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RESEARCH PAPER

Towards a viable control strategy for a model describing the dynamics of corruption

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Abstract

This paper explores the use of viability theory in the examination of corruption dynamics, using a susceptible-infected-recovered (SIR) model. In order to promote transparency, good governance, and sustainable economic growth, it is crucial to develop effective strategies for controlling corruption in society. Viability theory provides a framework for analyzing the long-term feasibility of different control policies by defining the set of constraints that define acceptable behavior for a given system. We use this framework to study the impact of different anti-corruption measures on the spread of corruption in a population. Our results show that a combination of measures targeting both the susceptible and corrupted populations can lead to significant reductions in corruption levels over time. We also discuss the challenges involved in applying viability theory to the study of corruption dynamics, including the need for reliable data and the limitations of simple models such as the SIR model. Our results highlight the potential of the viability theory as a valuable tool for promoting transparency, good governance, and sustainable development and suggest that further research in this area is needed to refine and improve the methods used. Our research offers a proof-of-concept for applying viability theory to manage the dynamics of corruption, paving the way for potential future research directions.

Keywords: Control theory; viability theory; SIR model; corruption dynamic

AMS 2020 Classification: 37N35; 93A30; 14L24

1 Introduction

Corruption is the improper use of power for individual benefit, eroding trust, weakening democracy, hindering economic development, and further exacerbating inequalities, poverty, social division, and the environmental crisis. [1]. Corruption is a pervasive problem that can have devastating effects on the stability and development of societies. It can undermine public trust, reduce

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economic growth, and lead to political instability. According to Transparency International's 2021 Corruption Perceptions Index [2], more than two-thirds of countries scored below 50 out of 100, indicating a high level of perceived corruption.

Traditionally, bribery or favor is used to distort or destroy integrity in the fulfillment of official obligations. More recently, corruption is defined as an inducement by persons, public or private, to indicate favor or act dishonestly or unfaithfully within the discharge of their duties. It is typically related to public officials, and therefore, the performance of public duties is impacted. However, it's now increasingly accepted that the act of corruption may apply to both public and personal individuals and will extend beyond bribery [3].

In the academic literature, the number of quantitative research studies on corruption is relatively low. Several researchers have explored the conceptual analogy between the spread of infectious diseases [4, 5] and the diffusion of corrupt behavior [6] within social networks. By adapting the epidemiological model to corruption dynamics, these studies aim to understand how corrupt practices propagate through networks of individuals and institutions. Abdulrahman [7] proposed a mathematical model with a constant recruitment rate and standard incidence for corruption as a disease with its transmission dynamics. Distributed numerical simulations showed that corruption can only be lowered to a very low degree, not completely eradicated. In [8], the authors suggested a mathematical model for corruption that took into account the awareness raised by anti-corruption campaigns and in-prison counseling. With the aid of differential equation stability theory, a nonlinear deterministic model illustrates and undergoes qualitative examination for the dynamics of corruption presented in [9]. Employing the next-generation matrix methodology, the basic reproduction number for the corruption-free equilibrium is determined. Additionally, they improved the model by adding one optimal control technique for optimal control. They came to the conclusion that if attempts to control corruption are strengthened and put into practice through the media and punishments, the degree of corruption in society may be lowered.

Efforts to control corruption have been the subject of much research and policy discussion. In recent years, there has been a growing interest in using mathematical modeling to understand the dynamics of corruption and develop effective strategies for controlling it. One such approach is the use of viability theory, or invariance [10].

The viability theory, which examines the evolution of dynamic systems under specified conditions, was formally introduced by Aubin [11]. Research on the efficacy of control systems primarily revolves around two key aspects. Firstly, it involves verifying the system's feasibility within a defined domain [12, 13] or devising an appropriate controller to ensure the closed-loop system's feasibility within that domain [14–16]. Secondly, it entails identifying the viability kernel (the largest controllable invariant manifold) of the system. Additionally, viability theory proves beneficial in addressing other control-related challenges such as the reachability problem [17], stabilization problem [15, 18], differential games [19], and safety behavior problem [20].

The ability to adjust and the multidisciplinary application of viability theory are demonstrated by its use in many domains. In order to assess the resilience and sustainability of economic systems, viability theory has been widely used in financial and economic modeling. Viability analysis has been used by researchers to examine sustainable development, financial stability, and macroeconomic dynamics. Research has, for instance, looked at the sustainability of resource allocation systems [21], the resilience of financial markets [22], and the viability of economic policies [23]. Viability theory has found significant use in ecological and environmental studies. Researchers have applied viability analysis to evaluate the resilience of ecosystems to environmental changes. Viability analysis can identify critical thresholds for biodiversity conservation [24, 25]. In control theory and engineering, viability theory has been employed to design robust control systems and ensure the feasibility of dynamic processes under constraints. Research in this area has focused on developing viability-based control algorithms [26, 27]. In biology and medicine, viability theory has been used to study the dynamics of biological systems, population viability, and disease control strategies. Researchers have used viability analysis to assess the viability of threatened species, model the spread of infectious diseases, and improve treatment protocols for chronic illnesses [28, 29]. For additional information on viability theory, see [11] or [30].

Motivated by the robust modeling capabilities of the viability theory, this study delves into the application of the SIR model to analyze corruption dynamics. Beginning with an exploration of the theoretical foundations, the research extends to practical investigations, leveraging viability theory to understand the propagation of corrupt behavior within societal frameworks and devise effective strategies for mitigating corruption within complex socio-economic systems.

