Fundamental Journal of Mathematics and Applications, 7(4) (2024), 203-211
Research Paper / Open Access

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1400093

Fixed Point Theorems for Almost α-ψ-Contractive Mappings in F-metric Spaces

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Article Information

Keywords: Almost contraction; α-Admissible mapping, Fixed point, F-Metric

AMS 2020 Classification: 47H10; 54H25

Abstract

In this paper, we introduce an almost α-ψ-contraction and a rational type α-ψ-contraction for α-admissible mappings in complete F-metric spaces which were introduced as a generalization of metric spaces. We prove the existence of a fixed point for these type contractions.

1. Introduction and preliminaries

The definition of metric spaces in 1905 and fixed point theory studies paved the way for important developments in both mathematics and the other sciences. Recently, some generalized metric spaces were introduced and fixed point theorems were proved. F-metric spaces were introduced as a generalization of metric spaces and Banach contraction principle were introduced in F-metric spaces in 2018 by Jleli *et al.* [1]. In F-metric spaces Hussain and Kanwal [2] proved coupled fixed point theorems, Mitrovic *et al.*[3], Laatefa and Ahmad [4] and Jahangir *et al.* [5] proved some generalized fixed point results. Altun and Erduran [6] proved fixed point results for single and multivalued mappings, Özturk [7] defined Ciric-Presic type contraction. Lateefa [8] and Zhou *et al.* [9] gave best proximity results in $F-$metric spaces. Alansari *et al.* [10] proved fuzzy fixed point theorems. Al-Mezel *et al.* [11] and Faraji *et al.*[12] defined $\alpha-\beta-$ admissible type contraction in $F-$metric spaces. Kanwal *et al.* [13] defined orthogonal $F-$metric spaces.

In recent years, several fundamental fixed point results have been extended and generalized by many authors in different directions. Samet *et al.* [14] introduced the concept of $\alpha-$admissible mappings on metric spaces. Many authors obtained some fixed point results using new concept and gave some applications [15, 16, 17, 18, 19, 20, 21, 22].

In this work, we define the concept of almost $\alpha-\psi-$contraction in $F-$metric spaces for $\alpha-$admissible mappings and we prove two main fixed point results.

Denoted by Ω the family of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying following properties;

$F_1)$ F is increasing,
$F_2)$ For each sequence $\{u_s\}_{s\in\mathbb{N}}$, $\lim_{s\to\infty} u_s=0 \Leftrightarrow \lim_{s\to\infty} F(u_s)=-\infty$.

Definition 1.1. *[1] Let $L\neq\varnothing$ be a set and $(F,t)\in\Omega\times[0,+\infty)$. Assume that $\sigma:L^2\to[0,\infty)$ a function satisfying the following:*

(d_1) $\sigma(u_1,u_2)=0\Leftrightarrow u_1=u_2$,
(d_2) $\sigma(u_1,u_2)=\sigma(u_2,u_1)$,

Received: 04-12-2023 **Revised:** 25-03-2024 **Accepted:** 27-05-2024 **Online:** 30-12-2024 **Published:** 31-12-2024

(d_3) for all $s \geq 2$ ($s \in \mathbb{N}$) *and for all* $\{u_s\} \subset L$ *such that if* $\sigma(u_1, u_s) > 0$, $F(\sigma(u_1, u_s)) \leq F\left(\Sigma_{i=1}^{s} \sigma(u_i, u_{i+1})\right) + t$ *for all* $u_1, u_2 \in L$.
Then (L, σ) *is named an* $F-$*metric space (shorty* $F-ms$*).*

Definition 1.2. *[1] Let* (L, σ) *be an* $F-ms$ *and* $\{u_s\}$ *be a sequence in* L.

i. $\{u_s\}$ *is named* $F-$*convergent if there is a* $\mu \in L$ *such that* $\sigma(u_s, \mu) \to 0$ *as* $s \to \infty$.
ii. $\{u_s\}$ *is named an* $F-$*Cauchy sequence if* $\sigma(u_s, u_v) \to 0$ *as* $s, v \to \infty$.
iii. (L, σ) *is named* $F-$*complete if each* $F-$*Cauchy sequence is* $F-$*convergent.*

Denoted by Φ the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying following properties

($\psi 1$) ψ is nondecreasing,
($\psi 2$) $\sum_{s=1}^{\infty} \psi^s(u) < \infty$.

