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A Generalization of The Prime Radicals of Rings

Research Article

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Abstract

Let R be a ring, I be an ideal of R and $\sqrt{I}$ be a prime radical of I. This study generalizes the prime radical of $\sqrt{I}$ which denotes by $\sqrt[n+1]{I}$, for $n \in \mathbb{Z}^+$. This generalization is called the n-prime radical of ideal I. Moreover, this paper demonstrates that R is isomorphic to a subdirect sum of ring H_i where H_i are n-prime rings. Furthermore, two open problems are presented.
Keywords: Prime ring, Prime ideal, Semiprime ideal, Prime radical.

1. INTRODUCTION

Let R be a ring and I be an ideal of R. The prime radical of the ideal I is

$$\sqrt{I} = \{r \in R : \text{for every } m - \text{system } M \text{ containing } r, M \cap I \neq \emptyset\}$$

and the radical of ring R is also defined as $\beta(R) = \sqrt{0}$ [11].

The reason for studying radicals is that a ring R is isomorphic to a subdirect sum of prime rings if and only if $\beta(R) = (0)$ [11]. Due to this feature, problems in rings can be transferred to prime rings by using the prime radical of a ring.

In generalizing the classical notion of the radical in a ring, different kinds of radicals have been defined by many authors, including Köthe [8], Baer [5], Levitzki [9], Jacobson [7], Brown-McCoy [6], Azumaya [4] and McCoy [10]. Moreover, different generalizations of the prime radical have also been tried to be accomplished [13], [12].

This study constructs a different generalization of the prime radical which is represented by $\sqrt[n+1]{I}$ and analyzes the properties of $\sqrt[n+1]{I}$. It also attempts to characterize the n-radical of a ring and denoted it by $\beta^n(R)$.

2. BASIC CONCEPTS AND NOTIONS

The current section provides the following basic definitions in [1, 11].

Definition 2.1. Let R be a ring and I be a semigroup ideal of R. If $aRb \subset I$ implies $a \in I$ or $b \in I$, then I is called a semigroup prime ideal.

Definition 2.2. Let I be an ideal of ring R. If $aRb \subset I$ implies $a \in I$ or $b \in I$, then I is called a prime ideal.

Definition 2.3. Let I be a semigroup ideal of ring R. I is called a semigroup semiprime ideal, if $aRa \subset I$ implies $a \in I$.

Definition 2.4. Let I be an ideal of ring R. I is a called semiprime ideal, if $aRa \subset I$ implies $a \in I$.

Definition 2.5. [3] Let R be a ring and $\emptyset \neq I$ be an ideal of R. In [1], the set $\mathcal{L}_R(I)$ is defined as follows:

$$\mathcal{L}_R(I) = \{a \in R : aRa \subset I\}$$

Motivated by this set,

$$\mathcal{L}_R^n(I) = \{a \in R : aRa \subset \mathcal{L}_R^{n-1}(I),\ n \in \mathbb{N}\}$$

Definition 2.6. [3] Let I be an ideal of ring R. I is called an n-prime ideal if $\mathcal{L}_R^n(I)$ is a semigroup prime ideal.

Definition 2.7. [3] Let I be an ideal of ring R. I is called an n-semiprime ideal if $\mathcal{L}_R^n(I)$ is a semigroup semiprime ideal.

Definition 2.8. [3] R is called an n-prime ring if $\mathcal{L}_R^n(0)$ is a semigroup prime ideal.

Definition 2.9. [3] R is called an n-semiprime ring if $\mathcal{L}_R^n(0)$ is a semigroup semiprime ideal.

3. A GENERALIZATION OF THE PRIME RADICAL

Key definitions and notations essential for the generalization of radicals are presented below.

Notation 1. Let I be a semigroup semiprime ideal. Then, from [3], $\mathcal{L}_R^n(I) = I$. Accordingly,

$$A^n(I) = \{\, J \subset R : J \text{ semigroup ideal and } \mathcal{L}_R^n(J) = I\} \tag{1}$$

is not an empty set because of $I \in A^n(I)$.

Lemma 3.1. Let J and $\bar{J}$ be two ideals of ring R. Then, $\mathcal{L}_R^n(J \cap \bar{J}) = \mathcal{L}_R^n(J) \cap \mathcal{L}_R^n(\bar{J})$.