The main contributions of this research are multifaceted. Firstly, it addresses a crucial challenge in the study of corruption dynamics by applying viability theory within the framework of the SIR model. Traditionally, ensuring the boundedness of trajectories in corruption studies, especially within the context of complex societal systems, poses significant difficulties. However, this paper overcomes this challenge by strategically incorporating viability theory, thus offering a novel avenue to guarantee trajectory boundedness and enhance the robustness of corruption modeling. Additionally, while viability theory inherently ensures constraint avoidance, the integration with the SIR model provides insights into stability aspects, thereby improving the overall understanding of corruption dynamics. This integration represents a theoretical innovation, enabling the analysis of corruption propagation and control in more complex socio-economic environments. Secondly, the study presents a methodological framework for constructing viable strategies to combat corruption using the SIR model, particularly in scenarios where corruption dynamics are influenced by various societal factors and constraints. Thirdly, the proposed approach is applied to real-world cases, such as studying corruption propagation within specific sectors or regions, thereby providing valuable insights for policymakers and practitioners aiming to tackle corruption effectively. The aim of this article is to provide an overview of the viability theory and its application to the control of corruption dynamics. Our approach considers both transient and asymptotic behavior. Rather than aiming for equilibrium or optimization, our goal is to identify policies that can restrict the number of corrupt individuals below a certain threshold at any given point in time. The article will discuss the concept of viability, the methods used to calculate the viability kernel, and the challenges involved in applying viability theory to the study of corruption dynamics. The article will also review recent research in this area and discuss the potential implications of this approach for policymakers and researchers working to address corruption.

2 The viability problem

The viability problem is dependent on the consistency of the acceptability constraints applied to the states and decisions of the system.

The dynamics

To describe the viability property, consider the following non-linear control system:

$$\dot{x}(t) = F(x(t), u(t)),$$

with state $x \in \mathbb{R}^n$, control $u \in \mathcal{U}$ and $F(x, u)$ is a Lipschitz function.

Definition 1 *Let K be a subset of the domain of F. A function $x(.)$ is said to be viable in K if and only if :*

$$\forall t \geq 0, \qquad x(t) \in K.$$

Definition 2 *The tangent cone to K at x denoted by $T_K(x)$, is the closed cone of elements v such that*

$$\liminf_{h \mapsto 0^+} \frac{d(x + hv, K)}{h} = 0.$$

Definition 3 *A subset K is a viability domain of a non-trivial set-valued map F if and only if the following condition is satisfied:*

$$\forall\ x\ \in\ K, \qquad F(x) \cap T_K(x) = \varnothing.$$

It is worth noting that the tangent cone $T_K(x) = \mathbb{R}^n$ for any interior point x within K, thereby satisfying the condition mentioned above. Therefore, it is sufficient to examine only the points on the boundary of set K to verify the condition.

Definition 4 *The viability kernel of K with respect to F, denoted by $Viab_F(K)$, is defined as the largest possible closed subset of K that is viable under F, which could potentially be an empty set.*

Definition 5 ***Viability Kernel*** *is the set of initial states x_0 from which a feasible path $(x(.); u(.))$ respecting the constraints (staying in K) at all times:*

$$Viab_F(K) = \{x_0 \in K \setminus \exists x(t) \quad such\ that \quad x(t) \in K, \quad \forall t \geq 0, \quad x(0) = x_0\}.$$

A set K would be viable for the dynamic (F, u) if the viability kernel $viab_F(K)$ coincides with the set of initial constraints K. Under appropriate assumptions on the dynamics, a closed set K is considered favorable when a control u can initiate a feasible trajectory within K from any state x in K, resulting in velocities $\dot{x} = F(x, u)$ that are either tangent or inward pointing with respect to the domain K.[1] The Hamiltonian formulation can be employed to express this idea. Specifically, we can consider the Hamiltonian function:

$$\mathcal{H}(x, q, u) = \sum_{i=1}^{n} q_i\, F_i(x, u).$$

In this case, the following statements are equivalent:

i K is viable for (F, U),

ii $Viab_F(K) = K$,

iii $\inf_{u \in U(x)} \mathcal{H}(x, q, u) \leq 0, \ \ \forall x \in K, \quad \forall q \in N_K(x)$,

with $N_K(x)$ is the cone normal to the set K at x.

3 Mathematical formulation and description of the problem

The SIR model can be adapted to study corruption dynamics by drawing an analogy between the spread of infectious diseases and the diffusion of corrupt behavior within social networks. In this adapted model, we consider a population split into three categories:

1 For example, if U is convex, closed, and bounded with F regular.

S is the number of susceptible,
C is the number of corrupts,
R is the number of individuals to recover,

in which we assume that the whole population: $N = S + C + R$.

We also suppose that:

- The population is fixed: no demographic phenomena (births, deaths, immigration and emigration).
- No longer being honest means necessarily becoming corrupt.
- Corrupt people are all infectious.
- Each person recovered is recovered forever.

