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BETA GENERATED SLASH DISTRIBUTION: DERIVATION, PROPERTIES AND APPLICATION TO LIFETIME DATA

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Abstract. In this paper, we introduce a new distribution called beta generated slash distribution by applying the slash construction idea to the existing beta distribution of first kind. The statistical properties of the distribution such as moments, skewness, kurtosis, median, moment generating function, mean deviations, Lorenz and Bonferroni curves, order statistics, Mills ratio, hazard rate functions have been discussed. The location-scale form of the beta generated slash distribution is also established. The hazard rate function is seen to assume different shapes depending upon the values of the parameters. The method of maximum likelihood is used to estimate the unknown parameters of beta generated slash distribution and a simulation study is conducted to check the performance of these estimates. Finally, the proposed distribution is applied to a real-life data set on failure times and the goodness-of-fit of the fitted distribution is compared with four other competing distributions to show its flexibility and advantage particularly in modeling heavy tailed data sets.

1. Introduction

The beta distribution is a continuous type of probability distribution. This distribution represents a family of probabilities and is a versatile way to represent outcomes for percentages or population. The basic beta distribution is called the beta distribution of first kind and is used in a range of disciplines including rule of succession, Bayesian statistics and task duration modelling. The probability density function of beta distribution of first kind is:

$$f(x,a,b)=\frac{1}{\beta(a,b)}x^{a-1}(1-x)^{b-1}, 0\leq x\leq 1 \tag{1}$$

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The shape of the distribution is controlled by the two shape parameters a and b. Beta distribution is more useful than the normal distribution if we need to model a behaviour that is obviously bounded. In (1), $\beta(a, b)$ is referred to as the beta function and with its help, incomplete beta function ratio, incomplete beta function, the members of several beta generalised distributions have been introduced. For example, Beta Normal distribution by Eugene et al. [1], Beta Gumbel distribution by Nadarajah and Kotz [2], Beta Exponential distribution by Nadarajah and Kotz [3], Beta Exponentiated Weibull distribution by Cordeiro et al. [4] and Beta - Dagum distribution by Domma and Condino [5].

The slash distribution is defined by Rogers and Tukey [6] as the ratio of standard normal random variable to the uniform random variable following the stochastic representation

$$Y = \frac{X}{U^{\frac{1}{q}}} \tag{2}$$

where $X \sim N(0,1)$ and $U \sim U(0,1)$. $q > 0$ is the shape parameter which controls the kurtosis of the distribution. The fundamental studies on slash distribution have focused on its properties and application to model heavy-tailed data. A modified version of slash distribution has been proposed by Reyes et al. [7] who considered the distribution of U in (2) as exponential distribution with parameter 2. Reyes et al. [8] have introduced generalised modified slash distribution by considering U in (2) to be distributed as two - parameter gamma distribution. The authors have showed that this generalised modified- slash distribution performs better than the existing slash distribution and modified-slash distribution in modelling heavy-tailed data. The logit slash distribution [9] is a new extension of slash distribution having support in (0,1). This distribution offers flexible forms depending on the values of the shape parameter q, thus making it useful for bounded heavy-tailed data. The slash distribution is particularly useful when models with heavy tails are neccesary to fit a real data set. This simple concept has launched a remarkable creativity among the reseachers. In the last decade, slash distribution for many popular parent distributions have been extensively explored. For example, the slashed versions for the epsilon half-normal has been established by Gui et al. [10] where a slash distribution is naturally defined with the help of an extension of half normal distribution, the extended slash distribution of sum of two independent logistic variables [11], the modified slash Birnbaum−Saunders distribution by Reyes et al. [12] where an extension of Birnbaum−Saunders has been introduced on the basis of modified slash distribution approach proposed by [7]. An extension of Akash distribution has been introduced by Gomez et al. [13] by using slash construction approach to make the kurtosis of the Akash distribution more flexible. Extensive works on multivariate slash distributions have also been carried out by several authors. For instance, the multivariate skew - slash distribution by Wang and Genton [14] where they discussed the multivariate skew version of the distribution and studied its properties and inferences and used it to fit some skewed data sets. The multivariate asymmetric slash Laplace distribution has been established by Punthumparambath [15]. An alternative to multivariate skew - slash distribution has been introduced by Arslan [16]. Genc established the generalisation of slash distribution by using the scale mixture of exponential power distribution [17]. A family of skew-slash distributions generated by normal and Cauchy kernels was established by Punthumparambath [18] [19]. The general properties of the canonical form of slash distribution have been studied by Rogers and Tukey [6] and Mosteller and Tukey [20]. The maximum likelihood estimators of the location and scale parameters of the standard slash distribution have been studied by Kafadar [21]. Both the discrete and continous structure of the uniform slash and α-slash distributions have been established by Jones and Higuchi [22]. A new family of modified slash distribution along with their applications has been studied by Reyes et al. [23] where type II modified slash distribution is introduced by considering the distribution of U in (2) to be Birnbaum−Saunders distribution.

An extensive review of the existing works on slash distribution revealed that the slash distribution is particulary useful when models with heavy tails are necessary to fit a real data set. In presence of extreme values, the heavy-tailed models are required to perform better modelling. Skewed models provide better prospect in modelling heavy - tailed data and slash distribution is one type of skewed distribution. The usual regression model which finds application across diverse fields of biology, sociology, economics, psychology, epidemiology, marketing etc may not conform to the normal probability law all the time. In such cases the error structures should be handled from the perspective of asymmetry or skewness. Also, slash distribution offers flexibility in modelling extreme events as it is associated with augmenting the kurtosis of the underlying data, thereby accomodating the outliers. Slash distribution has been more popular in robust statistical analysis. Slash distribution remains robust where traditional distribution may fail to adequately capture the tails of the data.

Heavy - tailed lifetime data often arise in real life which requires a flexible heavy-tailed probability model for describing its behaviour. One may also need to look for a probability model which is able to account for the outliers in lifetime data. A slash distribution being a flexible heavy-tailed model is equipped to handle such type of data. Further, most of the existing works focuses on establishing the slash version of random variables having support in the range $(-\infty, \infty)$ and $(0, \infty)$. However till now, not much work on developing the slash distribution for finitely bounded random variable has been carried out. This motivated us to carry out our work on constructing the slashed version of a finite bounded r.v. which is particulary applicable to lifetime data. In particular, the beta random variable of first kind has been considered for this research work.

Here, we introduce an extension of beta distribution through the slash construction idea and the proposed distribution has been named as the beta generated slash (BGSl) distribution. The newly proposed distribution is expected to be useful in modelling data with higher level of kurtosis, providing a more precise representation of extreme outcomes.