If ($\psi 1$) and ($\psi 2$) are satisfied, then

($\psi 3$) $\psi(u) < u$ for $u > 0$

holds.

Example 1.3. *[6] Let* $L = \mathbb{N}$ *and* $\sigma : L \times L \to [0, \infty)$ *be defined by*

$$\sigma(u, \mu) = \begin{cases} (u-\mu)^2, & \text{if } u, \mu \in \{1, 2, 3\} \\ 2|u-\mu|, & \text{other} \end{cases}.$$

Then (L, σ) *is an* $F-$*complete* $F-ms$ *with* $F(u) = \frac{-1}{u}$ *and* $t = \ln 3$. *But* (L, σ) *is not a metric space.*

Example 1.4. *[23] Let* $\Gamma = [0, \infty)$ *and* $\sigma : \Gamma \times \Gamma \to [0, \infty)$ *be defined by*

$$\sigma(u, \mu) = \begin{cases} (u-\mu)^2, & \text{if } (u, \mu) \in [0, 1] \times [0, 1] \\ |u-\mu|, & \text{if } (u, \mu) \notin [0, 1] \times [0, 1] \end{cases}.$$

Then (Γ, σ) *is an* $F-$*complete* $F-ms$ *with* $F(u) = \ln u$ *and* $t = \frac{1}{2}$. *But* (Γ, σ) *is not a metric space.*

2. Almost α-$\psi-$ contractions

Definition 2.1. *Let* (L, σ) *be an* $F-ms$. *Let* $T : L \to L$ *be a self-mapping on* L *and* $\alpha : L \times L \to [0, \infty)$ *be a function.* T *is named an* $\alpha-$*admissible mapping if*

$$\alpha(u_1, u_2) \geq 1 \Longrightarrow \alpha(Tu_1, Tu_2) \geq 1$$

for all $u_1, u_2 \in L$.

Definition 2.2. *Let* (L, σ) *be an* $F-ms$ *and* $T : L \to L$ *be an* $\alpha-$*admissible mapping satisfying*

$$\alpha(u_1, u_2) \geq 1 \Longrightarrow \sigma(Tu_1, Tu_2) \leq \psi(K(u_1, u_2)) + a \min\{\sigma(u_1, Tu_2), \sigma(u_2, Tu_1)\} \tag{2.1}$$

for all $u_1, u_2 \in L$, *where* $\psi \in \Phi$, $a \geq 0$ *and*

$$K(u_1, u_2) = \max\{\sigma(u_1, u_2), \sigma(u_1, Tu_1), \sigma(u_2, Tu_2)\}.$$

Then, T *is named to be an almost* $\alpha-\psi-$*contractive mapping.*

Theorem 2.3. *Let* (L, σ) *be an* $F-$*complete* $F-ms$ *and* $T : L \to L$ *be an almost* $\alpha-\psi-$*contractive mapping. Assume that*

i. T *is continuous or*
ii. *for a sequence* $\{u_s\}$ *in* L *if* $u_s \to u$ *and* $\alpha(u_s, u_{s+1}) \geq 1$ *for all* s, *then* $\alpha(u, Tu) \geq 1$

holds and there exists $u_0 \in L$ *such that* $\alpha(u_0, Tu_0) \geq 1$. *Then* T *has a fixed point (shortly FP).*

Proof. Let $u_0 \in L$ such that $\alpha(u_0, Tu_0) \geq 1$. Define a sequence $\{u_s\} \subset L$ by $u_s = Tu_{s-1}$ for all $s \in \mathbb{N}$. If $u_s = u_{s+1}$ for any $s \in \mathbb{N}$, then u_s is an FP of T. We suppose that $T(u) \neq T(u+1)$ for all $u \in L$. Since T is an $\alpha-$admissible mapping and $\alpha(u_0, Tu_0) \geq 1$, we deduce that $\alpha(u_1, u_2) = \alpha(Tu_0, T^2u_0)$. Continuing this process, we get $\alpha(u_s, u_{s+1}) \geq 1$ for all $s \geq 0$. Now by (2.1) we get