The model will have the following system of differential equations:

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \delta N - \frac{\lambda C_t S_t}{N} - \delta S_t,$$
$$\frac{\mathrm{d}C_t}{\mathrm{d}t} = \frac{\lambda C_t S_t}{N} - \nu C_t - \delta C_t,$$
$$\frac{\mathrm{d}R_t}{\mathrm{d}t} = \nu C_t - \delta R_t.$$

Table 1. Descriptions of the model's parameters

Parameters	Description
λ	Transmission rate of corruption from a corrupter person in a time period
δ	Birth and death rate, which are assumed to be equal
ν	The recovery rate

To simplify matters, we can redefine the occurrence as the proportions:

$$s_t = \frac{S_t}{N}, \qquad c_t = \frac{C_t}{N}, \qquad r_t = \frac{R_t}{N}.$$

We get

$$\frac{\mathrm{d}s_t}{\mathrm{d}t} = \delta - \lambda c_t s_t - \delta s_t,$$
$$\frac{\mathrm{d}c_t}{\mathrm{d}t} = \lambda c_t s_t - \nu c_t - \delta c_t,$$
$$\frac{\mathrm{d}r_t}{\mathrm{d}t} = \nu c_t - \delta r_t,$$

with initial conditions:

$$s(0) \geq 0, \qquad c(0) \geq 0, \qquad r(0) \geq 0.$$

Taking into account the overall population density, we have $s(t) + c(t) + r(t) = 1 \Rightarrow r(t) =$

$1 - s(t) - c(t)$. Thus, it is sufficient to consider

$$\begin{aligned} \frac{ds_t}{dt} &= \delta - \lambda c_t s_t - \delta s_t, \\ \frac{dc_t}{dt} &= \lambda c_t s_t - \nu c_t - \delta c_t. \end{aligned} \tag{1}$$

The set $\Delta = \{(s(t), c(t)) \in \mathbb{R}^2_+; \quad s(t) + c(t) \leq 1\}$ is positively invariant for system (1). System (1) has two equilibrium points that are given by the disease-free equilibrium point of the system $E^0 = (1, 0)$ and the endemic equilibrium point

$$E^* = \left(\frac{\nu + \delta}{\lambda}, \frac{\delta(\lambda - \nu - \delta)}{\lambda(\nu + \delta)} \right).$$

The endemic equilibrium point exists only when $\lambda > \nu + \delta$ i.e the transmission of corruption must be greater than the death rate of the corrupt individuals or $R_0 > 1$, where $R_0 = \frac{\lambda}{\nu+\delta}$ is known as reproduction number which determines the asymptotic behavior of the model.

Control problem

We will now introduce our second model, which incorporates honesty as an induced trait. The proposed model is as follows:

$$\begin{aligned} \frac{ds_t}{dt} &= \delta(1 - \alpha) - \lambda c_t s_t - \delta s_t, \\ \frac{dc_t}{dt} &= \lambda c_t s_t - \nu c_t - \delta c_t, \\ \frac{dr_t}{dt} &= \nu c_t - \delta r_t, \\ \frac{dh_t}{dt} &= \delta \alpha - \delta h_t. \end{aligned} \tag{2}$$

In this model, h represents the group in which honesty is implemented. Other parameters include α as the rate of honesty,
and: $s(t) + c(t) + r(t) + h(t) = 1 \Rightarrow r(t) = 1 - s(t) - c(t) - h(t)$.
Thus, it suffices to consider

$$\begin{aligned} \frac{ds_t}{dt} &= \delta(1 - \alpha) - \lambda c_t s_t - \delta s_t, \\ \frac{dc_t}{dt} &= \lambda c_t s_t - \nu c_t - \delta c_t, \\ \frac{dh_t}{dt} &= \delta \alpha - \delta h_t. \end{aligned} \tag{3}$$

The set $\Delta_1 = \{(s(t), c(t), h(t)) \in \mathbb{R}^3_+; \quad s(t) + c(t) + h(t) \leq 1\}$ is positively invariant for system (3).
System (3) has two equilibrium points that are given by the disease-free equilibrium point $E^0_1 = (1 - \alpha, 0, \alpha)$ and the endemic equilibrium point

$$E^*_1 = \left(\frac{\nu + \delta}{\lambda}, \frac{\delta(\lambda(1 - \alpha) - \nu - \delta)}{\lambda(\nu + \delta)}, \alpha \right).$$

The impact of honesty on the disease-free equilibrium point and endemic equilibrium point can be readily observed. The susceptible population is reduced by a factor of α (the honesty rate). Additionally, the reproduction number, which represents the number of secondary infections, is greatly affected. Following the introduction of honesty control in the model, the new reproduction number becomes $R_h = R_0(1-\alpha)$. The existence of an endemic equilibrium point is contingent on $R_h > 1$.

4 Viable control of corruption dynamics

Now, applying the viability theory to our dynamic, we focus our study on the following controlled dynamic system:

$$\begin{aligned}\frac{\mathrm{d}s_t}{\mathrm{d}t} &= \delta(1-\alpha) - \lambda c_t s_t - \delta s_t, \\ \frac{\mathrm{d}c_t}{\mathrm{d}t} &= \lambda c_t s_t - \nu c_t - \delta c_t, \\ \frac{\mathrm{d}h_t}{\mathrm{d}t} &= \delta\alpha - \delta h_t.\end{aligned} \tag{4}$$

Viability constraint

The main purpose is to determine the control α that keeps the number of corrupts below the c_m boundary, where the viability constraint expresses the honesty of a community as long as the viability constraint is achieved:

$$c_t < c_m, \quad \forall t \geq t_0, \tag{5}$$

with $0 < c_m \leq N$.
The presence of control essentially relies upon the underlying state $(s_{t_0}, c_{t_0}, h_{t_0})$ at the initial time t_0. We will currently concentrate on these initial states, likewise called the viability kernel [11].