We shall say that Y follows the BGSl distribution with parameters *a, b* and q or $Y \sim$ *BGSl(a,b,q)* if it can be stochastically expressed as

$$Y = \frac{X}{U^{\frac{1}{q}}}$$

where $X \sim$ *beta(a,b)* and $U \sim$ *U(0,1)* and are distributed independently of each other.

The rest of the paper is organised as follows. Section 2 introduces the density function of the proposed distribution. Expressions for pdf, cdf, various descriptive statistics are derived and and behaviour of the curve of the proposed distribution for varying values of the parameters graphically are shown in Section 3. The maximum likelihood estimation of the parameters of the distribution are dealt with in Section 4. In Section 5, some stochastic simulations are performed to illustrate the behaviour of the parameters of the proposed distribution. In Section 6, the proposed model is applied to data set on failure times to exhibit the potential of the distribution in modeling real-life data sets. Finally, the conclusions of this paper are given in Section 7 .

2. Definition and Derivation of the BGSl Distribution

Theorem 1. *Let* $Y \sim$ *BGSl(a,b,q). Then the pdf of* Y *is given by:*

$$f(y; a, b, q) = \begin{cases} \frac{q}{\beta(a,b) y^{q+1}} \beta(y; a+q, b), & 0 \le y < 1 \\ \\ \frac{q}{\beta(a,b) y^{q+1}} \beta(a+q, b), & 1 \le y < \infty \end{cases} \tag{3}$$

where a,b are the scale parameters, q is the shape parameter and $\beta(y; a+q, b)$ *is the incomplete beta function which is given by:*

$$\beta(y; a, b) = \int_0^y u^{a-1}(1-u)^{b-1} du$$

Proof. Let us consider X to be distributed as *Beta(a,b)*. Then the pdf of X is given by

$$f(x; a, b) = \frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}, \qquad 0 \le x \le 1$$

Let us now consider the following stochastic representation:

$$Y = \frac{X}{U^{\frac{1}{q}}}$$

where $U \sim U(0,1)$

Suppose

$$W = U \implies X = YW^{\frac{1}{q}}$$

Then the jacobian of the transformation is:

$$J = \begin{vmatrix} \frac{\partial X}{\partial Y} & \frac{\partial X}{\partial W} \\ \\ \frac{\partial U}{\partial Y} & \frac{\partial U}{\partial W} \end{vmatrix} = \begin{vmatrix} w^{\frac{1}{q}} & \frac{yw^{\frac{1}{q}-1}}{q} \\ \\ 0 & 1 \end{vmatrix} = w^{\frac{1}{q}}$$

$\therefore$

$$\begin{aligned} f_Y(y,w) &= f_{x,u}(yw^{\frac{1}{q}}, w)|J| \\ &= \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} w^{\frac{1}{q}} \\ &= \frac{1}{\beta(a,b)} y^{a-1} w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1} \end{aligned}$$

When 0$< X <$1 $\Longrightarrow$ 0$< yw^{\frac{1}{q}}$ $<$1 $\Longrightarrow$ 0$< y < \frac{1}{w^{\frac{1}{q}}}$

When 0$< U < 1 \implies$ 0$< W < 1$

$\therefore$ The required joint pdf is

$$f(y,w) = \begin{cases} \frac{1}{\beta(a,b)} y^{a-1} w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1}, 0 < y < \frac{1}{w^{\frac{1}{q}}}, 0 < w < 1 \\ 0, \qquad otherwise \end{cases} \tag{4}$$

Hence, the marginal distribution function of Y is given by:

$$f(y,w) = \begin{cases} f_1(y), & 0 \le y < 1 \\ f_2(y), & 1 \le y < \infty \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(y) &= \frac{y^{a-1}}{\beta(a,b)} \int_0^1 w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1} dw \\ &= \frac{q}{\beta(a,b)y^{q+1}} \beta(y; a+q, b) \end{aligned} \tag{6}$$

$\beta(y; a+b, q)$ being the incomplete beta function and

$$\begin{aligned} f_2(y) &= \frac{y^{a-1}}{\beta(a,b)} \int_0^{\frac{1}{y^q}} w^{\frac{a}{q}} (1-yw^{\frac{1}{q}})^{b-1} dw \\ &= \frac{q}{\beta(a,b)y^{q+1}} \beta(a+q, b) \end{aligned} \tag{7}$$

$\square$

The pdf of BGSl distribution for different values of parameters, is plotted in Figure 1. From the figure it is seen that the kurtosis of the distribution increases with an increase in the value of q.

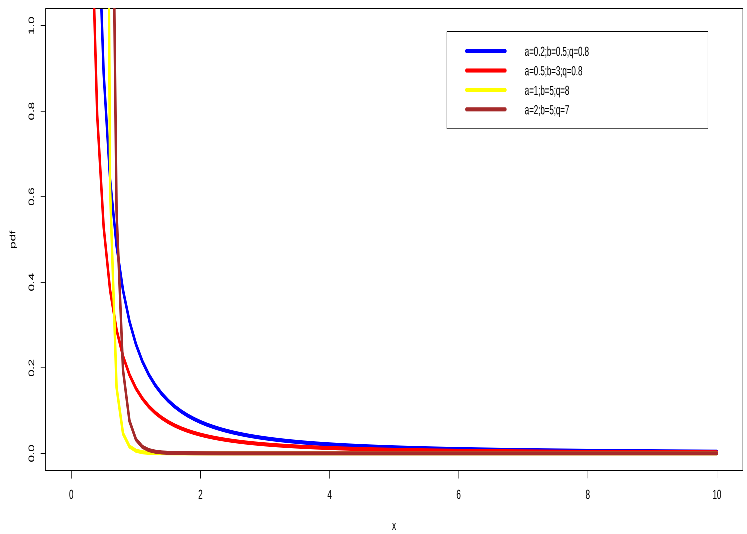

FIGURE 1. Probability density function plots of the BGSl distribution for different values of a,b and q

Again the cdf of Y is given by :

$$F(y) = \begin{cases} F_1(y), & 0 \leq y < 1 \\ F_2(y), & 1 \leq y < \infty \end{cases} \tag{8}$$

where

$$\begin{aligned} F_1(y) &= P(Y \leq y) \\ &= \int_0^y \frac{q}{\beta(a,b)t^{q+1}} \beta(t; a+q, b) dt \\ &= \frac{q}{\beta(a,b)} \int_0^y \beta(t; a+q, b) t^{-(q+1)} dt \\ &= \frac{\beta(y; a, b)}{\beta(a,b)} - y^{-q} \frac{\beta(y; a+q, b)}{\beta(a,b)} \end{aligned} \tag{9}$$

$$\begin{aligned} F_2(y) &= P(Y \leq y) \\ &= \int_0^1 f_1(t)dt + \int_1^y f_2(t)dt \\ &= 1 - \frac{\beta(a+q,b)}{\beta(a,b)} + \frac{\beta(a+q,b)}{\beta(a,b)}(1 - y^{-q}) \end{aligned} \tag{10}$$

The cdf plot for BGSl distribution is shown in Figure 2.