$$\begin{aligned} \sigma(u_{s+1}, u_{s+2}) &= \sigma(Tu_s, Tu_{s+1}) \leq \psi(K(u_s, u_{s+1}) + a\min\{\sigma(u_{s+1}, Tu_s), \sigma(u_s, Tu_{s+1})\} \\ &= \psi(K(u_s, u_{s+1})) + a\min\{\sigma(u_{s+1}, u_{s+1}), \sigma(u_s, u_{s+2})\} \\ &= \psi(K(u_s, u_{s+1})) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} K(u_s, u_{s+1}) &= \max\{\sigma(u_s, u_{s+1}), \sigma(u_s, Tu_s), \sigma(u_{s+1}, Tu_{s+1})\} \\ &= \max\{\sigma(u_s, u_{s+1}), \sigma(u_s, u_{s+1}), \sigma(u_{s+1}, u_{s+2})\} \\ &= \max\{\sigma(u_s, u_{s+1}), \sigma(u_{s+1}, u_{s+2})\}. \end{aligned} \tag{2.3}$$

If $\sigma(u_{s+1}, u_{s+2}) \geq \sigma(u_s, u_{s+1})$, then from (2.2) and (2.3), we have

$$\sigma(u_{s+1}, u_{s+2}) \leq \psi(\sigma(u_{s+1}, u_{s+2})) < \sigma(u_{s+1}, u_{s+2})$$

which is a contradiction. Thus, $\sigma(u_{s+1}, u_{s+2}) < \sigma(u_s, u_{s+1})$ for all s and so from (2.2) and (2.3), we have

$$\sigma(u_{s+1}, u_{s+2}) \leq \psi(\sigma(u_{s+1}, u_{s+2})).$$

By induction, we have

$$\sigma(u_{s+1}, u_{s+2}) \leq \psi^{s+1}(\sigma(u_0, u_1))$$

for all $s \geq 0$. Let $k > 0$ be fixed and $(F, t) \in \Omega \times [0, +\infty)$ be such that

$$0 < j < l \text{ implies } F(j) < F(k) - t. \tag{2.4}$$

For $0 < F\left(\sum_{s \geq s(k)} \psi^s(\sigma(u_0, u_1))\right) < l$ and for each $s, v \in \mathbb{N}, s > v,$ using (2.4) and (F1) we have

$$F\left(\sum_{i=v}^{s-1} \psi^i(\sigma(u_0, u_1))\right) \leq F\left(\sum_{v \geq v(k)}^{s-1} \psi^v(\sigma(u_0, u_1))\right) < F(k) - t.$$

From (d_3), we have

$$\begin{aligned} F(\sigma(u_s, u_v)) &\leq F\left(\sum_{i=v}^{s-1} \sigma(u_i, u_{i+1})\right) + t \\ &\leq F\left(\sum_{i=v}^{s-1} \psi^i(\sigma(u_0, u_1))\right) + t \\ &< F(k). \end{aligned}$$

Therefore, $\{u_s\} \subset L$ is an $F-$Cauchy sequence. Since L is $F-$complete, there exists $u \in L$ such that $u_s \to u$. If T is continuous, then we have

$$\lim_{s \to \infty} \sigma(Tu_s, Tu) = \lim_{s \to \infty} \sigma(u_{s+1}, u) = 0.$$

Hence, u is an FP of T.

Now, suppose (ii) is satisfied. Since $\alpha(u_s, u_{s+1}) \geq 1$ for all s and $u_s \to u$ as $s \to \infty$, we have $\alpha(u, Tu) \geq 1$.

From (2.1), we get

$$\begin{aligned} \sigma(u_{s+1}, Tu) &= \sigma(Tu_s, Tu) \\ &\leq \psi(K(u_s, u) + a\min\{\sigma(u, Tu_s), \sigma(u_s, Tu)\}) \end{aligned}$$

where $a \geq 0$ and

$$\begin{aligned} K(u_s, u) &= \max\{\sigma(u_s, u), \sigma(u_s, Tu_s), \sigma(u, Tu)\} \\ &= \max\{\sigma(u_s, u), \sigma(u_s, u_{s+1}), \sigma(u, Tu)\}. \end{aligned}$$

If $K(u_s,u)=\sigma(u_s,u)$, then

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t. \end{aligned}$$

Taking limit as $s\to\infty$, we have $\lim_{s\to\infty}F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty$. Hence, we get $\sigma(u,Tu)=0$.

