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Ostrowski Type Inequalities Including Riemann-Liouville Fractional Integrals for Two Variable Functions

Samet Erden[1*], Burçin Gökkurt Özdemir[2], Sevgi Kılıçer[1] and Canmert Demir[1]

[1]*Department of Mathematics, Faculty of Science, Bartın University, Bartın, Türkiye*
[2]*Department of Mathematics and Science Education, Faculty of Education, Bartın University, Bartın, Türkiye*
[*]*Corresponding author*

Abstract

The main purpose of this study is to establish new inequalities including Riemann-Liouville fractional integrals for various classes of functions with two variables. We first establish two identities involving Riemann-Liouville fractional integrals for higher-order partial differential functions. Then, some fractional Ostrowski type inequalities for functions of bounded variation of two variables are attained. Moreover, we present fractional integral inequalities for functions whose higher-order partial derivatives are elements of L_∞ and L_1, respectively. Some special cases and midpoint versions of our main results are also examined.

Keywords: *Bounded functions, fuctions of bounded variation, fractional integrals, Ostrowski type inequalities*
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1. Introduction

Many mathematicians with an interest in both pure and applied mathematics have devoted much of their effort over the past century to the study of various forms of integral inequalities. One of the many essential mathematical findings of A. M. Ostrowski [29] is the following classical integral inequality associated with the functions whose derivatives are bounded:

Theorem 1.1. *Supposing that $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then, one has the integral inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty ,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Applications of Ostrowski inequality can be found in quadrature, probability, and optimization theory, as well as in information, statistics, and integral operator theory. Until now, numerous scholarly articles and books regarding Ostrowski inequalities and their various applications have been published. For new results, the researchers examined refinements, counterparts and generalizations of classical Ostrowski inequality, and new Ostrowski-type inequalities under various assumptions for the functions. For example, Dragomir and Wang gave Ostrowski type results for functions whose first derivatives are elements of different Lebesgue spaces in [11] and [12]. Moreover, Barnett ve Dragomir [3] established an Ostrowski type inequality for double integrals. After that, Ostrowski type inequalities for functions whose partial derivatives are elements of Lebesgue p-norm were obtained by Dragomir et al. in [13].
There are problems where any-order derivatives of functions are required. Researchers working with such problems focused on Ostrowski type inequalities for higher-order differentiable functions. For illustrate, some mathematicians derived different results of Ostrowski type based on mappings whose higher-order derivatives are bounded ([23] and [2]). What's more, Ostrowski type inequalities for higher-order differentiable functions and their applications are presented by Cerone et al. in [7]. In addition, some researchers developed generalized integral inequalities for functions whose derivatives of any order belong to the p-norm or the infinite norm in [37] and [39]. In [18], Erden et al. improved weighted versions of inequalities involving higher-order derivatives, and they gave applications for moments of random

Email addresses: **erdensmt@gmail.com (S. Erden), gokkurtburcin@gmail.com (B. Gökkurt Özdemir), esevgi.math98@gmail.com (S. Kılıçer), canmert-demir2@gmail.com (C. Demir)**

variables using these inequalities. Also, some mathematicians examined the relationship between the real value and the approximate value of an integral with effective quadrature formulas that arise when investigating inequalities given for higher order derivatives ([26], [32] and [21]). In addition to all these studies, Ostrowski-type results for functions of two variables whose higher-order partial derivatives are bounded also worked ([8], [25] and [38]).
Another important issue within the scope of the article is Riemann-Liouville fractional integrals. It is important to be reminded definitions of Riemann-Liouville fractional integrals for one and two variables functions.

Definition 1.2. *Assume that $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

For more detailed information about fractional integrals, you can refer to the findamental books written in this field ([24], [30]). It should be noted that Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals are firstly given by Sarikaya et al. in [33]. In addition, Dragomir attained Ostrowski-type inequalities for functions of bounded variation, Hölder continuous functions, Lipschitzian functions, and absolutely continuous functions belonging to various Lebesgue norm spaces by means of identities equal to the sum of the right- and left-sided Riemann-Liouville integrals in [14]-[16]. Dragomir also provided some Ostrowski and trapezoid type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions by using bounded derivatives in [17]. Whereupon, Montgomery identity involving Riemann-Liouville fractional integrals and related Ostrowski type inequalities presented by Aglić Aljinović in [1]. In [31], Qayyum et al. provided extended Ostrowski type inequalities including Rieamnn-Liouville fractional integrals for higher-order derivatives. For more detailed studies on integral inequalities related to fractional calculus, the articles such as [22], [27], and [35] in the references can also be consulted
Now, we give the definitions Riemann-Liouville fractional integrals of two variable functions:

Definition 1.3. *[34] Let $f \in L_1([a,b] \times [c,d])$. The Riemann-Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta}$, $J_{a+,d-}^{\alpha,\beta}$, $J_{b-,c+}^{\alpha,\beta}$ and $J_{b-,d-}^{\alpha,\beta}$ are defined by*

$$J_{a+,c+}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s)dsdt, \quad x > a,\ y > c,$$

$$J_{a+,d-}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s)dsdt, \quad x > a,\ y < d,$$

$$J_{b-,c+}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s)dsdt, \quad x < b,\ y > c,$$

and

$$J_{b-,d-}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s)dsdt, \quad x < b,\ y < d.$$

Hermite-Hadamard inequality and Ostrowski inequality for fractional integrals of two variable functions are obtained in [34] and [28], respectively. Recently, Erden et al. [19] provided some Ostrowski type inequalities including Riemann-Liouville fractional integrals for functions in class of functions L_p, L_∞ and L_1, respectively. There are several papers on fractional Hermite-Hadamard and fractional Ostrowski type inequalities for two variable functions, you can find some of them in the references. For example, Sarıkaya developed Ostrowski type results involving Riemann-Liouville fractional integrals by using co-ordinated convex functions in [36].
We also recall functions of bounded variation with two variables to present a more understandable article.Mappings of bounded variation with two variables are defined as follows:

Definition 1.4. *[9] Assume that $f(x,y)$ is defined over the rectangle $Q = [a,b] \times [c,d]$. Let P be a partition of Q with*

$$P : a = x_0 < x_1 < ... < x_n = b, \text{ and } c = y_0 < y_1 < ... < y_m = d;$$

and for all i,j let

$$\Delta_{11}g(x_i,y_j) = g(x_{i-1},y_{j-1}) - g(x_{i-1},y_j) - g(x_i,y_{j-1}) + g(x_i,y_j).$$

The function $f(x,y)$ is said to be of bounded variation if the sum

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left|\Delta_{11}f(x_i,y_j)\right|$$

is bounded for all nets.