Viability analysis

We may use viability theory methods to analyze our dynamics and, in particular, calculate the viability kernel. This will allow us to see if corruption dynamics (4) are compatible with the viability constraint (5) at any given time t. The viability kernel is formally defined as follows:

Definition 6 *The viability kernel $Viab(c_m)$ is a set of initial states $(s_{t_0}, c_{t_0}, h_{t_0})$ for which an honesty rate $t \mapsto \alpha(t) \in [0,1]$ exists so that the dynamic system (4) solution meets the viability constraint (5).*

$$Viab(c_m) = \left\{ (s_{t_0}, c_{t_0}, h_{t_0}) \mid \begin{array}{c} \textit{there exist a control } \alpha_t(.), \textit{ so that the solution to } (4) \\ \textit{that starts from } (s_{t_0}, c_{t_0}, h_{t_0}) \textit{ satisfies the constraint } (5) \end{array} \right\}. \tag{6}$$

Note that our unconstrained domain of research is the positively invariant set

$$\{(s_t, c_t, h_t) \mid 0 \leq s_t, 0 \leq c_t, s_t + c_t + h_t \leq 1\}.$$

Since the initial point should fulfill the viability constraint (5), The rectangle $[0,1] \times [0, c_m[\times [0,1]$ must contain the viability kernel $Viab(c_m)$.
The constraint set $\mathbb{V}$ is the intersection of the unconstrained domain of study and the cuboid

$[0,1] \times [0, c_m[\times [0,1]$ (see Figure 1)

$$\mathbb{V} := \{(s_t, c_t, h_t) \mid 0 \leq s \,,\, 0 \leq c_t < c_m \,, 0 \leq h \;, s_t + c_t + h_t \leq 1\}. \tag{7}$$

That is,

$$Viab(c_m) \subset \mathbb{V}.$$

As they are a fundamental step in defining the viability kernel, we describe and give a geometrical

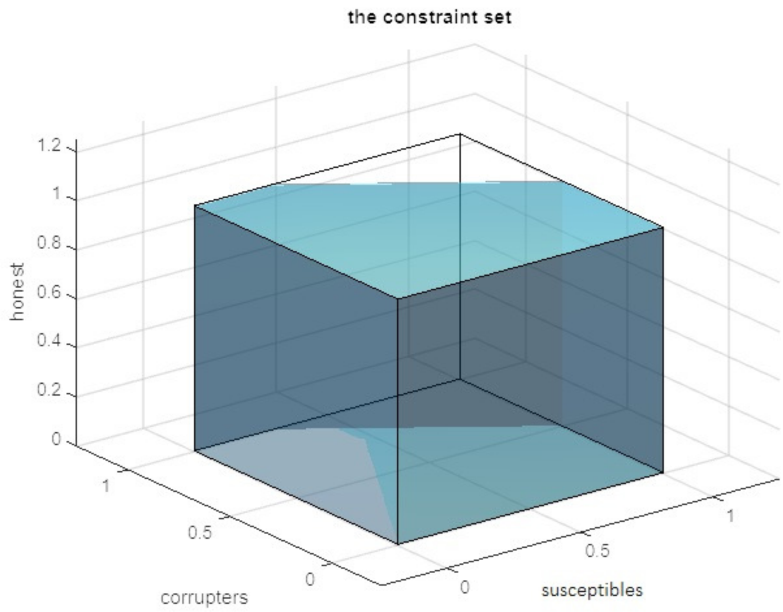


Figure 1. The constraint set $\mathbb{V}$

characterization of the so-called viability domains of system (4).

Definition 7 *A viability domain for the system (4) is a subset K of the set of states $[0;1] \times [0; c_m[\times [0;1]$ if there exists a control $\alpha_t(.)$ such that the solution to (4), which starts from $(s_{t_0}, c_{t_0}, h_{t_0})$, remains inside K for every $t \geq 0$.*

The viability kernel is linked to the viability domains in the following way:

Theorem 1 *[11] The viability kernel is the constraint set's largest viability domain.*

We associate the vector field (u_s, u_c, u_h) with system (4):

$$\begin{pmatrix} u_s \\ u_c \\ u_h \end{pmatrix} = \begin{pmatrix} -\lambda c_t s_t + \delta(1-\alpha) - \delta s_t \\ \lambda c_t s_t - \nu c_t + \delta c_t \\ \delta\alpha - \delta h_t \end{pmatrix},$$

the system (4) is equivalent to:

$$\begin{cases} \dot{s} &= u_s(s(t), c(t), \alpha(t)), \\ \dot{c} &= u_c(s(t), c(t)), \\ \dot{h} &= u_h(h(t), \alpha(t)). \end{cases} \tag{8}$$

Using the vector field u, we provide a geometric description of the viability regions of the system with control.

Proposition 1 *[11] For a Marchaud controlled system, a closed subset U is considered viable if the family of vectors formed by the vector field when the control varies is assured to have at least one vector contained within the tangent cone at any point in U.*

In our situation, the geometric characterization of viability domains is as follows:

Proposition 2 *Consider a closed subset K If there is a control $\alpha_t \in [0;1]$ such that (u_s, u_c, u_h) is an inward-pointing vector, then the set K is a viability domain for the system (4) whenever (s, c, h) varies along the frontier δK of the set K.*
To be considered a viability domain for system (4), the scalar product of the vector (u_s, u_c, u_h) and an outward-pointing normal vector (relative to set K) must be less than or equal to zero for a closed subset K with a piecewise smooth boundary δK.