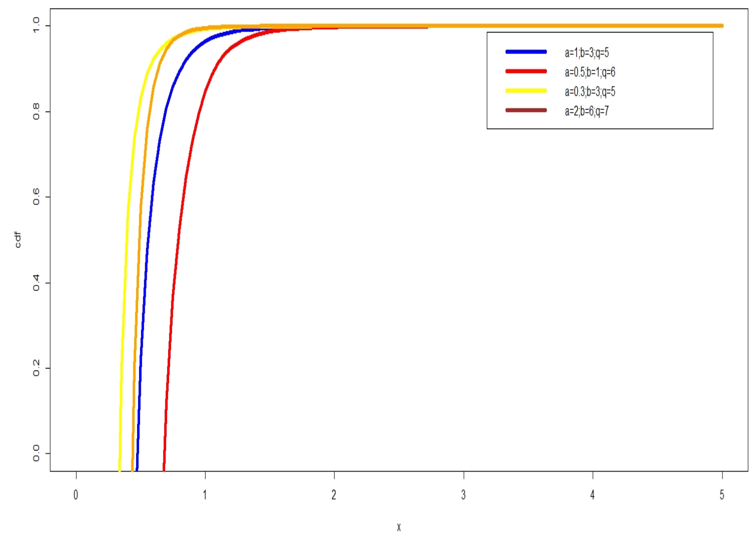

FIGURE 2. Cumulative distribution function plots of the BGSl distribution for different values of a,b and q

2.1. **Location Scale form of BGSl(a,b,q).** Another form of beta generated slash distribution is the location - scale form. By applying the well known location - scale transformation and considering the general form of BGSl distribution, we get the location - scale transformed BGSl variate as

$$T = \mu + \sigma \frac{X}{U^{\frac{1}{q}}} \tag{11}$$

where $X \sim$ Beta(a,b) and $U \sim U(0,1)$ are independent, $q > 0$, $0 < \mu < \infty$ and $\sigma > 0$. μ and σ are the location and scale parameters respectively. The location-scale form of BGSl distribution has the following pdf:

$$f(t;a,b,q) = \begin{cases} \frac{q\sigma^q (t-\mu)^{-(q+1)}}{\beta(a,b)} \beta(\frac{t-\mu}{\sigma}; a+q, b), & \mu < T < \mu + \sigma \\ \\ \frac{q\sigma^q (t-\mu)^{-(q+1)}}{\beta(a,b)} \beta(a+q, b), & \mu + \sigma \leq T < \infty \end{cases} \tag{12}$$

It is denoted by $T \sim BGSlLS(a,b,q,\mu,\sigma)$.

2.1.1. *Special cases of BGSlLS(a, b, q, μ,σ):*

- If $\mu = 0$, $\sigma = 1$, then BGSlLS(a,b,q,μ,σ) reduces to BGSl(a,b,q).
- If $q \to \infty$ then BGSlLS(a,b,q,μ,σ) tends to $\beta(a,b,\mu,\sigma)$ which is the location scale form of beta distribution.

3. Properties of BGSL(a,b,q)

3.1. **Moments and Other Descriptive Measures.** If $Y \sim BGSl(a,b,q)$, then the r^{th} raw moment of Y is given by:

$$\begin{aligned}
\mu_r^{/} &= E(Y^r) \\
&= \int_0^{\infty} y^r f(y)dy \\
&= \int_0^{1} y^r f_1(y)dy + \int_1^{\infty} y^r f_2(y)dy
\end{aligned}$$

where r =*1, 2, 3......* and $q>0$.

In particular,

$$\begin{aligned}
\mu_1^{/} &= \frac{a}{(a+b)}\frac{q}{(q-1)}, \quad q>1 \\
\mu_2^{/} &= \frac{a(a+1)}{(a+b)(a+b+1)}\frac{q}{(q-2)}, \quad q>2 \\
\mu_3^{/} &= \frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}\frac{q}{(q-3)}, \quad q>3 \\
\mu_4^{/} &= \frac{a(a+1)(a+2)(a+3)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}\frac{q}{(q-4)}, \quad q>4
\end{aligned}$$

The measures of skewness and kurtosis denoted by γ_1 and γ_2 respectively, are defined as:

$$\begin{aligned}
\gamma_1 &= \frac{\mu_3^{/} - 3\mu_3^{/}\mu_1^{/} + 2\mu_1^{/3}}{(\mu_2^{/} - \mu_1^{/2})^{\frac{3}{2}}}, \quad q>3 \\
\gamma_2 &= \frac{\mu_4^{/} - 4\mu_3^{/}\mu_1^{/} + 6\mu_2^{/}\mu_1^{/2} - 3\mu_1^{/4}}{(\mu_2^{/} - \mu_1^{/2})^2}, \quad q>4 \qquad (13)
\end{aligned}$$

In Table 1, the skewness and kurtosis values for some selected values of a,b and q are displayed. From Table 1, it is observed that skewness decreases and kurtosis increases with an increase in *q*. When b is fixed for some values, skewness and kurtosis increase for $a<0.5$ but kurtosis decreases slowly as $a>0.5$.

TABLE 1. Skewness and Kurtosis measurements of BGSl(a,b,q) distribution for different values of a, b and q

a	q	b	Skewness	Kurtosis
0.25	5	0.5	1.286	69.346
		1	1.339	109.617
		2	1.894	130.506
	6	0.5	1.017	58.723
		1	1.707	64.635
		2	2.442	97.201
	10	0.5	0.765	53.397
		1	1.465	56.188
		2	2.185	81.890
0.5	5	0.5	0.860	128.556
		1	1.339	109.617
		2	1.894	140.506
	6	0.5	0.488	130.7417
		1	1.025	139.622
		2	1.603	160.215
	10	0.5	0.126	141.748
		1	0.736	139.866
		2	1.336	157.930
1	5	0.5	0.9102	348.533
		1	1.069	235.526
		2	1.411	210.333
	6	0.5	0.285	352.952
		1	0.610	218.938
		2	1.037	183.11
	10	0.5	0.384	402.169
		1	0.158	223.903
		2	0.684	174.225

3.2. **Median.** The median (M) of a probability distribution is the value which divides the total area under the probability curve into two equal halves. Since the area under the probability curve of BGSl distribution is different in the range $[0, 1)$ and $[1, \infty)$, so the median of the proposed distribution can appear in either one of the two ranges - $[0, 1)$ or $[1, \infty)$. To find the median, the following steps are used:

(1) Compute F(1)=$\int_0^1 f_1(y)dy$.