Therefore, one can define the concept of total variation of a function of variables, as follows:
Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left|\Delta_{11}f(x_i,y_j)\right|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f):=\bigvee_{c}^{d}\bigvee_{a}^{b}(f):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of f on Q.
There are many of papers on inequalities for functions of bounded variation of one variable, the cornerstone article that can be cited as an example about this topic is the work done by Dragomir [10]. There are also some paper on inequalities for functions of bounded variation with two variables ([4]-[6]). However there is a few papers fractional integral inequalities for functions of bounded variation with two variables. As an example, Erden et al. presented recent fractional Ostrowski and Trapezoid type integral inequalities by using functions of bounded variation with two variables in [20].
In this paper, Ostrowski type fractional integral inequalities for two variables functions that have higher order partial derivatives will be examined. The fractional integral identities will be first established by means of higher-order derivatives. By integration by parts and elementary analysis operations will be used to establish identities involving Riemann-Liouville fractional integrals. After that, the inequalities involving fractional integrals for three different classes of functions consisting of bounded, bounded variation and L_1 space will be given by using equalities obtained in section 2.

2. Some Identities for Double Integrals

This section will present the identities needed to obtain the main results. First of all, some notations will be defined to make the expressions more understandable as follows.

$$F(x,t,y,s):=\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}-\frac{\partial^{n+m}f(t,y)}{\partial t^n\partial y^m}-\frac{\partial^{n+m}f(x,s)}{\partial x^n\partial s^m}+\frac{\partial^{n+m}f(x,y)}{\partial x^n\partial y^m},\tag{2.1}$$

$$\mathscr{J}(f)=(-1)^{n+m}J^{\alpha,\beta}_{b-,d-}f(a,c)+(-1)^nJ^{\alpha,\beta}_{b-c+}f(a,d)+(-1)^mJ^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+}f(b,d),\tag{2.2}$$

$$M_{n,m}(f)=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\frac{(b-a)^{k+\alpha}(d-c)^{j+\beta}}{\Gamma(k+\alpha+1)\Gamma(j+\beta+1)}\left[\frac{\partial^{k+j}f(a,c)}{\partial t^k\partial s^j}+(-1)^{n+k}\frac{\partial^{k+j}f(b,c)}{\partial t^k\partial s^j}+(-1)^{m+j}\frac{\partial^{k+j}f(a,d)}{\partial t^k\partial s^j}+(-1)^{m+n+k+j}\frac{\partial^{k+j}f(b,d)}{\partial t^k\partial s^j}\right],\tag{2.3}$$

$$\begin{aligned}N_{n,m}(f)&=\sum_{j=0}^{m-1}\frac{(d-c)^{j+\beta}}{\Gamma(j+\beta+1)}\left[(-1)^{n+m+j}J^{\alpha}_{b-}\frac{\partial^j f(a,d)}{\partial s^j}+(-1)^nJ^{\alpha}_{b-}\frac{\partial^j f(a,c)}{\partial s^j}+(-1)^{m+j}J^{\alpha}_{a+}\frac{\partial^j f(b,d)}{\partial s^j}+J^{\alpha}_{a+}\frac{\partial^j f(b,c)}{\partial s^j}\right]\\&+\sum_{k=0}^{n-1}\frac{(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}\left[(-1)^{n+m+k}J^{\beta}_{d-}\frac{\partial^k f(b,c)}{\partial t^k}+(-1)^{n+k}J^{\beta}_{c+}\frac{\partial^k f(b,d)}{\partial t^k}+(-1)^mJ^{\beta}_{d-}\frac{\partial^k f(a,c)}{\partial t^k}+J^{\beta}_{c+}\frac{\partial^k f(a,d)}{\partial t^k}\right]\end{aligned}\tag{2.4}$$

$$\begin{aligned}P_{n,m}(f)&=\frac{2(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\times\sum_{j=0}^{m-1}\frac{(d-c)^{j+\beta}}{\Gamma(j+\beta+1)}\left[(-1)^{m+j}\frac{\partial^{n+j}f(x,d)}{\partial x^n\partial s^j}+\frac{\partial^{n+j}f(x,c)}{\partial x^n\partial s^j}\right]\\&+\frac{2(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\sum_{k=0}^{n-1}\frac{(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}\left[(-1)^{n+k}\frac{\partial^{k+m}f(b,y)}{\partial t^k\partial y^m}+\frac{\partial^{k+m}f(a,y)}{\partial t^k\partial y^m}\right]\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}L(f)&=\frac{4(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\frac{(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\frac{\partial^{n+m}f(x,y)}{\partial x^n\partial y^m}-\frac{2(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\left((-1)^nJ^{\alpha}_{b-}\frac{\partial^m f(a,y)}{\partial y^m}+J^{\alpha}_{a+}\frac{\partial^m f(b,y)}{\partial y^m}\right)\\&-\frac{2(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\left((-1)^mJ^{\beta}_{d-}\frac{\partial^n f(x,c)}{\partial x^n}+J^{\beta}_{c+}\frac{\partial^n f(x,d)}{\partial x^n}\right).\end{aligned}\tag{2.6}$$

It is obtained two double integral identities involving Riemann-Liouville fractional integrals as follows. These equalities are the main material of inequalities developed throughout the article.