Viability kernel

Proposition 3 *The viability kernel in (6) is as follows:*

$$Viab\,(c_m) = \mathbb{V} \cap \{(s, c, h) \mid 0 \leq h\ , s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}. \tag{9}$$

With $\mathfrak{C}(s)$ is the set of applications such as s the solution $s \in [s_m, 1] \mapsto \mathfrak{C}(s)$ to the differential equation:

$$\begin{aligned} -u_s(s(t), \mathfrak{C}(s), \alpha(t))\mathfrak{C}'(s) + u_c(s(t), \mathfrak{C}(s)) = 0, \\ \mathfrak{C}(s_m) = c_m. \end{aligned} \tag{10}$$

Proof We show that the set $Viab\,(c_m)$ is a viable set. Since $\mathbb{V}$ is an invariant set, we can focus on the boundary of the set: $\{(s, c, h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}$.
The boundary can be obtained by considering the conditions where each of the three inequalities in the set definition is satisfied with equality.
First, considering $h = 0$, we get the lower boundary:

$$\partial(\mathbb{V} \cap \{(s, c, h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}) = \{(s, c) \mid s_m \leq s \leq 1,\ c < \mathfrak{C}(s),\ s + c < 1\}.$$

Second, for $s = s_m$, we have:

$$\partial(\mathbb{V} \cap \{(s, c, h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}) = \{(s_m, c, h) \mid 0 \leq h,\ c < c_m,\ c + h < 1 - s_m\}.$$

Finally, for $c = \mathfrak{C}(s)$, we get the upper boundary:

$$\partial(\mathbb{V} \cap \{(s, c, h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}) = \{(s, c, h) \mid s_m \leq s \leq 1,\ c = \mathfrak{C}(s),\ 0 \leq h\}.$$

To examine the scalar product of u with the normal vector to the boundaries of $Viab(c_m)$, we first need to find the normal vectors to each boundary.
Boundary: $h = 0$ The normal vector to this boundary is $\mathbf{n} = (0, 0, 1)$.
Boundary: $s = s_m$ The normal vector to this boundary is $\mathbf{n} = (-1, 0, 0)$.
Boundary: $c = \mathfrak{C}(s)$ The normal vector to this boundary is $\mathbf{n} = \left(-\frac{\partial \mathfrak{C}}{\partial s}, 1, 0\right)$.
Next, we examine the scalar product of u with these normal vectors:

1 Boundary: $h = 0$

$$u.n = u_h(0,0,0).1 = -1 < 0.$$

Since the scalar product is negative, this implies that the vector u points into the domain $Viab(c_m)$.

2 Boundary: $s = s_m$

$$u.n = u_s(s_m, c, h).(-1) < 0.$$

Since the scalar product is negative, this implies that the vector u points into the domain $Viab(c_m)$.

3 Boundary: $c = \mathfrak{C}(s)$

Consider any point (s, c, h) on the boundary with $c = \mathfrak{C}(s)$. Let $u = (u_s, u_c, u_h)$ be any outward pointing normal vector at this point. Since u is outward pointing, we have $u \cdot (-\nabla c, 1, 0) > 0$ where $-\nabla c = (-\frac{\partial \mathfrak{C}}{\partial s}, -\frac{\partial \mathfrak{C}}{\partial c}, 0) = (-\mathfrak{C}'(s)u_s, 1 - u_c, 0)$. This implies that

$$u_s \mathfrak{C}'(s) u_s - (1 - u_c) > 0.$$

Since u is normal to the boundary, we also have $u \cdot (0,0,1) = u_h > 0$. Combining these two inequalities, we get

$$u_s \mathfrak{C}'(s) u_s > 1 - u_c,$$

and hence

$$u_s \mathfrak{C}'(s) u_s - (1 - u_c) + u_h^2 > u_h^2.$$

This can be written as

$$\mathbf{u} \cdot \begin{pmatrix} -\mathfrak{C}'(s)u_s \\ 1 - u_c \\ 2u_h \end{pmatrix} > 0.$$

Since u is outward pointing, we must have

$$-\mathfrak{C}'(s) u_s u_s - (1 - u_c) + 2u_h^2 \leq 0.$$

Using $c = \mathfrak{C}(s)$ and $u_h > 0$, this can be rewritten as $-u_s u_c + u_h^2 \leq 0$. Since u is a unit vector, we have $u_s^2 + u_c^2 + u_h^2 = 1$, which implies $u_h^2 \leq 1 - u_s^2 - u_c^2$. Substituting this inequality in the above expression, we get $u_s^2 + u_c^2 \leq 1$. Therefore, all outward-pointing normal vectors at the boundary satisfy $u_s^2 + u_c^2 \leq 1$, which implies that all inward-pointing vectors satisfy $u_s^2 + u_c^2 \geq 1$.