(2) If F(1) ≥ 0.5 then the median will lie in $[0,1)$ and M is obtained by solving the following equation:

$$\int_0^M f_1(y)dy = 0.5$$

$$\implies \frac{\beta(M;a,b)}{\beta(a,b)} - \frac{M^{-q}\beta(M;a+q,b)}{\beta(a,b)} = 0.5$$

(3) If F(1) < 0.5 then the median will lie in $[1,\infty)$ and M is obtained by solving the following equation:

$$\int_0^1 f_1(y)dy + \int_1^M f_2(y)dy = 0.5$$

$$\implies 1 - \frac{\beta(a+q,b)}{\beta(a,b)} + \frac{\beta(a+q,b)}{\beta(a,b)}\left[1 - M^{-q}\right] = 0.5 \tag{14}$$

The median values for different set of parameters are given in Table 2:

TABLE 2. Median values for diffrent set of parameters

Parameters	Median
(0.9,0.3,2)	0.75938
(1,1.5,2)	0.56557
(2,0.3,0.5)	4.23087
(0.9,0.3,0.5)	9.96179

3.3. **Moment Generating Function.** For a random variable Y with pdf *f(y)*, the moment generating function is given by:

$$M_Y(t) = E(e^{ty})$$

Hence the moment generating function of BGSL distribution is given by:

$$\begin{aligned}
M_Y(t) &= E\left(e^{ty}\right) \\
&= \int_0^1 e^{ty} f_1(y)dy + \int_1^\infty e^{ty} f_2(y)dy \\
&= \int_0^1 e^{ty}\frac{q\beta(y;a+q,b)}{\beta(a,b)y^{q+1}}dy + \int_1^\infty e^{ty}\frac{q\beta(a+q,b)}{\beta(a,b)y^{q+1}}dy \\
&= \frac{q}{\beta(a,b)}\int_0^1 e^{ty}y^{-(q+1)}\beta(y;a+q,b)dy + \frac{q\beta(a+q,b)}{\beta(a,b)}\int_0^1 e^{ty}y^{-(q+1)}dy \\
&= \frac{q}{\beta(a,b)}\int_0^1 \left(\sum_{k=0}^{\infty}\frac{(ty)^k}{k!}\right) y^{-(q+1)}\beta(y;a+q,b)dy+
\end{aligned}$$

$$\frac{q\beta(a+q,b)}{\beta(a,b)}\int_1^\infty\left(\sum_{k=0}^\infty\frac{(ty)^k}{k!}\right)y^{-(q+1)}dy$$

$$= \frac{q}{\beta(a,b)}\sum_{k=0}^\infty\frac{t^k}{k!}\int_0^1 y^{k-q-1}\beta(y;a+q,b)dy + \frac{q\beta(a+q,b)}{\beta(a,b)}\sum_{k=0}^\infty\frac{t^k}{k!}\int_1^\infty y^{k-q-1}dy$$

$$= \frac{q}{\beta(a,b)}\sum_{k=0}^\infty\frac{t^k}{k!}\frac{1}{k-q}\left\{\beta(a+q,b)-\beta(a+k,b)\right\} + \frac{q\beta(a+q,b)}{\beta(a,b)}\sum_{k=0}^\infty\frac{t^k}{k!}\frac{1}{q-k}$$

$$= \frac{q}{\beta(a,b)}\frac{t^0}{0!}\frac{1}{(-q)}\left\{\beta(a+q,b)-\beta(a,b)\right\} + \frac{q}{\beta(a,b)}\sum_{k=1}^\infty\frac{t^k}{k!}\frac{1}{k-q}$$

$$\left\{\beta(a+q,b)-\beta(a+k,b)\right\} + \frac{q\beta(a+q,b)}{q\beta(a,b)} + \sum_{k=1}^\infty\frac{t^k}{k!(q-k)}\left\{\frac{q\beta(a+q,b)}{\beta(a,b)}\right\}$$

$$= 1 - \frac{q}{\beta(a,b)}\sum_{k=1}^\infty\frac{t^k}{k!(k-q)}\beta(a+k,b) \tag{15}$$

3.4. **Additive Property.** In this section, the additive property of BGSl(a,b,q) is discusssed through the following theorem.

Theorem 2. *$BGSl(a,b,q)$ does not satisfy the additive property i.e., if $X\sim BGSl(a_1, b_1,q_1)$ and $Y\sim BGSl(a_2,b_2,q_2)$, then $(X+Y)$ does not follow the BGSl distribution.*

Proof. The m.g.f. of $BGSl(a,b,q)$ is given by:

$$\begin{aligned} M_Y(t) &= E\left(e^{ty}\right) \\ &= \int_0^1 e^{ty}f_1(y)dy + \int_1^\infty e^{ty}f_2(y)dy \\ &= 1 - \frac{q}{\beta(a,b)}\sum_{k=1}^\infty\frac{t^k}{k!(k-q)}\beta(a+k,b) \end{aligned} \tag{16}$$

Let $Z = X+Y$ where $X\sim$ BGSl(a_1, b_1,q_1) and $Y\sim$ BGSl(a_2,b_2,q_2) and are independently distributed of each other. Then the m.g.f. of Z is

$$\begin{aligned} M_Z(t) &= M_{X+Y}(t) \\ &= M_X(t)M_Y(t) \\ &= \left(1-\frac{q_1}{\beta(a_1,b_1)}\sum_{k=1}^\infty\frac{t^k}{k!(k-q_1)}\beta(a_1+k,b_1)\right) \\ &\times\left(1-\frac{q_2}{\beta(a_2,b_2)}\sum_{k=1}^\infty\frac{t^k}{k!(k-q_2)}\beta(a_2+k,b_2)\right) \end{aligned} \tag{17}$$

which is not the m.g.f. of BGSl distribution.
Thus, $X+Y$ does not follow BGSl distribution or in other words, the BGSl distribution does not satisfy the additive property. □

3.5. **Mean Deviation About Mean.** The mean deviation about mean of a population measure the amount of scatter in a population to some extent. For a random variable Y with pdf *f(y)*, cdf *F(Y)*, mean μ = *E(Y)*, the mean deviation about mean is defined by:

$$\begin{aligned}\delta_1(y) &= \int_0^\infty |y-\mu| f(y)dy \\ &= \int_0^\infty (\mu-y)f(y)dy + \int_\mu^\infty (y-\mu)f(y)dy \\ &= \mu F(\mu) - \int_0^\infty yf(y)dy - \mu\left[1-F(\mu)\right] + \int_\mu^\infty yf(y)dy \\ &= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty yf(y)dy \\ &= 2\mu F(\mu) - 2\int_0^\mu yf(y)dy \qquad (18)\end{aligned}$$