Lemma 2.1. . *Let $f:[a,b]\times[c,d]=:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be an absolutely continuous function such that the partial derivatives $\frac{\partial^{k+l}f(t,s)}{\partial t^k\partial s^l}$ exists and are continuous on Δ for $k=0,1,2,...,n$, $l=0,1,2,...,m$ with $n,m\in\mathbb{N}^+$. Then, for any $(x,y)\in\Delta$, we have*

$$\begin{aligned}&\frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}+(b-t)^{n+\alpha-1}\right]\times\left[(s-c)^{m+\beta-1}+(d-s)^{m+\beta-1}\right]F(x,t,y,s)dsdt\right\}\\&=\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}(f)+L(f)\end{aligned}\tag{2.7}$$

where $F(x,t,y,s)$, $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}(f)$ and $L(f)$ are defined as in (2.1)-(2.6), respectively.

Proof. There are four different integrals that need to be calculated. For the first integrals, by the definition of $F(x,t,y,s)$ and fundamental analysis operations, it is easy to see that

$$\begin{aligned}\int_a^b\int_c^d \frac{(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}}{\Gamma(n+\alpha)\Gamma(m+\beta)}F(x,t,y,s)\mathrm{d}s\mathrm{d}t &= \int_a^b\int_c^d \frac{(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}}{\Gamma(n+\alpha)\Gamma(m+\beta)}\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\mathrm{d}s\mathrm{d}t \\ &\quad -\int_a^b \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}\frac{\partial^{n+m}f(t,y)}{\partial t^n\partial y^m}dt\left(\int_c^d \frac{(s-c)^{m+\beta-1}}{\Gamma(m+\beta)}ds\right) \\ &\quad -\left(\int_a^b \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}dt\right)\int_c^d \frac{(s-c)^{m+\beta-1}}{\Gamma(m+\beta)}\frac{\partial^{n+m}f(x,s)}{\partial x^n\partial s^m}ds \\ &\quad +\frac{\partial^{n+m}f(x,y)}{\partial x^n\partial y^m}\left(\int_a^b \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}dt\right)\left(\int_c^d \frac{(s-c)^{m+\beta-1}}{\Gamma(m+\beta)}ds\right).\end{aligned} \tag{2.8}$$

It is observed that

$$\frac{1}{\Gamma(n+\alpha)}\int_a^b (t-a)^{n+\alpha-1}\frac{\partial^{n+m}f(t,y)}{\partial t^n\partial y^m}dt = (-1)^n J_{b-}^{\alpha}\frac{\partial^m f(a,y)}{\partial y^m} - \sum_{k=0}^{n-1}\frac{(-1)^{n+k}(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}\frac{\partial^{k+m}f(b,y)}{\partial t^k\partial y^m},$$

$$\frac{1}{\Gamma(m+\beta)}\int_c^d (s-c)^{m+\beta-1}\frac{\partial^{n+m}f(x,s)}{\partial x^n\partial s^m}ds = (-1)^m J_{d-}^{\beta}\frac{\partial^n f(x,c)}{\partial x^n} - \sum_{j=0}^{m-1}\frac{(-1)^{m+j}(d-c)^{j+\beta}}{\Gamma(j+\beta+1)}\frac{\partial^{n+j}f(x,d)}{\partial x^n\partial s^j}$$

and

$$\begin{aligned}\int_a^b\int_c^d \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}\frac{(s-c)^{m+\beta-1}}{\Gamma(m+\beta)}\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\mathrm{d}s\mathrm{d}t &= \int_a^b \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}\left(\int_c^d \frac{(s-c)^{m+\beta-1}}{\Gamma(m+\beta)}\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\mathrm{d}s\right)\mathrm{d}t \\ &= \frac{(-1)^m}{\Gamma(\beta)}\int_c^d (s-c)^{\beta-1}\left(\int_a^b \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}\frac{\partial^n f(t,s)}{\partial t^n}\mathrm{d}t\right)\mathrm{d}s \\ &\quad -\sum_{j=0}^{m-1}\frac{(-1)^{m+j}(d-c)^{j+\beta}}{\Gamma(j+\beta+1)}\int_a^b \frac{(t-a)^{n+\alpha-1}}{\Gamma(n+\alpha)}\frac{\partial^{n+j}f(t,d)}{\partial t^n\partial s^j}dt \\ &= \frac{(-1)^{n+m}}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (t-a)^{\alpha-1}(s-c)^{\beta-1}f(t,s)\mathrm{d}s\mathrm{d}t \\ &\quad -\sum_{k=0}^{n-1}\frac{(-1)^{n+k}(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}\frac{(-1)^m}{\Gamma(\beta)}\int_c^d (s-c)^{\beta-1}\frac{\partial^k f(b,s)}{\partial t^k}\mathrm{d}s \\ &\quad -\sum_{j=0}^{m-1}\frac{(-1)^{m+j}(d-c)^{j+\beta}}{\Gamma(j+\beta+1)}\frac{(-1)^n}{\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}\frac{\partial^j f(t,d)}{\partial s^j}\mathrm{d}t \\ &\quad +\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\frac{(-1)^{n+m+k+j}(b-a)^{k+\alpha}(d-c)^{j+\beta}}{\Gamma(k+\alpha+1)\Gamma(j+\beta+1)}\frac{\partial^{k+j}f(b,d)}{\partial t^k\partial s^j}.\end{aligned}$$

Substituting the results of the above integrals in (2.8), owing to the definitions of Riemann Liouville fractional integrals, it is found that

