}(\xi^v, X^c) & = & 0, \\
\tilde{C}(X^c, Y^v) & = & 0, & \tilde{C}(\xi^v, Y^c) & = & 0, \\
\tilde{C}(Y^c, X^c) & = & 0, & \tilde{C}(\xi^v, X^v) & = & 0, \\
\tilde{C}(Y^c, Y^c) & = & -\sigma^c, & \tilde{C}(\xi^v, Y^v) & = & 0.
\end{array}
$$

6. Conclusion

In this paper, we saw that some differential geometrical properties of level hypersurfaces of the function f are preserved in this discussion. In addition to Tani's work [10], within the framework of this complete lift of Rimannian metrical structure, the other way of prolongation of hypersurfaces is described. Again, in this article, we noticed that, unlike [13], a level hypersurface of f^v is always lightlike, i.e it doesn't depend on any additional condition.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

M. Y. and A. Ö. contributed to the research, to the analysis of the results and to the writing of the manuscript.

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