Hence the mean deviation for BGSl distribution is given by:

$$\begin{aligned}\delta_1(y) = I_{[0,1)}(y)&\left[\frac{2aq}{(a+b)(q-1)}\left\{\frac{\beta(\mu;a,b)}{\beta(a,b)} - \mu^{-q}\beta\left(\mu;a+q,b\right)\right\}\right. \\ &\left. -\left\{\frac{1}{q-1}\beta\left(\mu,a+1,b\right) - \frac{\mu^{1-q}}{q-1}\beta(\mu;a+q,b)\right\}\right] \\ +(1-I_{[0,1)})(y)&\left[\frac{2q\beta(a+q,b)}{\beta(a,b)(1-q)}\left\{(1-\mu^{-q}) - \frac{\beta(a+q,b)}{\beta(a,b)}\right\}\right] \qquad (19)\end{aligned}$$

where

$$I_{[0,1)}(y) = \begin{cases} 1, & if \quad 0 \le y < 1 \\ 0, & if \quad 1 \le y < \infty \end{cases}$$

3.6. **Mills Ratio.** The Mills Ratio is the ratio of complementary cumulative distribution function to the probability density function. Mills ratio can be used in regression analysis to take account of a possible selection bias. Mills Ratio for BGSl distribution is :

$$\begin{aligned}M(y) &= \frac{1-F_1(y)}{f_1(y)} + (1-I_{[0,1)}(y))\frac{1-F_2(y)}{f_2(y)} \\ &= I_{[0,1)}(y)\left[\frac{1-\frac{\beta(y;a,b)}{\beta(a,b)} + y^{-q}\beta\left(y;a+q,b\right)}{\frac{q\beta(y;a+q,b)}{\beta(a,b)y^{q+1}}}\right]\end{aligned}$$

$$+ (1 - I_{[0,1)})(y)\left[\frac{1 - \frac{\beta(a+q,b)}{\beta(a,b)}(1 - y^{-q})}{\frac{q}{\beta(a,b)y^{q+1}}\beta\left(a+q,b\right)}\right]$$

$$= I_{[0,1)}(y)\left[\frac{y^{q+1}\beta\left(a,b\right) - y^{q+1}\beta\left(y;a,b\right) - y\beta(y;a+q,b)}{q\beta(y;a+q,b)}\right]$$

$$+ (1 - I_{[0,1)})(y)\left[\frac{y^{q+1}\left\{\beta(a,b) - \beta(a+q,b)(1 - y^{-q})\right\}}{q\beta(a+q,b)}\right] \tag{20}$$

3.7. **Order Statistics.** Consider a random sample $y_1,\ y_2,......,y_n$ of size n drawn from BGSl(a,b,q). Further, let $y_{(1)} < y_{(2)} < .... < y_{(n)}$ denote the order statistics corresponding to this sample. Then the probability density function of the k^{th} order statistic is given by

$$f_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}\left[F(y)\right]^{k-1}\left[1 - F(y)\right]^{n-k} f(y)$$

Hence the density of k^{th} order statistic for BGSl(a,b,q) is

$$f_{(k)}(y) = I_{[0,1)}(y)\left[\frac{n!}{(k-1)!(n-k)!}\left\{\frac{\beta(y;a,b)}{\beta(a,b)} - y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{k-1}\right.$$
$$\left.\left\{1 - \frac{\beta(y;a,b)}{\beta(a,b)} + y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{n-k}\frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b)\right]$$
$$+ (1 - I_{[0,1)}(y))\left[\left\{\frac{\beta(a+q,b)}{\beta(a,b)}(1 - y^{-q})\right\}^{k-1}\left\{1 - \frac{\beta(a+q,b)}{\beta(a,b)}(1 - y^{-q})\right\}^{n-k}\right.$$
$$\left.\frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b)\right] \tag{21}$$

The p.d.f of the smallest order statistic $y_{(1)}$ is

$$f_{(1)}(y) = I_{[0,1)}(y)\left[n\left\{1 - \frac{\beta(y;a,b)}{\beta(a,b)} + y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{n-1}\right.$$
$$\left.\frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b)\right] + (1 - I_{[0,1)}(y))$$
$$\times\left[\left\{1 - \frac{\beta(a+q,b)}{\beta(a,b)}(1 - y^{-q})\right\}^{n-1}\frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b)\right] \tag{22}$$

The pdf of the largest order statistic $y_{(n)}$ is

$$f_{(n)}(y) = I_{[0,1)}(y)\left[n\left\{\frac{\beta(y;a,b)}{\beta(a,b)} - y^{-q}\frac{\beta(y;a+q,b)}{\beta(a,b)}\right\}^{n-1}\frac{q}{\beta(a,b)y^{q+1}}\beta(y;a+q,b)\right] + (1-I_{[0,1)}(y))\left[n\left\{\frac{\beta(a+q,b)}{\beta(a,b)}(1-y^{-q})\right\}^{n-1}\frac{q}{\beta(a,b)y^{q+1}}\beta(a+q,b)\right] \tag{23}$$

3.8. **Lorenz and Bonferroni Curve.** The Bonferroni and Lorenz Curve are the most used tools in income inequality measurement. These two curves are widely used in the field of reliability, demography, medicine and insurance. The Bonferroni and Lorenz curves are defined as:

$$L(F(y)) = I_{[0,1)}(y)\left[\frac{1}{\mu}\int_0^y tf_1(t)dt\right] + (1-I_{[0,1)}(y))\left[\frac{1}{\mu}\int_0^y tf_2(t)dt\right] \tag{24}$$

$$\begin{aligned} B(F(y)) &= I_{[0,1)}(y)\left[\frac{1}{\mu F_1(y)}\int_0^y tf_1(t)dt\right] + (1-I_{[0,1)}(y))\left[\frac{1}{\mu F_2(y)}\int_0^y tf_2(t)dt\right] \\ &= I_{[0,1)}(y)\left[\frac{L(F_1(y))}{F_1(y)}\right] + (1-I_{[0,1)}(y))\left[\frac{L(F_2(y))}{F_2(y)}\right] \end{aligned} \tag{25}$$