$$\begin{aligned}\int_a^b\int_c^d \frac{(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}}{\Gamma(n+\alpha)\Gamma(m+\beta)}F(x,t,y,s)\mathrm{d}s\mathrm{d}t &= (-1)^{n+m}J_{b-d-}^{\alpha,\beta}f(a,c) + \sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\frac{(-1)^{n+m+k+j}(b-a)^{k+\alpha}(d-c)^{\beta+j}}{\Gamma(k+\alpha+1)\Gamma(\beta+j+1)}\frac{\partial^{k+j}f(b,d)}{\partial t^k\partial s^j} \\ &\quad -\sum_{j=0}^{m-1}\frac{(-1)^{m+n+j}(d-c)^{\beta+j}}{\Gamma(\beta+j+1)}J_{b-}^{\alpha}\left(\frac{\partial^j f(a,d)}{\partial s^j}\right) \\ &\quad -\sum_{k=0}^{n-1}\frac{(-1)^{m+n+k}(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}J_{d-}^{\beta}\left(\frac{\partial^k f(b,c)}{\partial t^k}\right) \\ &\quad -\frac{(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\left\{(-1)^n J_{b-}^{\alpha}\left(\frac{\partial^m f(a,y)}{\partial y^m}\right) - \sum_{k=0}^{n-1}\frac{(-1)^{n+k}(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}\frac{\partial^{k+m}f(b,y)}{\partial t^k\partial y^m}\right\} \\ &\quad -\frac{(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\left\{(-1)^m J_{d-}^{\beta}\left(\frac{\partial^n f(x,c)}{\partial x^n}\right) - \sum_{j=0}^{m-1}\frac{(-1)^{m+j}(d-c)^{\beta+j}}{\Gamma(\beta+j+1)}\frac{\partial^{n+j}f(x,d)}{\partial x^n\partial s^j}\right\} \\ &\quad +\frac{(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\frac{(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\frac{\partial^{n+m}f(x,y)}{\partial x^n\partial y^m}.\end{aligned}$$

If we similarly observe the other integrals and later we add all these identities side by side, then the desired equality can be attained. □

Remark 2.2. *Under the same assumption of Lemma 2.1 with $n=m=0$, because the sum symbols disappear and $n=m=0$ must be written in the remaining expressions, one has the identity*

$$\frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}+(b-t)^{n+\alpha-1}\right]\times\left[(s-c)^{m+\beta-1}+(d-s)^{m+\beta-1}\right]F(x,t,y,s)\,dsdt\right\} \tag{2.9}$$
$$= \Big|J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)-2\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right]$$
$$-2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+4\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Big|$$

which was proved Erden et al. in [20].

Lemma 2.3. *Let $f:[a,b]\times[c,d]=:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be an absolutely continuous function such that the partial derivatives $\frac{\partial^{k+l}f(t,s)}{\partial t^k\partial s^l}$ exists and are continuous on Δ for $k=0,1,2,...,n+1$, $l=0,1,2,...,m+1$ with $n,m\in\mathbb{N}^+$. Then, for any $(x,y)\in\Delta$, we have*

$$\frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}+(b-t)^{n+\alpha-1}\right]\times\left[(s-c)^{m+\beta-1}+(d-s)^{m+\beta-1}\right]\left(\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}dvdu\right)dsdt\right\}$$
$$= \mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}(f)+L(f) \tag{2.10}$$

where $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}(f)$ and $L(f)$ are defined as in (2.2)-(2.6), respectively.

Proof. Owing to the conditions of the Theorem, it is easy to see that

$$\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}dvdu = \frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}-\frac{\partial^{n+m}f(t,y)}{\partial t^n\partial y^m}-\frac{\partial^{n+m}f(x,s)}{\partial x^n\partial s^m}+\frac{\partial^{n+m}f(x,y)}{\partial x^n\partial y^m}$$
$$= :F(x,t,y,s).$$

The proof of this Lemma follows the same strategy which was used in the proof of the previous Lemma by considering the above equality. □

Remark 2.4. *Under the same assumption of Lemma 2.3 with $n=m=0$, because the sum symbols disappear and $n=m=0$ must be written in the remaining expressions, one has the identity*

$$\frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}+(b-t)^{n+\alpha-1}\right]\times\left[(s-c)^{m+\beta-1}+(d-s)^{m+\beta-1}\right]\left(\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}dvdu\right)dsdt\right\}$$
$$= \Big|J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)-2\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right] \tag{2.11}$$
$$-2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+4\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Big|$$

which was proved Erden et al. in [19].

3. Double Integral Inequalities for Functions of Bounded Variations

In this section, we present new Ostrowski type inequalities involving Riemann-Liouville Fractional integrals for functions of two variables with bounded variation.

Theorem 3.1. *Suppose that all the assumptions of Lemma 2.1 hold. If $\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}$ is of bounded variation on Δ, for any $(x,y)\in\Delta$, then we have the inequalities*

$$\begin{aligned}\left|\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}(f)+L(f)\right| &\leq \frac{1}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\times\Bigg\{A_n(x)C_m(y)\bigvee_a^x\bigvee_c^y\left(f^{(n+m)}\right)+A_n(x)D_m(y)\bigvee_a^x\bigvee_y^d\left(f^{(n+m)}\right)\\ &\quad +B_n(x)C_m(y)\bigvee_x^b\bigvee_c^y\left(f^{(n+m)}\right)+B_n(x)D_m(y)\bigvee_x^b\bigvee_y^d\left(f^{(n+m)}\right)\Bigg\}\\ &\leq \frac{1}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\left[(b-a)^{n+\alpha}+\left|(x-a)^{n+\alpha}-(b-x)^{n+\alpha}\right|\right]\\ &\quad\times\left[(d-c)^{m+\beta}+\left|(y-c)^{m+\beta}-(d-y)^{m+\beta}\right|\right]\bigvee_a^b\bigvee_c^d\left(f^{(n+m)}\right)\end{aligned} \tag{3.1}$$

where $A_n(x)$, $B_n(x)$, $C_m(y)$ and $D_m(y)$ are defined by

$A_n(x)=(b-a)^{n+\alpha}-(b-x)^{n+\alpha}+(x-a)^{n+\alpha}$,

$B_n(x) = (b-a)^{n+\alpha} - (x-a)^{n+\alpha} + (b-x)^{n+\alpha}$,

$C_m(y) = (d-c)^{m+\beta} - (d-y)^{m+\beta} + (y-c)^{m+\beta}$

and

$D_m(y) = (d-c)^{m+\beta} - (y-c)^{m+\beta} + (d-y)^{m+\beta}$,

respectively. Here, $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}(f)$ *and* $L(f)$ *are also defined as in (2.2)-(2.6).*