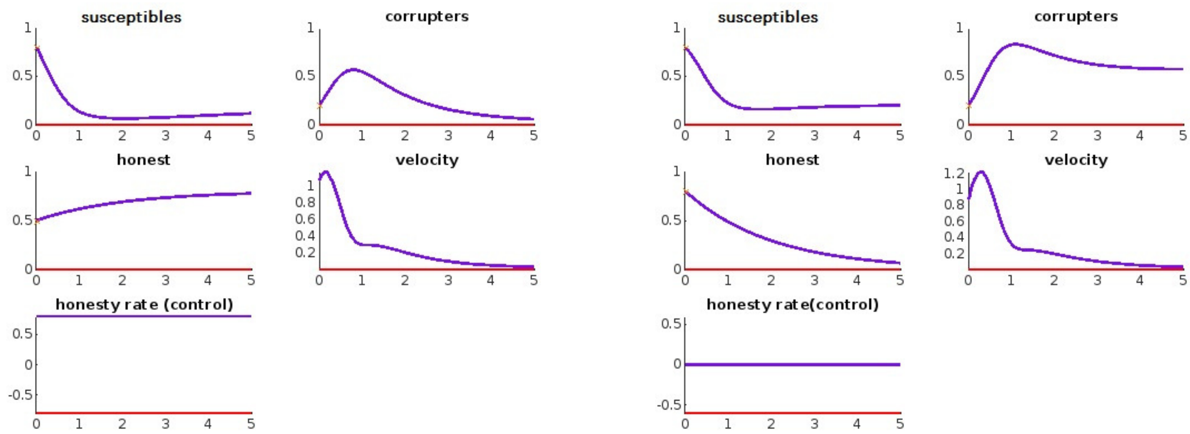
We have shown that the set $\mathbb{V} \cap \{(s, c, h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}$ is forward invariant and that its boundary is also invariant. Moreover, we have shown that there exists a control α such that any vector u in the interior of the set $\mathbb{V} \cap \{(s, c, h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}$ satisfies $u \cdot v \leq 0$ where v is the outward normal to the boundary of the set.

Therefore, by definition, this set is viable. This means that starting from any initial condition in this set, there exists a control strategy $\alpha(t)$ that will keep the system inside the set for all future times. In other words, the set $\mathbb{V} \cap \{(s,c,h) \mid 0 \leq h,\ s_m \leq s \leq 1 \text{ and } c < \mathfrak{C}(s)\}$ represents a region of safe and sustainable operation for the system.

5 Simulation results and discussion

We simulated the corruption dynamic using the constraints $c_m = 0.7$ and the parameters $\lambda = 1.98$, $\delta = 0.5$, and $\nu = 0.5$.

The simulation represents the progression of the corruption dynamic over time with honesty control. The results reveal that corrupts initially rise significantly, but subsequently begin to drop due to the impact of the honesty control. The honest fraction of the population gradually increases, demonstrating that the control technique is effective. Therefore, the simulation shows that the honesty control is successful in reducing corruption in a population.



(a) maximum control **(b)** zerocontrol

Figure 2. Time profiles for the dynamics of corruption starting from $x(0) = (0.8, 0.2, 0.5)$

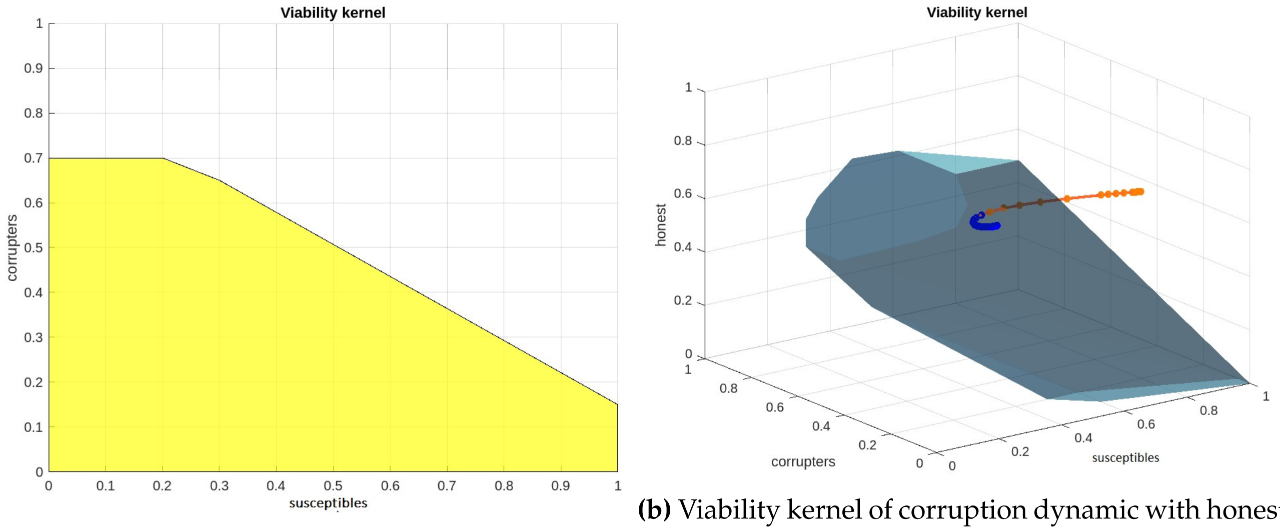
As shown in the Figure 2a, the maximum control method is used, causing the proportion of corrupt persons to climb and peak at a low level before gradually declining to zero over time. On the other hand, the proportion of honest people grows with time and eventually approaches one. This shows that the honesty control technique is effective over time in eliminating corruption in the community.

While in Figure 2b, the honesty control strategy is set to zero, implying that no control is applied to the system. As a result, we see that the proportion of corrupt people increases over time and reaches a high level. At the same time, the proportion of honest people falls to zero. This behavior implies that the corruption dynamic is not viable under this strategy, and the system will eventually collapse due to the significant level of corruption.

Figure 3a shows the viability kernel of the corruption dynamic. The yellow area within the viability kernel shows the initial conditions under which the system can remain viable, whereas the white area outside the viability kernel represents the starting conditions under which the system will eventually collapse into complete corruption.

The figure indicates that, for the given corruption dynamics, the system can stay viable for a limited range of beginning conditions (i.e., within the viability kernel) but will eventually collapse into total corruption for initial conditions outside the viability kernel.