Proof. Since $\mathcal{L}_R^0(J) = J$ and $\mathcal{L}_R^0(\bar{J}) = \bar{J}$, for $n = 0$, the proof is obtained evidently. Let $n = 1$. If $x \in \mathcal{L}_R(J) \cap \mathcal{L}_R(\bar{J})$, then $xrx \in \mathcal{L}_R^0(J) = J$ and $xrx \in \mathcal{L}_R^0(\bar{J}) = \bar{J}$, for all $r \in R$. Hence, $xrx \in J \cap \bar{J} = \mathcal{L}_R^0(J \cap \bar{J})$. Therefore, $x \in \mathcal{L}_R(J \cap \bar{J})$. Accordingly, $\mathcal{L}_R(J) \cap \mathcal{L}_R(\bar{J}) \subseteq \mathcal{L}_R(J \cap \bar{J})$. Conversely, if $x \in \mathcal{L}_R(J \cap \bar{J})$, then $xrx \in J \cap \bar{J} = \mathcal{L}_R^0(J \cap \bar{J})$. From here, $xrx \in \mathcal{L}_R^0(J) = J$ and $xrx \in \mathcal{L}_R^0(\bar{J}) = \bar{J}$, for all $r \in R$. This means $x \in \mathcal{L}_R(J) \cap \mathcal{L}_R(\bar{J})$. Consequently, $\mathcal{L}_R(J \cap \bar{J}) \subseteq \mathcal{L}_R(J) \cap \mathcal{L}_R(\bar{J})$. Assume that

$$\mathcal{L}_R^n(J \cap \bar{J}) = \mathcal{L}_R^n(J) \cap \mathcal{L}_R^n(\bar{J})$$

for an arbitrary $n \in \mathbb{N}$.

Let $x \in \mathcal{L}_R^{n+1}(J) \cap \mathcal{L}_R^{n+1}(\bar{J})$. Then, $xrx \in \mathcal{L}_R^n(J)$ and $xrx \in \mathcal{L}_R^n(\bar{J})$, for all $r \in R$. Therefore, $xrx \in \mathcal{L}_R^n(J) \cap \mathcal{L}_R^n(\bar{J})$ for all $r \in R$. From here, $xrx \in \mathcal{L}_R^n(J \cap \bar{J})$. As a result, $x \in \mathcal{L}_R^{n+1}(J \cap \bar{J})$. Thus, $\mathcal{L}_R^{n+1}(J) \cap \mathcal{L}_R^{n+1}(\bar{J}) \subseteq \mathcal{L}_R^{n+1}(J \cap \bar{J})$. The converse is similar. Hence, $\mathcal{L}_R^{n+1}(J \cap \bar{J}) = \mathcal{L}_R^{n+1}(J) \cap \mathcal{L}_R^{n+1}(\bar{J})$.

Corollary 3.2. $\{J_i\}_{i\in\Lambda}$ is a set of ideals of ring R. Then, $\mathcal{L}_{\mathcal{R}}{}^n(\bigcap_{i\in\Lambda} J_i) = \bigcap_{i\in\Lambda} \mathcal{L}_{\mathcal{R}}{}^n(J_i)$.

Proof. The proof is evident from induction.

Theorem 3.3. Let I be a semigroup semiprime ideal. $J \cap \bar{J} \in A^n(I)$, for every $J, \bar{J} \in A^n(I)$.

Proof. If $J, \bar{J} \in A^n(I)$, then $\mathcal{L}_R^n(J) = \mathcal{L}_R^n(\bar{J}) = I$. From Lemma 3.1,

$$I = \mathcal{L}_R^n(J) \cap \mathcal{L}_R^n(\bar{J}) = \mathcal{L}_R^n(J \cap \bar{J})$$

Then, $J \cap \bar{J} \in A^n(I)$.

Lemma 3.4. Let I be a semigroup semiprime ideal of ring R. Then, $A^n(I) \subset A^{n+1}(I)$, for all $n \in \mathbb{N}$.

Proof. Let $J \in A^n(I)$. Thus, $\mathcal{L}_R^{n+1}(J) = \mathcal{L}_R(\mathcal{L}_R^n(J)) = \mathcal{L}_R(I) = I$. Therefore, $J \in A^{n+1}(I)$ and $A^n(I) \subset A^{n+1}(I)$.

Notation 2. Let I_α be a semigroup semiprime ideal, for all $\alpha \in \Lambda$. Consider the set

$$\left\{ J = \bigcap_{\alpha \in \Lambda} J_\alpha : J_\alpha \in A^n(I_\alpha), \alpha \in \Lambda \right\}$$

From Notation 1, since $I_\alpha \in A^n(I_\alpha)$, for all $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} I_\alpha$ is an element of this set. Let's symbolize this set with $\overline{\bigcap_{\alpha \in \Lambda}} A^n(I_\alpha)$.