Proof. If we take absolute value of both sides of the equality (2.7), owing to the well-known triangle inequality, we possess

$$\begin{aligned} \left|\mathscr{J}(f)+M_{k,j}(f)-N_j(f)+P_k(f)+L(f)\right| &\leq \frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d \left[(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}\right.\right. \\ &\quad (t-a)^{n+\alpha-1}(d-s)^{m+\beta-1}+(b-t)^{n+\alpha-1}(s-c)^{m+\beta-1} \\ &\quad \left.\left.+(b-t)^{n+\alpha-1}(d-s)^{m+\beta-1}\right]\left|F(x,t,y,s)\right| \mathrm{d}s\mathrm{d}t\right\} \end{aligned} \tag{3.2}$$

for any $(x,y)\in\Delta$.
Seeing that $\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}$ is of bounded variation on $[a,x]\times[c,y]$, it follows that

$$\begin{aligned} |F(x,t,y,s)| &= \left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}-\frac{\partial^{n+m}f(t,y)}{\partial t^n\partial y^m}-\frac{\partial^{n+m}f(x,s)}{\partial x^n\partial s^m}+\frac{\partial^{n+m}f(x,y)}{\partial x^n\partial y^m}\right| \\ &\leq \bigvee_a^x\bigvee_c^y\left(f^{(n+m)}\right), \end{aligned}$$

and similar inequalities can be formulated for other intervals. In this case, if the first integral in the right hand side of the statement (3.2) is calculated by taking into account the above inequality, then we can easily conclude that

$$\begin{aligned} \int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}\left|F(x,t,y,s)\right|dsdt\right. &\leq \bigvee_a^x\bigvee_c^y\left(f^{(n+m)}\right)\int_a^x\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt \\ &\quad +\bigvee_a^x\bigvee_y^d\left(f^{(n+m)}\right)\int_a^x\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt \\ &\quad +\bigvee_x^b\bigvee_c^y\left(f^{(n+m)}\right)\int_x^b\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt \\ &\quad +\bigvee_x^b\bigvee_y^d\left(f^{(n+m)}\right)\int_x^b\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt. \end{aligned}$$

And calculating the above four integrals, then one has the result

$$\begin{aligned} \int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}\left|F(x,t,y,s)\right|dsdt\right. &\leq \frac{(x-a)^{n+\alpha}}{n+\alpha}\frac{(y-c)^{m+\beta}}{m+\beta}\bigvee_a^x\bigvee_c^y\left(f^{(n+m)}\right) \\ &\quad +\frac{(x-a)^{n+\alpha}}{n+\alpha}\frac{(d-c)^{m+\beta}-(y-c)^{m+\beta}}{m+\beta}\bigvee_a^x\bigvee_y^d\left(f^{(n+m)}\right) \\ &\quad +\frac{(b-a)^{n+\alpha}-(x-a)^{n+\alpha}}{n+\alpha}\frac{(y-c)^{m+\beta}}{m+\beta}\bigvee_x^b\bigvee_c^y\left(f^{(n+m)}\right) \\ &\quad +\frac{(b-a)^{n+\alpha}-(x-a)^{n+\alpha}}{n+\alpha}\frac{(d-c)^{m+\beta}-(y-c)^{m+\beta}}{m+\beta}\bigvee_x^b\bigvee_y^d\left(f^{(n+m)}\right). \end{aligned}$$

Should the other integrals are also observed by taking account of the fact that $f:\Delta\to\mathbb{R}$ is of bounded variation on $[a,x]\times[y,d]$, $[x,b]\times[c,y]$ and $[x,b]\times[y,d]$, one can readily attain the first inequality in (3.1). The second inequality is obvious from the facts that

$$\begin{aligned} \max\{ac,ad,bc,bd\} &= \max\{a,b\}\max\{c,d\}, \end{aligned} \tag{3.3}$$

$$\max\{a^n,b^n\} = (\max\{a,b\})^n = \left(\frac{a+b+|a-b|}{2}\right)^n$$

for $a,b,c,d,n>0$. This completes the proof.

□

Remark 3.2. *If we choose $n=m=0$ in the inequality (3.1), then, for any $(x,y)\in\Delta$, we have*

$$\begin{aligned}
&\Big|J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)-2\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right]\\
&-2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+4\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Big|\\
\leq\;&\left\{A_0(x)C_0(y)\bigvee_a^x\bigvee_c^y(f)+A_0(x)D_0(y)\bigvee_a^x\bigvee_y^d(f)+B_0(x)C_0(y)\bigvee_x^b\bigvee_c^y(f)+B_0(x)D_0(y)\bigvee_x^b\bigvee_y^d(f)\right\}\\
\leq\;&\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[(b-a)^{\alpha}+\left|(x-a)^{\alpha}-(b-x)^{\alpha}\right|\right]\times\left[(d-c)^{\beta}+\left|(y-c)^{\beta}-(d-y)^{\beta}\right|\right]\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

which was given by Erden et al. in [20].

Corollary 3.3. *Under the assumptions of Theorem 3.1 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the Midpoint inequality*

$$\left|\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)+L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)\right|\leq\frac{(b-a)^{n+\alpha}(d-c)^{m+\beta}}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\bigvee_a^b\bigvee_c^d\left(f^{(n+m)}\right)$$

where $P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)$ and $L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)$ are defined by

$$\begin{aligned}
P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)&=\frac{2(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\sum_{j=0}^{m-1}\frac{(d-c)^{j+\beta}}{\Gamma(j+\beta+1)}\left[(-1)^{m+j}\frac{\partial^{n+j}f\left(\frac{a+b}{2},d\right)}{\partial t^n\partial s^j}+\frac{\partial^{n+j}f\left(\frac{a+b}{2},c\right)}{\partial t^n\partial s^j}\right]\\
&\quad+\frac{2(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\sum_{k=0}^{n-1}\frac{(b-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}\left[(-1)^{n+k}\frac{\partial^{k+m}f\left(b,\frac{c+d}{2}\right)}{\partial t^k\partial s^m}+\frac{\partial^{k+m}f\left(a,\frac{c+d}{2}\right)}{\partial t^k\partial s^m}\right]
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)&=\frac{4(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\frac{(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\frac{\partial^{n+m}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t^n\partial s^m}-\frac{2(d-c)^{m+\beta}}{\Gamma(m+\beta+1)}\left((-1)^nJ^{\alpha}_{b-}\frac{\partial^m f\left(a,\frac{c+d}{2}\right)}{\partial s^m}+J^{\alpha}_{a+}\frac{\partial^m f\left(b,\frac{c+d}{2}\right)}{\partial s^m}\right)\\
&-\frac{2(b-a)^{n+\alpha}}{\Gamma(n+\alpha+1)}\left((-1)^mJ^{\beta}_{d-}\frac{\partial^n f\left(\frac{a+b}{2},c\right)}{\partial t^n}+J^{\beta}_{c+}\frac{\partial^n f\left(\frac{a+b}{2},d\right)}{\partial t^n}\right),
\end{aligned}\tag{3.5}$$

respectively. Also, $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$ are defined as in (2.2), (2.3), (2.4), respectively.