If $K(u_s,u)=\sigma(u_s,u_{s+1})$, we have

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u_{s+1}))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t. \end{aligned}$$

Taking limit as $s\to\infty$ we have

$$\lim_{s\to\infty}F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty.$$

So, we get $\sigma(u,Tu)=0$.

If $K(u_s,u)=\sigma(u,Tu)$, then by (d_3),

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u,Tu))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u,u_s)+\sigma(u_s,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t. \end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t=-\infty.$$

Thus, we have $F(\sigma(u,Tu))=0$. Therefore, $\sigma(u,Tu)=0$ and so T has an FP. □

Example 2.4. *Let $L=\mathbb{N}$ and $\sigma:L\times L\to[0,\infty)$ be defined by*

$$\sigma(u,\mu)=\begin{cases} (u-\mu)^2, & \text{if } u,\mu\in\{1,2,3\} \\ 2|u-\mu|, & \text{other} \end{cases}.$$

Let $T:L\to L$ be defined by

$$T(u)=\begin{cases} u/2, & \text{if } u \text{ is even} \\ (u+1)/2, & \text{if } u \text{ is odd} \end{cases}$$

and $\alpha:L\times L\to[0,\infty)$ be defined by $\alpha(u,\mu)=1$. Then, T is $\alpha-$admissible and continuous. Assume $\psi:[0,\infty)\to[0,\infty)$ be defined by $\psi(u)=\frac{u}{2}$ and $a>0$.

If $u=1$ and $\mu=2$, or u and μ are consecutive natural numbers with $\mu<u$, then $\sigma(Tu,T\mu)=0$. So, the proof is clear.

If u is even and μ is odd and $u,\mu\geq 4$, then

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma\left(\frac{u}{2},\frac{\mu+1}{2}\right)=2\left|\frac{u}{2}-\frac{\mu+1}{2}\right|=|u-\mu-1| \\ &\leq \frac{2|u-\mu|}{2}+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\} \\ &\leq \psi(K(u,\mu))+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}. \end{aligned}$$

If u and μ are even, then

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma\left(\frac{u}{2},\frac{\mu}{2}\right)=2\left|\frac{u}{2}-\frac{\mu}{2}\right|=|u-\mu| \\ &\leq \frac{2|u-\mu|}{2}+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\} \\ &\leq \psi(K(u,\mu))+a\min\{\sigma(u,T\mu),\sigma(Tu,\mu)\}. \end{aligned}$$

If u and μ are odd, then

$$\begin{aligned} \sigma(Tu,T\mu) &= \sigma\left(\frac{u+1}{2},\frac{\mu+1}{2}\right)=2\left|\frac{u}{2}-\frac{\mu}{2}\right|=|u-\mu| \\ &\leq \frac{2|u-\mu|}{2}+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\} \\ &\leq \psi(K(u,\mu))+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}. \end{aligned}$$

Hence, all the conditions of Theorem 2.3 are satisfied. 1 is a unique FP of T.

In this example, if $u=1$ and $\mu=3$, then T is not an $\alpha-\psi$ contractive mapping. Therefore, T is an almost $\alpha-\psi$ weak contractive mapping for $a=2$.

If we take $\psi(u)=u$ in Theorem 2.3, we have the following corollary.

Corollary 2.5. *Let (L,σ) be an $F-$complete $F-ms$ and $T:L\to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2)\geq 1 \Longrightarrow \sigma(Tu_1,Tu_2)\leq K(u_1,u_2)+a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\}$$

for all $u_1,u_2\in L$, where $a\geq 0$ and

$$K(u_1,u_2)=\max\{\sigma(u_1,Tu_1),\sigma(u_2,Tu_2),\sigma(u_1,u_2)\}.$$

Assume that

i. T is continuous or
ii. for a sequence $\{u_s\}\subset L$ if $u_s\to u$ and $\alpha(u_s,u_{s+1})\geq 1$ for all $s\in\mathbb{N}$, then $\alpha(u,Tu)\geq 1$

holds and there exists $u_0\in L$ such that $\alpha(u_0,Tu_0)\geq 1$. Then T has an FP.