Figure 3b illustrates the trajectory of a system in state space with three variables. The blue zone



(a) Viability kernel of corruption dynamic

(b) Viability kernel of corruption dynamic with honest people

Figure 3. Viability kernel for the dynamic of corruption with control

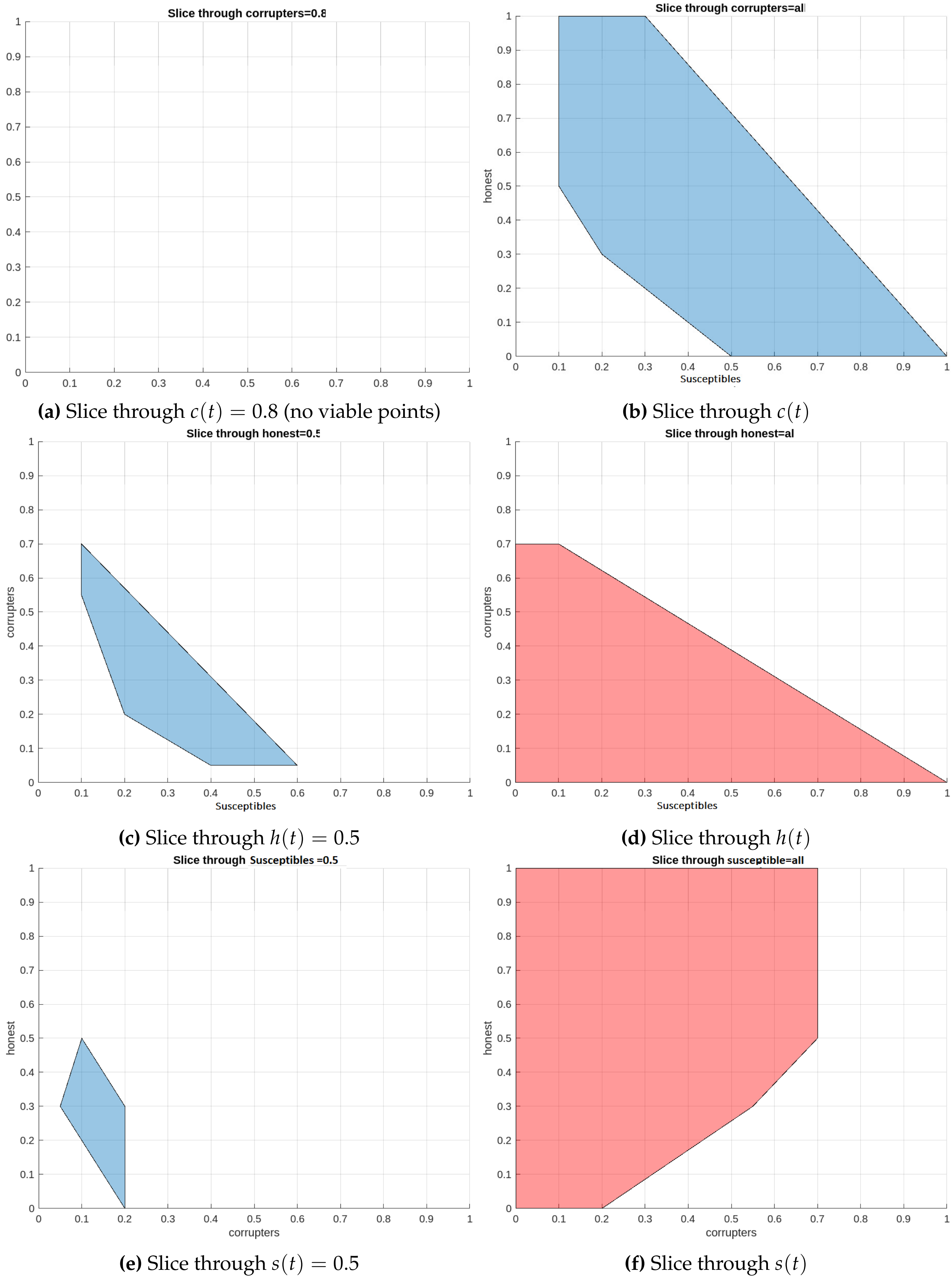
represents the initial set of states from which the system can progress to the desired goal set. Because there is a set of initial conditions from which the system can be steered towards the desired goal while avoiding obstacles, the figure shows that the system is controllable. The system's trajectory is determined by its beginning conditions, and the vector field shows how the system will evolve over time. The graphic provides a good visual representation of the system's behavior in state space.

Figure 4a illustrates a slice through the three-dimensional viability kernel of the corruption dynamic with control. The slice is taken at the corrupts value of 0.8, and it indicates that the viability kernel is empty beyond this value of corrupts.

In Figure 4b, we can show the corruption dynamic's viability kernel for different values of susceptible and honest people. The region of initial states where there exists a feasible path that stays inside the constraint set, regardless of the value of control α is shown by the slice through all the values of corrupts. We can observe that with large values of susceptible and honest persons, practically all initial states are viable, showing that controlling corruption in a society with low corruption levels is quite easy. However, as corruption levels rise, the viable zone narrows, and for really high levels of corruption, the viability kernel becomes empty, indicating that controlling corruption from such beginning conditions is impossible. This emphasizes the critical need to prevent corruption from reaching dangerously high levels in the first place.