By using similar methods, special results involving Riemann-Liouville fractional integals for partial derivatives of different orders and their midpoint versions can be examined.

4. Double Integral Inequalities for $L_\infty[a,b]$

We now examine how inequalities will come out when functions whose higher-order partial derivatives are bounded are considered.

Theorem 4.1. *Suppose that all the assumptions of Lemma 2.3 hold. If the partial derivative of order $n+m+2$ of f exists and is bounded, i.e.,*

$$\left\|f^{(n+m+2)}\right\|_\infty=\sup_{(u,v)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}\right|<\infty,$$

then, for any $(x,y)\in\Delta$, one has the inequalities

$$\begin{aligned}
\left|\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}(f)+L(f)\right|\;&\leq\;\frac{1}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)}\times\Big\{G_\alpha(a,b,x;n)G_\beta(c,d,y;m)\left\|f^{(n+m+2)}\right\|_{[a,x]\times[c,y],\infty}\\
&+G_\alpha(a,b,x;n)H_\beta(c,d,y;m)\left\|f^{(n+m+2)}\right\|_{[a,x]\times[y,d],\infty}\\
&+H_\alpha(a,b,x;n)G_\beta(c,d,y;m)\left\|f^{(n+m+2)}\right\|_{[x,b]\times[c,y],\infty}\\
&+H_\alpha(a,b,x;n)H_\beta(c,d,y;m)\left\|f^{(n+m+2)}\right\|_{[x,b]\times[y,d],\infty}\Big\}\\
&\leq\;\frac{1}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)}\left\|f^{(n+m+2)}\right\|_{[a,b]\times[c,d],\infty}\\
&\left[(n+\alpha-1)(b-a)^{n+\alpha+1}+2(b-x)^{n+\alpha+1}+2(x-a)^{n+\alpha+1}\right]\\
&\times\left[(m+\beta-1)(d-c)^{m+\beta+1}+2(d-y)^{m+\beta+1}+2(y-c)^{m+\beta+1}\right]
\end{aligned}\tag{4.1}$$

where

$$\begin{aligned}
G_{\alpha}(a,b,x;n) &= (x-a)^{n+\alpha+1}+(b-x)^{n+\alpha+1}+(b-a)^{n+\alpha}\left[(n+\alpha)(x-a)-(b-x)\right],\\
H_{\alpha}(a,b,x;n) &= (x-a)^{n+\alpha+1}+(b-x)^{n+\alpha+1}+(b-a)^{n+\alpha}\left[(n+\alpha)(b-x)-(x-a)\right],\\
G_{\beta}(c,d,y;m) &= (y-c)^{m+\beta+1}+(d-y)^{m+\beta+1}+(d-c)^{m+\beta}\left[(m+\beta)(y-c)-(d-y)\right],
\end{aligned}$$

and

$$H_{\beta}(c,d,y;m) = (y-c)^{m+\beta+1}+(d-y)^{m+\beta+1}+(d-c)^{m+\beta}\left[(m+\beta)(d-y)-(y-c)\right].$$

Here, $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}(f)$ and $L(f)$ are defined as in (2.2)-(2.6), respectively.

Proof. Taking modulus of both sides of the equality (2.10), on account of the triangle inequality, we find that

$$\begin{aligned}
\left|\mathscr{J}(f)+M_{k,j}(f)-N_j(f)+P_k(f)+L(f)\right| &\leq \frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}+(b-t)^{n+\alpha-1}\right]\right.\\
&\quad\times\left[(s-c)^{m+\beta-1}+(d-s)^{m+\beta-1}\right]\left|\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}\mathrm{d}v\mathrm{d}u\right|\mathrm{d}s\mathrm{d}t.
\end{aligned}\tag{4.2}$$

If the bounded function property is applied to each subinterval by taking into account the assumption of the function f in the theorem for the first integral that needs to be calculated, then one has

$$\begin{aligned}
\int_a^b\int_c^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}\left|\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}\mathrm{d}v\mathrm{d}u\right|\mathrm{d}s\mathrm{d}t &\leq \left\|f^{(n+m+2)}\right\|_{[a,x]\times[c,y],\infty}\int_a^x\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t\\
&\quad+\left\|f^{(n+m+2)}\right\|_{[a,x]\times[y,d],\infty}\int_a^x\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t\\
&\quad+\left\|f^{(n+m+2)}\right\|_{[x,b]\times[c,y],\infty}\int_x^b\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t\\
&\quad+\left\|f^{(n+m+2)}\right\|_{[x,b]\times[y,d],\infty}\int_x^b\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t
\end{aligned}\tag{4.3}$$

for any $(x,y)\in\Delta$. Calculating integals in the right hand of the the inequality (4.3), it follows that

$$\int_a^x\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t=\frac{(x-a)^{n+\alpha+1}}{(n+\alpha)(n+\alpha+1)}\frac{(y-c)^{m+\beta+1}}{(m+\beta)(m+\beta+1)},$$

$$\int_a^x\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t=\frac{(x-a)^{n+\alpha+1}}{(n+\alpha)(n+\alpha+1)}\times\left[\frac{(d-c)^{m+\beta}\left[(m+\beta)(d-y)-(y-c)\right]}{(m+\beta)(m+\beta+1)}+\frac{(y-c)^{m+\beta+1}}{(m+\beta)(m+\beta+1)}\right],$$