Definition 2.6. *Let (L,σ) be an $F-ms$ and $T:L\to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2)\geq 1 \Longrightarrow \sigma(Tu_1,Tu_2)\leq \psi(K(u_1,u_2))+a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\} \tag{2.5}$$

for all $u_1,u_2\in L$, where $\psi\in\Phi$, $a\geq 0$ and

$$K(u_1,u_2)=\max\left\{\frac{\sigma(u_1,Tu_1)\sigma(u_2,Tu_2)}{\sigma(u_1,u_2)+1},\sigma(u_1,u_2)\right\}.$$

Then T is said to be an almost $\alpha-\psi-$rational type contractive mapping.

Theorem 2.7. *Let (L,σ) be an $F-$complete $F-ms$ and $T:L\to L$ be an almost $\alpha-\psi-$rational type contractive mapping. Assume that*

i. T is continuous or
ii. for a sequence $\{u_s\}\subset L$ if $u_s\to u$ and $\alpha(u_s,u_{s+1})\geq 1$ for all $s\in\mathbb{N}$, then $\alpha(u,Tu)\geq 1$

holds and there exists $u_0\in L$ such that $\alpha(u_0,Tu_0)\geq 1$. Then T has an FP.

Proof. Let $u_0\in L$ such that $\alpha(u_0,Tu_0)\geq 1$. Define a sequence $\{u_s\}\subset L$ by $u_s=Tu_{s-1}$ for all $s\in\mathbb{N}$. If $u_s=u_{s+1}$ for some $s\geq 0$, then u_s is an FP of T. We suppose that $u_s\neq u_{s+1}$ for all $s\in\mathbb{N}$. Since T is an $\alpha-$admissible mapping and $\alpha(u_0,Tu_0)\geq 1$, we deduce that $\alpha(u_1,u_2)=\alpha(Tu_0,T^2u_0)$. Continuing this process, we get $\alpha(u_s,u_{s+1})\geq 1$ for all $s\in\mathbb{N}$. Now, by (2.5) we get

$$\begin{aligned}\sigma(u_{s+1},u_{s+2}) &= \sigma(Tu_s,Tu_{s+1})\leq\psi(K(u_s,u_{s+1})+a\min\{\sigma(u_{s+1},Tu_s),\sigma(u_s,Tu_{s+1})\}\\ &= \psi(K(u_s,u_{s+1})+a\min\{\sigma(u_{s+1},u_{s+1}),\sigma(u_s,u_{s+2})\}\\ &= \psi(K(u_s,u_{s+1}))\end{aligned}$$

where

$$\begin{aligned}K(u_s,u_{s+1}) &= \max\left\{\frac{\sigma(u_s,Tu_s)\sigma(u_{s+1},Tu_{s+1})}{D(u_s,u_{s+1})+1},D(u_s,u_{s+1})\right\}\\ &= \max\left\{\frac{\sigma(u_s,u_{s+1})\sigma(u_{s+1},u_{s+2})}{\sigma(u_s,u_{s+1})+1},\sigma(u_s,u_{s+1})\right\}\\ &= \max\{\sigma(u_{s+1},u_{s+2}),\sigma(u_s,u_{s+1})\}.\end{aligned}$$

If $\sigma(u_{s+1},u_{s+2})\geq\sigma(u_s,u_{s+1})$, then we have

$$\sigma(u_{s+1},u_{s+2})\leq\psi(\sigma(u_{s+1},u_{s+2}))<\sigma(u_{s+1},u_{s+2})$$

which is a contradiction. Thus $\sigma(u_{s+1},u_{s+2})<\sigma(u_s,u_{s+1})$ for all s. So, we have

$$\sigma(u_{s+1},u_{s+2})\leq\psi(\sigma(u_s,u_{s+1})).$$

By induction, we get

$$\sigma(u_{s+1},u_{s+2})\leq\psi^{s+1}(\sigma(u_0,u_1)).$$

Let $k>0$ be fixed and $(F,t)\in\Omega\times[0,+\infty)$ be such that

$$0<j<l \text{ implies } F(j)<F(k)-t.$$

For $0<F\left(\sum_{s\geq s(k)}\psi^{s}(\sigma(u_0,u_1))\right)<l$ and for each $s,v\in\mathbb{N}, s>v$, using (2.5) and (F1) we have

$$F\left(\sum_{i=v}^{s-1}\psi^{i}(\sigma(u_0,u_1))\right)\leq F\left(\sum_{v\geq v(k)}^{s-1}\psi^{s}(\sigma(u_0,u_1))\right)<F(k)-t.$$