In Figure 4c, the slice is taken through $h(t) = 0$. The blue zone indicates the corruption dynamic's viability kernel under the stated limitations. In this situation, the viability kernel is a subset of the region with a large number of susceptible and a low number of corrupts. Because there is less pressure on individuals to participate in corrupt activity when the number of susceptible is high. Similarly, when the number of corrupt population is few, corrupt behavior has less of a chance of spreading. The boundary of the viability kernel is represented by the black line within the blue region when the tangent cone condition is satisfied. The tangent cone condition is not satisfied outside of the viability kernel, which means that there are no viable trajectories that remain within the feasible region at all times.

In Figure 4d, we observe a visualization of the corruption dynamic's viability kernel for various values of the honest fraction $h(t)$. The susceptible (non-corrupt persons) are represented by the x-axis, while the corrupts are represented by the y-axis (corrupt individuals). As we slice across $h(t)$, we are simply fixing the value of $h(t)$ and seeing how the viability kernel changes in relation



(a) Slice through $c(t) = 0.8$ (no viable points)

(b) Slice through $c(t)$

(c) Slice through $h(t) = 0.5$

(d) Slice through $h(t)$

(e) Slice through $s(t) = 0.5$

(f) Slice through $s(t)$

Figure 4. Slices of the viability kernel for the corruption problem with honesty as a control

to the other two variables, susceptible and corrupts. For example, if we take a slice across a certain value of $h(t)$ (e.g., $h(t) = 0.6$), we may examine the related viability kernel for that value of $h(t)$ and compare it to the viability kernels for other $h(t)$ values.Thus, the image sheds light on the viability of the corruption dynamic under many situations, emphasizing the relevance of the honest percentage in determining system stability.

This slice of the corrupted dynamic graphic Figure 4e and Figure 4f shows the system's behavior when the susceptible population's initial value is set to 0.5. The color map illustrates the viability kernel. We can see from this slice that for low values of the honest percentage, there is a large region of starting states that are not feasible, as indicated by the white color. This region increases as the honest fraction grows, and more beginning states become viable. Yet, even at high levels of the honest fraction, some initial states are not viable, as seen by the remaining white patches. The corrupt axis depicts the system's level of corruption, and we can see that the viable zone shrinks as the level of corruption increases. The honest axis indicates the system's level of honesty, and we can see that as the level of honesty increases, so does the viable region.

We may say that the viability theory is effective in providing a clear knowledge of the set of initial conditions that lead to a feasible trajectory that respects the restrictions over time in our simulation of the corruption dynamic. It enables us to find the system's viability kernel, which is the biggest closed subset of the state space that is viable under the system's dynamics. We can obtain an understanding of the system's behavior and the impact of different parameters on the dynamics by evaluating distinct slices of the viability kernel.

Furthermore, the viability theory is a powerful tool for developing control mechanisms that ensure the system's viability and adherence to time limits. We can avoid constructing control techniques that lead to such regions of the state space by identifying them. Furthermore, the viability theory can be used to analyze the resilience of control systems by determining if the system remains viable in the presence of uncertainties or disturbances.

To summarize, the viability theory is a useful tool for comprehending and devising control techniques for complicated systems with restrictions, such as the corruption dynamic in our simulation.

6 Conclusion

This study introduces an innovative approach to analyzing corruption dynamics using the SIR model, integrating two innovative methodologies: viability theory and epidemic modeling. Diverging from conventional methods, the proposed approach does not necessitate prior knowledge of specific system dynamics.

The approach we have developed represents, to the best of our knowledge, a novel methodology in the mathematical modeling of corruption dynamics. Unlike traditional approaches focused on achieving equilibrium or optimization, our method prioritizes the design of policies aimed at consistently keeping the number of corrupt individuals below a specified threshold over time.

After first setting a briber's level of C_m, we used a non-stationary technique to find all the starting states where the largest number of bribers at the peak may stay below C_m. We've also found potential answers and offered examples of techniques for managing the highest limit of corrupts at peak and asymptotically reducing the number of corrupts to zero.

The core concept was given using a simple SIR model of corruption dynamics with the honesty rate as a control. On the one hand, our model can be enhanced to be more accurate by defining an upper constraint on the control $\alpha < 1$, preventing a full honesty rate that was either impossible or highly expensive. However, this approach can be utilized in various other models that incorporate different controls.

Inspired by the complexities inherent in corruption studies, the research presents a novel framework for understanding and controlling corruption propagation within societal systems. Initially, a pioneering application of viability theory is demonstrated to ensure the viability of corruption trajectories, addressing challenges of boundedness typically encountered in corruption modeling. Subsequently, the viability-theory-based approach is extended to analyze a broader spectrum of corruption dynamics, encompassing diverse socio-economic contexts and constraints. Furthermore, the practical utility of the proposed methodology is exemplified through its application to real-world scenarios, such as studying corruption within specific sectors or regions. Comparative analyses underscore the effectiveness of the proposed approach in elucidating corruption dynamics and informing evidence-based interventions. Nonetheless, the study acknowledges potential limitations, such as the assumption of viability for certain sets. The accuracy and usefulness of the viability kernel will depend on the quality of the data and models used, and further research is needed to refine the methods and develop more robust approaches. Nevertheless, the viability kernel concept offers a valuable tool for addressing corruption and promoting sustainable development.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

No Data associated with the manuscript.

Ethical approval (optional)

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

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Author's contributions

H.A.: Methodology, Conceptualization, Validation, Software, Data Curation, Writing - Original Draft. M.E.K.: Writing-Review Editing, Supervision. All authors have read and agreed to the published version of the manuscript.

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