$$\int_x^b\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t=\frac{(y-c)^{m+\beta+1}}{(m+\beta)(m+\beta+1)}\times\left[\frac{(b-a)^{n+\alpha}\left[(n+\alpha)(b-x)-(x-a)\right]}{(n+\alpha)(n+\alpha+1)}+\frac{(x-a)^{n+\alpha+1}}{(n+\alpha)(n+\alpha+1)}\right]$$

and

$$\begin{aligned}
\int_x^b\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}|t-x||s-y|\,\mathrm{d}s\mathrm{d}t &= \left[\frac{(b-a)^{n+\alpha}\left[(n+\alpha)(b-x)-(x-a)\right]}{(n+\alpha)(n+\alpha+1)}+\frac{(x-a)^{n+\alpha+1}}{(n+\alpha)(n+\alpha+1)}\right]\\
&\quad\times\left[\frac{(d-c)^{m+\beta}\left[(m+\beta)(d-y)-(y-c)\right]}{(m+\beta)(m+\beta+1)}+\frac{(y-c)^{m+\beta+1}}{(m+\beta)(m+\beta+1)}\right]
\end{aligned}$$

If results of these four integrals are substituted in the inequality (4.3), then the exact expression of the inequality (4.3) is found. If the other three integrals deriving from the expression (4.2) are calculated by using similar methods, the desired inequalities (4.1) can be obtained. □

Remark 4.2. *If we choose $n=m=0$ in the inequali (3.1), then, for any $(x,y)\in\Delta$, we have*

$$\begin{aligned}
&\Big|J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)-2\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right]\\
&-2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+4\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Big|\\
\leq\;&\frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2)}\Big\{G_{\alpha}(a,b,x;0)\,G_{\beta}(c,d,y;0)\left\|f^{(2)}\right\|_{[a,x]\times[c,y],\infty}+G_{\alpha}(a,b,x;0)\,H_{\beta}(c,d,y;0)\left\|f^{(2)}\right\|_{[a,x]\times[y,d],\infty}\\
&+H_{\alpha}(a,b,x;0)\,G_{\beta}(c,d,y;0)\left\|f^{(2)}\right\|_{[x,b]\times[c,y],\infty}+H_{\alpha}(a,b,x;0)\,H_{\beta}(c,d,y;0)\left\|f^{(2)}\right\|_{[x,b]\times[y,d],\infty}\Big\}\\
\leq\;&\frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2)}\left[(\alpha-1)(b-a)^{\alpha+1}+2(b-x)^{\alpha+1}+2(x-a)^{\alpha+1}\right]\\
&\times\left[(\beta-1)(d-c)^{\beta+1}+2(d-y)^{\beta+1}+2(y-c)^{\beta+1}\right]\left\|f^{(2)}\right\|_{[a,b]\times[c,d],\infty}
\end{aligned}$$

which was provided Erden et al. in [19]. Here, $\left\|f^{(2)}\right\|_{\infty}$ is defined by

$$\left\|f^{(2)}\right\|_{\infty}=\sup_{(u,v)\in(a,b)\times(c,d)}\left|\frac{\partial^{2}f(u,v)}{\partial u\partial v}\right|<\infty.$$

Corollary 4.3. *Suppose that all the assumptions oftheorem 3.1 hold. If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we possess the midpoint inequalities*

$$\begin{aligned}
&\left|\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)+L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)\right|\\
\leq\;&\frac{(b-a)^{n+\alpha+1}(d-c)^{m+\beta+1}}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)}\left[\frac{1}{2^{n+\alpha}}+\frac{n+\alpha-1}{2}\right]\left[\frac{1}{2^{m+\beta}}+\frac{m+\beta-1}{2}\right]\\
&\times\left\{\left\|f^{(n+m+2)}\right\|_{[a,\frac{a+b}{2}]\times[c,\frac{c+d}{2}],\infty}+\left\|f^{(n+m+2)}\right\|_{[a,\frac{a+b}{2}]\times[\frac{c+d}{2},d],\infty}+\left\|f^{(n+m+2)}\right\|_{[\frac{a+b}{2},b]\times[c,\frac{c+d}{2}],\infty}+\left\|f^{(n+m+2)}\right\|_{[\frac{a+b}{2},b]\times[\frac{c+d}{2},d],\infty}\right\}\\
\leq\;&4\left[\frac{1}{2^{n+\alpha}}+\frac{n+\alpha-1}{2}\right]\left[\frac{1}{2^{m+\beta}}+\frac{m+\beta-1}{2}\right]\\
&\times\frac{(b-a)^{n+\alpha+1}(d-c)^{m+\beta+1}}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)}\left\|f^{(n+m+2)}\right\|_{[a,b]\times[c,d],\infty}
\end{aligned}$$

where $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)$ and $L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)$ are also defined as in (2.2), (2.3), (2.4), (3.4) and (3.5), respectively.

5. Double Integral Inequalities for $L_1[\Delta]$

Some results based on fractional integrals for functions whose higher-order derivatives are elements of L_1-norm are observed in this part. Some special cases and midpoint versions of our main results are also given.

Theorem 5.1. *Suppose that all the assumptions of Lemma 2.3 hold. If If the partial derivative of order $n+m+2$ of f exists and is element of $L_1[\Delta]$, i.e.,*

$$\left\|f^{(n+m+2)}\right\|_{1}=\int_{a}^{b}\int_{c}^{d}\left|\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}\right|\mathrm{d}v\mathrm{d}u<\infty$$

then, for any $(x,y)\in\Delta$, we have the inequalities

$$\begin{aligned}
\left|\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}(f)+L(f)\right|\;\leq\;&\frac{1}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\Big\{A_{n}(x)C_{m}(y)\left\|f^{(n+m+2)}\right\|_{[a,x]\times[c,y],1}\\
&+A_{n}(x)D_{m}(y)\left\|f^{(n+m+2)}\right\|_{[a,x]\times[y,d],1}\\
&+B_{n}(x)C_{m}(y)\left\|f^{(n+m+2)}\right\|_{[x,b]\times[c,y],1}+B_{n}(x)D_{m}(y)\left\|f^{(n+m+2)}\right\|_{[x,b]\times[y,d],1}\Big\}\\
\leq\;&\frac{\left[(b-a)^{n+\alpha}+\left|(x-a)^{n+\alpha}-(b-x)^{n+\alpha}\right|\right]}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\\
&\times\left[(d-c)^{m+\beta}+\left|(y-c)^{m+\beta}-(d-y)^{m+\beta}\right|\right]\left\|f^{(n+m+2)}\right\|_{[a,b]\times[c,d],1}
\end{aligned}\tag{5.1}$$

where $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}(f)$ and $L(f)$ are defined as in (2.2)-(2.6), respectively.