Theorem 3.5. Let I_α be a semigroup semiprime ideal, for all $\alpha \in \Lambda$. Then,

$$\overline{\bigcap_{\alpha \in \Lambda}} A^n(I_\alpha) \subset A^n\left(\bigcap_{\alpha \in \Lambda} I_\alpha\right)$$

Proof. Let $J \in \overline{\bigcap_{\alpha \in \Lambda}} A^n(I_\alpha)$. Then, $J = \bigcap_{\alpha \in \Lambda} J_\alpha$ where $J_\alpha \in A^n(I_\alpha)$, for all $\alpha \in \Lambda$. Hence, from Corollary 3.2,

$$\mathcal{L}_R^n(J) = \mathcal{L}_R^n\left(\bigcap_{\alpha \in \Lambda} J_\alpha\right) = \bigcap_{\alpha \in \Lambda} \mathcal{L}_R^n(J_\alpha) = \bigcap_{\alpha \in \Lambda} I_\alpha$$

Accordingly, $J \in A^n(\bigcap_{\alpha \in \Lambda} I_\alpha)$.

Example 1. Let $(F, +, .)$ be a field and $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, d, e \in F \right\}$ be a ring.

$I_1 = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, e \in F \right\}$, $I_2 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : b, c, d, e \in F \right\}$,

$I_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, \in F \right\}$, $I_4 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} : b, c, d \in F \right\}$,

$I_5 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b, c \in F \right\}$, $I_6 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} : c, d \in F \right\}$ and

$I_7 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \in F \right\}$ are semigroup semiprime ideals of R. From here,

$\mathcal{L}_R(I_1) = I_1$, $\mathcal{L}_R(I_2) = I_4$, $\mathcal{L}_R(I_3) = I_1$, $\mathcal{L}_R(I_4) = I_4$, $\mathcal{L}_R(I_5) = I_4$, $\mathcal{L}_R(I_6) = I_4$, $\mathcal{L}_R(I_7) = I_4$ and $\mathcal{L}_R(0) = I_5 \cup I_6$. Besides that, I_1 and I_2 are prime ideals. Moreover, $I_1 \cap I_2 = I_4$ is a semiprime ideal. Thus, $A(I_1) = \{I_1\}$and$A(I_2) = \{I_2, I_3\}$.

Consider with $\Lambda = \{1,2\}$, $\overline{\bigcap_{\alpha \in \Lambda}} A(I_\alpha) = \{I_4, I_5\}$.

Because of $\bigcap_{\alpha \in \Lambda} I_\alpha = I_1 \cap I_2 = I_4$,

$$A\left(\bigcap_{\alpha\in\Lambda} I_\alpha\right) = \{I_4, I_5, I_6, I_7\}$$

is provided.
Therefore,

$$\overline{\bigcap_{\alpha\in\Lambda}} A(I_\alpha) \subset A\left(\bigcap_{\alpha\in\Lambda} I_\alpha\right).$$

Notation 3. Let I be a semigroup semiprime ideal. Then,

$$\mathcal{L}_R^{-n}(I) = \bigcap_{J\in A^n(I)} J.$$

Theorem 3.6. If I is a semigroup semiprime ideal, then $\mathcal{L}_R^{-n}(I) \in A^n(I)$.

Proof. Let I be a semigroup semiprime ideal.

$$\mathcal{L}_R^{n}(\mathcal{L}_R^{-n}(I)) = \mathcal{L}_R^{n}\left(\bigcap_{J\in A^n(I)} J\right) = \bigcap_{J\in A^n(I)} \mathcal{L}_R^{n}(J) = I$$

Hence, $\mathcal{L}_R^{-n}(I) \in A^n(I)$. Since I is a semigroup semiprime ideal, $\mathcal{L}_R^{-n}(I) \in A^n(I)$. Therefore, $\mathcal{L}_R^{-n}(I) \subset J$ for all $J \in A^n(I)$.

Definition 3.7. If I is a semigroup semiprime ideal of R, then $\mathcal{L}_R^{-n}(I)$ is called an n-minimal semigroup semiprime ideal of $A^n(I)$.

Lemma 3.8. [3] Let R and S be two rings and $\varphi: R \to S$ be an endomorphism and P be an ideal with $\mathrm{Ker}\varphi = K \subset P$. If P is a n-semiprime ideal of ring R, then $\varphi(P)$ is a n-semiprime ideal of ring R.

Lemma 3.9. [3] Let R and S be two rings and $\varphi: R \to S$ be an endomorphism and P be an ideal with $\mathrm{Ker}\varphi = K \subset P$. If P is a n-prime ideal of ring R, then $\varphi(P)$ is a n-prime ideal of ring R.