After simplifications,

$$L(F(y)) = I_{[0,1)}(y)\left[\frac{\left[\beta(y;a+1,b) - t^{1-q}\beta(y;a+q,b)\right](a+b)}{a\beta(a,b)}\right] - (1-I_{[0,1)}(y))\left[\frac{\beta(a+q,b)(a+b)y^{1-q}}{a\beta(a,b)}\right] \tag{26}$$

$$\begin{aligned} B(F(y)) &= I_{[0,1)}(y)\frac{\left[\beta(y;a+1,b) - t^{1-q}\beta(y;a+q,b)\right](a+b)}{a\beta(a,b)} \\ &\times \left[\frac{\beta(y;a,b)}{\beta(a,b)} - y^{-q}\beta(y;a+q,b)\right] \\ &+ (1-I_{[0,1)}(y))\left[\frac{\beta(a+q,b)(a+b)y^{1-q}}{a\left(\beta(a,b) - \beta(a+q,b) + \beta(a+q,b)(1-y^{-q})\right)}\right] \end{aligned} \tag{27}$$

3.9. **Hazard Rate Function.** The hazard rate function is a very important tool in understanding about the failure mechanism of a lifetime distribution. Hazard rate function can be used to postulate life distributions in the presence of several competing risk factors. It measures the instantaneous rate at which a system or component is likely to fail, given that it has survived up to a certain time. The hazard rate function of BGSl(a,b,q) is obtained by using the following formula:

$$h(y) = I_{[0,1)}(y)\frac{f_1(y)}{1-F_1(y)} + (1-I_{(0,1)})\frac{f_2(y)}{1-F_2(y)}$$

$$= I_{[0,1)}(y)\left[\frac{q\beta(y,a+q,b)}{y^{q+1}\left\{\beta(a,b)-\beta(y,a,b)+y^{-q}\beta(y,a+q,b)\right\}}\right]$$
$$+ (1-I_{[0,1)})(y)\left[\frac{q\beta(y,a+q,b)}{y^{q+1}\left\{1-(1-y^{-q}\beta(a+q,b))\right\}}\right] \tag{28}$$

The HRF plots of the BGSl distribution for various values of the parameters are shown in Figure 3. For all the combinations of a,b and q, the initial hazard is high which decreases consistently. This shows the flexibility of the hazard rate function of the beta generated slash distribution, thereby indicating that various real-life situations can be suitably modeled using this distribution. For example, for a patient who has undergone a surgery, the hazard (probability of death because of post-surgical complications in this case) is high for a specific period post the procedure. The hazard keeps on decreasing with time and after a certain period of time,i.e. after full recovery, the hazard drops to approximately 0 and remains constant thereafter.

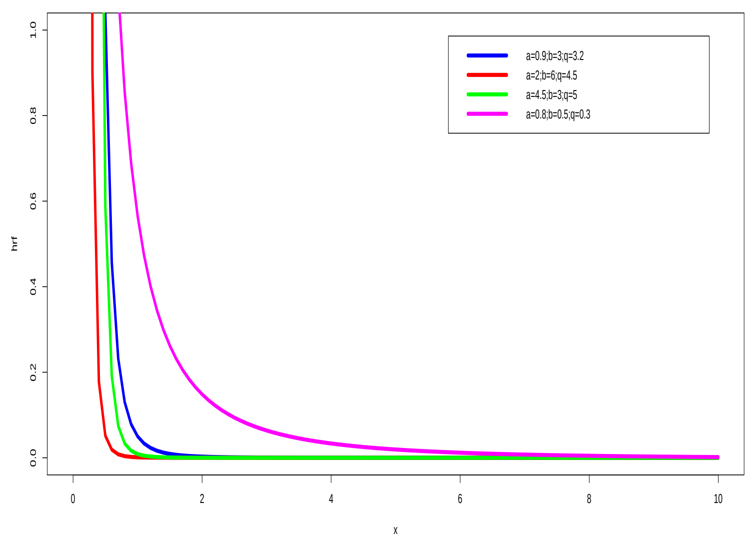

FIGURE 3. HRF plots of the BGSl distribution for different values of a,b and q

We shall now discuss the marginal probability of the variate obtained via the conditional distribution of the location-scale form of beta distribution given a Uniform$(0, 1)$ variate, which is presented in Theorem 3.

Theorem 3. *If $Z|U \sim \beta(a,b,0,\ \sigma\ u^{-\frac{1}{q}})$ where $\beta(a,b,0,\ \sigma\ u^{-\frac{1}{q}})$ is location -scale form of beta distribution and $U \sim U(0,1)$ then $Z \sim BGSl(a,b,q,0,\sigma\ u^{-\frac{1}{q}})$.*

Proof.

$$\begin{aligned} P(Z|U=u) &= f(z|u) \\ &= \frac{1}{\sigma u^{\frac{1}{q}}}\left(\frac{z}{\sigma}\right)^{a-1}\left(1-\frac{z}{\sigma}\right)^{b-1} \end{aligned}$$

$\therefore$

$$\begin{aligned} f_z &= \int_0^1 f(z|u)f(u)du \\ &= \int_0^1 \frac{u^{\frac{1}{q}}}{\sigma\beta(a,b)}\left(u^{\frac{1}{q}}\frac{z}{\sigma}\right)^{a-1}\left(1-\frac{z}{\sigma}\right)^{b-1}du \\ &= \frac{q\sigma^q}{z^{q+1}\beta(a,b)}\beta(\frac{z}{\sigma},a+q,b) \end{aligned}$$

□

4. Estimation

In this section, we discuss the maximum likelihood method of estimation for the unknown model parameters of BGSl(a,b,q). Let $y_1,y_2,......,y_n$ be a random sample of size n from BGSl(a,b,q). Then the log - likelihood function is obtained as:

$$\begin{aligned} L(a,b,q,\boldsymbol{y}) &= \prod_{i=1}^{n} f(y_i,a,b,q) \\ &= L_1(a,b,q,\boldsymbol{y}) * L_2(a,b,q,\boldsymbol{y}) \end{aligned}$$

where

$$\begin{aligned} L_1(a,b,q,\boldsymbol{y}) &= \textstyle\prod_{i=1}^{n} f_1^{I_{(0,1)}(y_i)} \\ &= f_1^{\sum_{i=1}^{n} I_{(0,1)}(y_i)} \end{aligned}$$

$$\log L_1(a,b,q,\boldsymbol{y}) = \sum_{i=1}^{n} I_{[0,1)}(y_i)\Big[\log q + \log\beta(y_i,a+q,b) \\ - \log\beta(a,b) - (q+1)\log y_i\Big] \tag{29}$$