Proof. Taking modulus of both sides of the equality (**??**), due to the triangle inequality, we find that

$$\left|\mathscr{J}(f)+M_{k,j}(f)-N_j(f)+P_k(f)+L(f)\right| \leq \frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{n+\alpha-1}+(b-t)^{n+\alpha-1}\right]\right. \times\left[(s-c)^{m+\beta-1}+(d-s)^{m+\beta-1}\right]\left|\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}dvdu\right|dsdt. \tag{5.2}$$

Considering the first integral that needs to be calculated in the right hand side of the above inequality, seeing that the partial derivative of order $n+m+2$ of f is element of $L_1[\Delta]$, it follows that

$$\begin{aligned}\int_a^b\int_c^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}\left|\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}dvdu\right|dsdt &\leq \left\|f^{(n+m+2)}\right\|_{[a,x]\times[c,y],1}\int_a^x\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt\\ &+\left\|f^{(n+m+2)}\right\|_{[a,x]\times[y,d],1}\int_a^x\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt\\ &+\left\|f^{(n+m+2)}\right\|_{[x,b]\times[c,y],1}\int_x^b\int_c^y(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt\\ &+\left\|f^{(n+m+2)}\right\|_{[x,b]\times[y,d],1}\int_x^b\int_y^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}dsdt\end{aligned}$$

for any $(x,y)\in\Delta$.And so, we conclude that

$$\begin{aligned}\int_a^b\int_c^d(t-a)^{n+\alpha-1}(s-c)^{m+\beta-1}\left|\int_x^t\int_y^s\frac{\partial^{n+m+2}f(u,v)}{\partial u^{n+1}\partial v^{m+1}}dvdu\right|dsdt &\leq \frac{(x-a)^{n+\alpha}}{n+\alpha}\frac{(y-c)^{m+\beta}}{m+\beta}\left\|f^{(n+m+2)}\right\|_{[a,x]\times[c,y],1}\\ &+\frac{(x-a)^{n+\alpha}}{n+\alpha}\frac{(d-c)^{m+\beta}-(y-c)^{m+\beta}}{m+\beta}\left\|f^{(n+m+2)}\right\|_{[a,x]\times[y,d],1}\\ &+\frac{(b-a)^{n+\alpha}-(x-a)^{n+\alpha}}{n+\alpha}\frac{(y-c)^{m+\beta}}{m+\beta}\left\|f^{(n+m+2)}\right\|_{[x,b]\times[c,y],1}\\ &+\frac{(b-a)^{n+\alpha}-(x-a)^{n+\alpha}}{n+\alpha}\frac{(d-c)^{m+\beta}-(y-c)^{m+\beta}}{m+\beta}\left\|f^{(n+m+2)}\right\|_{[x,b]\times[y,d],1}.\end{aligned}$$

In the same manner that we calculate the first integral, the other tree integrals that need to be calculated in the inequality (5.2) can be found. If we substitute the results of these four integals in the inequality (5.2), the requaried expression (5.1) can be attained. The proof is thus completed. □

Remark 5.2. *Taking $n=m=0$ in the inequali (5.1), then, for any $(x,y)\in\Delta$, one has the inequalities*

$$\begin{aligned}&\left|J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)-2\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right]\right.\\ &\left.-2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+4\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\right|\\ \leq\ &\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left\{A_0(x)C_0(y)\left\|f^{(2)}\right\|_{[a,x]\times[c,y],1}+A_0(x)D_0(y)\left\|f^{(2)}\right\|_{[a,x]\times[y,d],1}+B_0(x)C_0(y)\left\|f^{(2)}\right\|_{[x,b]\times[c,y],1}\right.\\ &\left.+B_0(x)D_0(y)\left\|f^{(2)}\right\|_{[x,b]\times[y,d],1}\right\}\\ \leq\ &\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left\|f^{(2)}\right\|_{[a,b]\times[c,d],1}\end{aligned}\tag{5.3}$$

where $\left\|f^{(2)}\right\|_1$ is defined by

$$\left\|f^{(2)}\right\|_1=\int_a^b\int_c^d\left|\frac{\partial^2 f(u,v)}{\partial u\partial v}\right|dvdu<\infty.$$

Corollary 5.3. *Suppose that all the assumptions of Theorem 5.1 hold. If we choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the Midpoint inequality*

$$\left|\mathscr{J}(f)+M_{k,j}(f)-N_{k,j}(f)+P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)+L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)\right|\leq\frac{(b-a)^{n+\alpha}(d-c)^{m+\beta}}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\left\|f^{(n+m+2)}\right\|_{[a,b]\times[c,d],1} \tag{5.4}$$

where $\mathscr{J}(f)$, $M_{k,j}(f)$, $N_{k,j}(f)$, $P_{k,j}\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)$ and $L\left(f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)$ are defined as in (2.2), (2.3), (2.4), (3.4) and (3.5), respectively.

Remark 5.4. *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in the inequalities (5.3), then we possess*

$$\begin{aligned}
&\Big| J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)-2\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha}_{a+}f\left(b,\frac{c+d}{2}\right)\right]\\
&-2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f\left(\frac{a+b}{2},c\right)+J^{\beta}_{c+,}f\left(\frac{a+b}{2},d\right)\right]+4\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Big|\\
\leq\;& \frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left\|f^{(2)}\right\|_{[a,b]\times[c,d],1}.
\end{aligned}$$

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