From (d_3) of $F-$metric,

$$\begin{aligned} F(\sigma(u_s,u_v) &\leq F\left(\sum_{i=v}^{s-1}\sigma(u_i,u_{i+1})\right)+t \\ &\leq F\left(\sum_{i=v}^{s-1}\psi^{i}((\sigma(u_0,u_1))\right)+t \\ &< F(k). \end{aligned}$$

Therefore, $\{u_s\}$ is an $F-$Cauchy sequence in L. Since L is $F-$complete, there exists $u\in L$ such that $u_s\to u$. If T is continuous, we have

$$\lim_{s\to\infty}\sigma(Tu_s,Tu)=\lim_{s\to\infty}\sigma(u_{s+1},u)=0.$$

So, u is an FP of T.

Now, suppose (ii) is satisfied. Since $\alpha(u_s,u_{s+1})\geq 1$ for all $s\geq 0$ and $u_s\to u$ as $u\to\infty$, we have $\alpha(u,Tu)\geq 1$.

From (2.5) we have

$$\begin{aligned} \sigma(u_{s+1},Tu) &= \sigma(Tu_s,Tu) \\ &\leq \psi(K(u_s,u))+a\min\{\sigma(u,Tu_s),\sigma(u_s,Tu)\} \end{aligned}$$

where $a\geq 0$ and

$$\begin{aligned} K(u_s,u) &= \max\left\{\frac{\sigma(u_s,Tu_s)\sigma(u,Tu)}{\sigma(u_s,u)+1},\sigma(u_s,u)\right\} \\ &= \max\left\{\frac{\sigma(u_s,u_{s+1})\sigma(u,Tu)}{\sigma(u_s,u)+1},\sigma(u_s,u)\right\}. \end{aligned}$$

If $K(u_s,u)=\sigma(u_s,u)$, then

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t. \end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u_s,u)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty.$$

Thus, we get $F(\sigma(u,Tu))=0$. If $K(u_s,u)=\sigma(u_s,u_{s+1})$, we have

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u_s,u_{s+1}))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t. \end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u_s,u_{s+1})+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t=-\infty.$$

Thus, using (F2) we get $F(\sigma(u,Tu))=0$. If $K(u_s,u)=\sigma(u,Tu)$, then by (d_3),

$$\begin{aligned} F(\sigma(u_{s+1},Tu)) &\leq F(\psi(\sigma(u,Tu))+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+t \\ &\leq F(\sigma(u,u_s)+\sigma(u_s,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t. \end{aligned}$$

Taking limit as $s\to\infty$, we have

$$\lim_{s\to\infty}F(\sigma(u,Tu)+a\min\{\sigma(u,u_{s+1}),\sigma(u_s,Tu)\})+2t=-\infty.$$

Hence, we get $F(\sigma(u,Tu))=0$. Therefore, $\sigma(u,Tu)=0$ and so T has an FP. □

Example 2.8. *Let $\Gamma=[0,\infty)$ and $\sigma:\Gamma\times\Gamma\to[0,\infty)$ be defined by*

$$\sigma(u,\mu)=\begin{cases}(u-\mu)^2, & \text{if } (u,\mu)\in[0,1]\times[0,1]\\ |u-\mu|, & \text{if } (u,\mu)\notin[0,1]\times[0,1]\end{cases}.$$

Let $T:\Gamma\to\Gamma$ be defined by

$$T(u)=\begin{cases}2u, & \text{if } u\in[0,1]\\ \frac{4u+2}{3}, & \text{if } u\in(1,\infty)\end{cases}$$

and
$\alpha:\Gamma\times\Gamma\to[0,\infty)$ be defined by

$$\alpha(u,\mu)=\begin{cases}1, & \text{if } (u,\mu)\in[0,1]\times[0,1]\\ 0, & \text{if } (u,\mu)\notin[0,1]\times[0,1]\end{cases}.$$