Again,

$$\begin{aligned} L_2(a,b,q,\boldsymbol{y}) &= \prod_{i=1}^{n} f_2^{1-I_{(0,1)}(y_i)} \\ &= f_2{}^{\left(n-\sum_{i=1}^{n} I_(0,1)(y_i)\right)} \end{aligned}$$

$$\log L_2(a,b,q,\boldsymbol{y}) = \sum_{i=1}^{n} \left(n - I_{[0,1)}(y_i)\right) \Big[\log q + \log \beta(a+q,b) - \log \beta(a,b) - (q+1)\log y_i\Big] \tag{30}$$

$$\log L(a,b,q,\boldsymbol{y}) = \sum_{i=1}^{n} I_{[0,1)}(y_i)\Big[\log q + \log \beta(y_i,a+q,b) - \log \beta(a,b) - (q+1)\log y_i\Big] + \sum_{i=1}^{n} \left(n - I_{[0,1)}(y_i)\right) \Big[\log q + \log \beta(a+q,b) - \log \beta(a,b) - (q+1)\log y_i\Big] \tag{31}$$

The maximum likelihood estimates (MLE) of the parameters are computed by solving the maximum likelihood equations, which are given by

$$\frac{\partial \log L}{\partial a} \implies \sum_{i=1}^{n} I_{[0,1)}(y_i)\left[\frac{1}{\beta(y_i,a+q,b)}\frac{d}{da}\beta(y_i,a+q,b) - \{\psi_0(a) - \psi_0(a+b)\}\right] + \left(n - I_{[0,1)}(y_i)\right)\left[\frac{1}{\beta(a+q,b)}\frac{d}{da}\beta(a+q,b) - \{\psi_0(a) - \psi_0(a+b)\}\right] = 0 \tag{32}$$

$$\frac{\partial \log L}{\partial b} \implies \sum_{i=1}^{n} I_{[0,1)}(y_i)\left[\frac{1}{\beta(y_i,a+q,b)}\frac{d}{db}\beta(y_i,a+q,b) - \{\psi_0(b) - \psi_0(a+b)\}\right] + \left(n - I_{[0,1)}(y_i)\right)\left[\frac{1}{\beta(a+q,b)}\frac{d}{db}\beta(a+q,b) - \{\psi_0(b) - \psi_0(a+b)\}\right] = 0 \tag{33}$$

$$\frac{\partial \log L}{\partial q} \implies \sum_{i=1}^{n} I_{[0,1)}(y_i)\left[\frac{1}{q} + \frac{1}{\beta(y_i,a+q,b)}\frac{d}{dq}\beta(y_i,a+q,b) - logy_i\right] + \left(n - I_{[0,1)}(y_i)\right)\left[\frac{1}{q} + \frac{1}{\beta(a+q,b)}\frac{d}{dq}\beta(a+q,b) - logy_i\right] = 0 \tag{34}$$

The above maximum likelihood Equations 32- 34 are not in closed form and so, they are difficult to be solved analytically. Hence, we shall use a suitable numerical technique to solve the above equations for a, b and q.

Here all the calculations have been carried out using the R software version 3.6.3. The maxLik package is used to obtain the maximum likelihood estimates of the parameters, the rootSolve package is used to generate random variables from BGSl(a,b,q) and zipfR package is used to evaluate the incomplete beta function.

5. Simulation

In this section, generation of random numbers from BGSl(a,b,q) is discussed. For different values of the parameters a,b and q, we generate random samples of size 50, 100, 200 and 500 from BGSl(a,b,q) . Then the MLEs of the parameters are obtained for each of the generated samples. Finally, the average values of bias and mean squared error (MSE) of these estimates are calculated by using the Monte Carlo approximation technique, taking $N = 1{,}000$ replicates. The algorithm used in this simulation study is shown below:

(1) Simulate $X \sim Beta(a, b)$
(2) Simulate $U \sim U(0, 1)$
(3) Compute $Y = \frac{X}{U^{\frac{1}{q}}}$

Y thus generated is a random number from the BGSl(a,b,q). To calculate the average bias and MSE of the likelihood estimates, we use the formulae as shown below :

Let the true value of the parameter a be a^* and estimate be $\hat{a}$. Then the bias and mean square error (MSE) of $\hat{a}$ in estimating a is given by:

$$Bias(\hat{a}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{a}_i - a*)$$

$$MSE(\hat{a}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{a}_i - a*)^2$$

where N is the number of replications and $\hat{a}_i$ is the MLE of $\hat{a}$ obtained in the i^{th} replicate. Similarly, the bias and MSE of b and q are calculated. It is well known that an estimate is consistent if the bias and MSE decrease (approaches to zero) with an increase in the sample size. Table 3 shows the results of the simulation studies.

From Table 3, it has been found that the parameters are well estimated and the bias and MSE of all the estimators approaches towards zero with an increase in the sample size. Hence, the estimates of the parameters can be believed to be consistent.

6. Application

To show the flexibility of the proposed distribution over some existing distributions in modeling heavy - tailed data we apply these distributions to a real life data set. The dataset is taken from Proschan [24] which describes the times among airconditioning equipment consecutive failures in a Boeing 720 airplane. The data set comprises of the observations:
74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326.
The histogram of the data set exhibits a right skewed behavior, which may be aptly

TABLE 3. Average bias and RMSE of BGSl(a,b,q) distribution.

parameters	n	$\hat{a}$			$\hat{b}$			$\hat{q}$		
		Mean	Bias($\hat{a}$)	RMSE($\hat{a}$)	Mean	Bias($\hat{b}$)	RMSE($\hat{b}$)	Mean	Bias($\hat{q}$)	RMSE($\hat{q}$)
	30	0.32641	0.02643	0.01384	0.59527	0.03287	0.08526	0.13275	0.00457	0.00042
a=0.3	50	0.31664	0.01658	0.00763	0.54859	0.02548	0.06785	0.10034	0.00331	0.00030
b=0.5	100	0.30603	0.00609	0.00368	0.53248	0.00825	0.05964	0.00957	0.00275	0.00018
q=0.9	200	0.30304	0.00327	0.00181	0.53008	0.00803	0.04327	00932	0.00187	0.00004
	500	0.30238	0.00235	0.00072	0.51635	0.00629	0.03219	0.00854	0.00104	0.00001
	30	0.17883	0.27816	0.07949	0.05267	0.05034	00034	1.91939	0.80460	0.00896
a=1.2	50	0.59274	0.09273	0.02966	0.53127	0.04520	0.00028	1.87617	0.78347	0.00725
b=0.5	100	0.52979	0.02979	0.01298	0.52958	0.04278	0.00023	1.86524	0.76219	0.00658
q=0.8	200	0.52421	0.02421	0.00664	0.51653	0.40595	0.00002	1.85928	0.75727	0.00568
	500	0.50996	0.00996	0.00252	0.50216	0.03863	0.00001	1.84872	0.73527	0.00504
	30	0.87878	0.07873	0.03837	1.23645	0.49152	2.96958	1.99152	0.87210	1.28645
a=0.8	50	0.83583	0.03583	0.02111	0.96879	0.00630	1.12792	1.94481	0.73594	1.00257
b= 1.3	100	0.82729	0.02729	0.01039	0.00435	0.00952	1.71888	0.70267	0.36485	0.99854
q=1.8	200	0.80816	0.00816	0.00498	0.94876	0.00380	0.00634	1.69458	0.65468	0.97380
	500	0.80684	0.00684	0.00203	0.92135	0.00303	0.00439	1.54896	0.58642	0.93276

modelled by the proposed distribution. Since beta generated slash distribution is an extended distribution having support on positive real line, we compare its fit to the considered data sets with some other extended distributions, namely Modified Slash Lindley (MSL) distribution, Generalised beta distribution of first kind (GB1), Generalised Exponential distribution(GE) and Generalised Gamma(GG) distribution distributed on the range $(0,\infty)$. The various values of log-likelihood, AIC and BIC statistic for BGSl and its competing distributions are shown in Table 4

From the Table 4, it is seen that the BGSl distribution has maximum likelihood and minimum AIC, BIC statistics. Hence the BGSl distribution performs better than the other competing distributions. Furthermore, Figure 4 and Figure 5 show the histogram of the data set along with the fitted densities and the empirical cdf versus fitted cdfs respectively for the times among airconditioning equipment consecutive failures in a Boeing 720 airplane. These figures confirm the best fit of BGSL(a, b, q) as compared to the other competing distributions.

7. Conclusion

This paper introduces a new distribution called the beta generated slash distribution having three parameters, which is obtained from the beta distribution by applying slash construction idea. The various distributional aspects such as moments, skewness, kurtosis, median, moment generating function, mean deviation, mills ratio, order statistics, Lorenz and Bonferroni curves are studied. The method of maximum likelihood is used to estimate the parameters and a simulation study is performed to study the finite sample behaviour of the ML estimates. The MLE's

TABLE 4. Estimated parameters and discrimination criteria of *BGSl(a,b,q)* distribution, *Modified Slash Lindley (MSL) distribution*, *Generalised beta distribution of first kind (GB1)*, *Generalised Exponential distribution(GE)* and *Generalised Gamma(GG) distribution* fitted to the data on failure times

Distribution	MLE	log-likelihood	AIC	BIC
BGSl(a,b,q)	$\hat{a}$=0.01976			
	$\hat{b}$=7.9395	-16.9600	39.92008	37.4483
	$\hat{q}$=4.05135			
MSL	$\hat{}\lambda$=0.05035			
	$\hat{q}$=1.196	-26.0660	56.1321	54.4843
GB1	$\hat{a}$=0.4980			
	$\hat{b}$=0.0271	-74.2625	156.5254	153.2293
	$\hat{p}$=0.5655			
	$\hat{q}$=0.0513			
GE	$\hat{a}$=0.4980153			
	$\hat{\lambda}$=0.02710	-76.7627	157.5254	155.8773
GG	$\hat{a}$=97.9396			
	$\hat{d}$=0.6999	-84.4641	174.9282	172.4565
	$\hat{p}$=0.5655			

are found to be consistent and precise in estimating the true value of the parameters. To show the application of the proposed distribution, it is applied to a dataset consisting of failure times and its fit is compared with that of Modified Slash Lindley (MSL) distribution, Generalised beta distribution of 1st kind (GB1), Generalised Gamma distribution (GG)and Generalised Exponential (GE) distribution using log - likelihood measure, Akaike information criterion(AIC) and Bayesian information criterion(BIC). It is observed that the BGSl distribution is a better fit to the data as compared to the others. Thus it can be concluded that the proposed distribution is more flexible and has advantage in modeling right skewed heavy - tailed datasets occuring in $[0, \infty)$ or any subset of it.

Author Contribution Statements The authors have made equal contributions in this work.

Declaration of Competing Interests The authors declares that they have no conflict of interest about the publication of this paper.

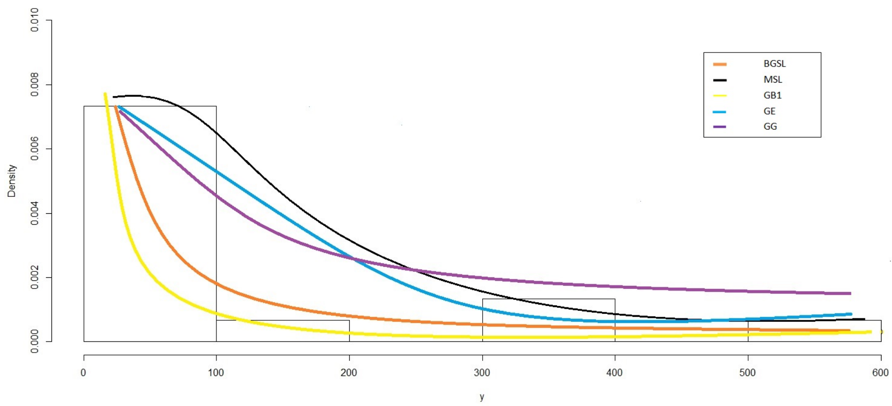

FIGURE 4. Histogram of and fitted densities to the data on air-conditioning equipment consecutive failure times in a Boeing 720 airplane

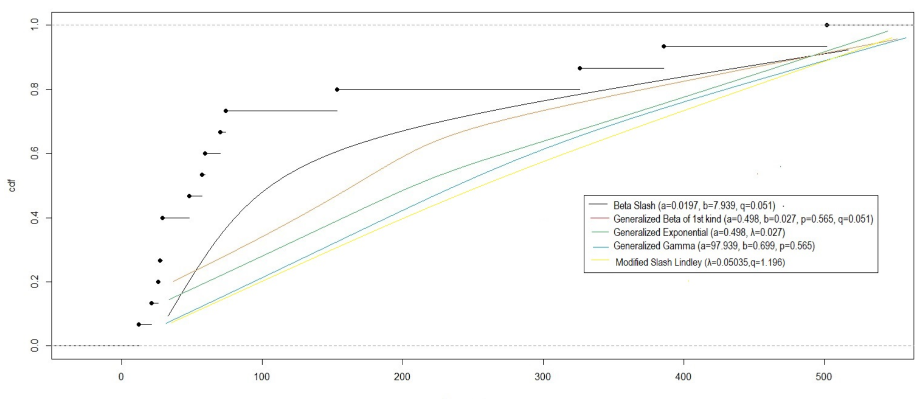

FIGURE 5. CDF plot of the observed data and fitted distributions

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