Then, T is $\alpha-$admissible for $(u,\mu)\in[0,1]\times[0,1]$ and continuous. Assume $\psi:[0,\infty)\to[0,\infty)$ be defined by $\psi(u)=2u$. If $u,\mu\le\frac{1}{2}$ and $a=100$, we have

$$\begin{aligned}\sigma(Tu,T\mu) &= \sigma(2u,2\mu)=|2u-2\mu|\\ &\le 2|u-\mu|^2+a\min\{|u-2\mu|^2,|\mu-2u|^2\}\\ &\le \psi(\max\left\{\frac{\sigma(u,Tu),\sigma(\mu,T\mu)}{1+\sigma(u,\mu)},\sigma(u,\mu)\right\})+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}.\end{aligned}$$

If $1\ge u,\mu>\frac{1}{2}$ and $a=100$, we have

$$\begin{aligned}\sigma(Tu,T\mu) &= \sigma(2u,2\mu)=|2u-2\mu|\\ &\le 2|u-\mu|^2+a\min\{|u-2\mu|^2,|\mu-2u|^2\}\\ &\le \psi(\max\left\{\frac{\sigma(u,Tu),\sigma(\mu,T\mu)}{1+\sigma(u,\mu)},\sigma(u,\mu)\right\})+a\min\{\sigma(u,T\mu),\sigma(\mu,Tu)\}.\end{aligned}$$

Hence, 0 is an FP of T.

Corollary 2.9. *Let (L,σ) be an $F-$complete $F-ms$ and $T:L\to L$ be an $\alpha-$admissible mapping satisfying*

$$\alpha(u_1,u_2)\ge1\Longrightarrow\sigma(Tu_1,Tu_2)\le\psi(\sigma(u_1,u_2))+a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\}$$

for all $u_1,u_2\in L$, where $\psi\in\Phi$, $a\ge0$. Assume that

- *i. T is continuous or*
- *ii. for a sequence $\{u_s\}\subset L$ if $u_s\to u$ and $\alpha(u_s,u_{s+1})\ge1$ for all $s\in\mathbb{N}$, then $\alpha(u,Tu)\ge1$.*

If there exists $u_0\in L$ such that $\alpha(u_0,Tu_0)\ge1$, then T has an FP.

If we take $\psi(u)=u$ in Theorem 2.7, we get the following corollary.

Corollary 2.10. *Let (L,σ) be an $F-$complete $F-ms$ and $T:L\to L$ be an α-admissible mapping satisfying*

$$\alpha(u_1,u_2)\ge1\Longrightarrow\sigma(Tu_1,Tu_2)\le K(u_1,u_2)+a\min\{\sigma(u_1,Tu_2),\sigma(u_2,Tu_1)\}$$

for all $u_1,u_2\in L$ and $a\ge0$ and

$$K(u_1,u_2)=\max\left\{\frac{\sigma(u_1,Tu_1)\sigma(u_2,Tu_2)}{\sigma(u_1,u_2)+1},\sigma(u_1,u_2)\right\}.$$

Assume that

- *i. T is continuous or*
- *ii. for a sequence $\{u_s\}\subset L$ if $u_s\to u$ and $\alpha(u_s,u_{s+1})\ge1$ for all $s\in\mathbb{N}$, then $\alpha(u,Tu)\ge1$*

holds and there exists $u_0\in L$ such that $\alpha(u_0,Tu_0)\ge1$. Then T has an FP.

Declarations

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions

Author's Contributions: Conceptualization, C.A.; methodology C.A. and V.Ö.; validation, V.Ö. investigation, C.A.; resources, C.A.; data curation, C.A.; writing—original draft preparation, C.A.; writing—review and editing, V.Ö.; supervision, V.Ö. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

Use of AI tools: The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: C. Acar and V. Öztürk, *Fixed point theorems for almost α-ψ-contractive mappings in F-metric spaces*, Fundam. J. Math. Appl., **7**(4) (2024), 203-211. DOI 10.33401/fujma.1400093