Theorem 3.10. Let I_α be a semigroup semiprime ideal, for all $\alpha \in \Lambda$. Then,

$$\mathcal{L}_R^{-n}(\bigcap_{\alpha\in\Lambda} I_\alpha) \subset \bigcap_{\alpha\in\Lambda} \mathcal{L}_R^{-n}(I_\alpha)$$

Proof. From Theorem 3.5,

$$\mathcal{L}_R^{-n}(\bigcap_{\alpha\in\Lambda} I_\alpha) = \bigcap_{J\in A^n(\bigcap_{\alpha\in\Lambda} I_\alpha)} J \subset \bigcap_{J\in\overline{\bigcap_{\alpha\in\Lambda}} A^n(I_\alpha)} J.$$

Therefore,

$$\bigcap_{J\in\overline{\bigcap_{\alpha\in\Lambda}} A^n(I_\alpha)} J = \bigcap_{\alpha\in\Lambda} \mathcal{L}_R^{-n}(I_\alpha).$$

Example 2. Adopting the Example 1,

$$\mathcal{L}_R^{-1}(I_1) = \bigcap_{J \in A(I_1)} J = I_2,$$

$$\mathcal{L}_R^{-1}(I_2) = \bigcap_{J \in A(I_2)} J = I_5,$$

$$\bigcap_{\alpha \in \Lambda} \mathcal{L}_R^{-1}(I_\alpha) = \mathcal{L}_R^{-1}(I_1) \cap \mathcal{L}_R^{-1}(I_2) = I_5$$

and

$$\mathcal{L}_R^{-1}(\bigcap_{\alpha \in \Lambda} I_\alpha) = \mathcal{L}_R^{-1}(I_1 \cap I_2) = I_7.$$

Therefore,

$$\mathcal{L}_R^{-1}(\bigcap_{\alpha \in \Lambda} I_\alpha) = I_7 \subset I_5 = \bigcap_{\alpha \in \Lambda} \mathcal{L}_R^{-1}(I_\alpha).$$

This means that equality may not be achieved.

Definition 3.11. Let I be an ideal of ring R and $\sqrt{I}$ be the prime radical of I. Then, the n-prime radical of I is characterized as

$$\sqrt[n+1]{I} = \mathcal{L}_R^{-(n-1)}(\sqrt{I}), \qquad \text{for } n \in \mathbb{Z}^+$$

where 1-prime radical of I is equivalent to $\sqrt{I} = \mathcal{L}_R^{0}(\sqrt{I})$. Moreover, the n-prime radical of ring R can be defined as n-radical of the ring

$$\beta^n(R) = \sqrt[n+1]{(0)}$$

where $n \geq 1$, $n \in \mathbb{N}$, and 1-radical of ring R is equivalent to $\beta(R)$.

Theorem 3.12. Let I be an ideal of ring R. Then,

$$\sqrt[n+1]{I} \subset \sqrt[n]{I}$$

for $n \in \{2,3,\ldots\}$.

Proof. Let I be an ideal of R. Then, from Lemma 3.2, $A^{n-1}(\sqrt{I}) \subset A^n(\sqrt{I})$. Accordingly,

$$\sqrt[n+1]{I} = \mathcal{L}_R^{-(n-1)}(\sqrt{I}) = \bigcap_{J \in A^{n-1}(\sqrt{I})} J \subset \bigcap_{J' \in A^{n}(\sqrt{I})} J' = \mathcal{L}_R^{-(n)}(\sqrt{I}) = \sqrt[n]{I}.$$

Corollary 3.13. Let I be an ideal of ring R. Then,

$$\ldots \subset \sqrt[n+1]{I} \subset \sqrt[n]{I} \subset \ldots \subset \sqrt[4]{I} \subset \sqrt[3]{I} \subset \sqrt{I} \subset I$$

Proof. The proof is obvious from induction.

Corollary 3.14. Let I be an ideal of ring R. Then,

$$\sqrt[n+1]{I} = \bigcap_{I \subset P, P\text{prime}} \mathcal{L}_R^{-(n-1)}(P).$$

Proof. $\sqrt{I} = \bigcap_{I \subset P, P\text{prime}} P$. Then,

$$\sqrt[n+1]{I} = \mathcal{L}_R^{-(n-1)}(\sqrt{I}) = \mathcal{L}_R^{-(n-1)}(\bigcap_{I\subset P, P prime} P) = \bigcap_{I\subset P, P prime} \mathcal{L}_R^{-(n-1)}(P).$$

Example 3. For the $\mathbb{Z}_{36}$ ring, let's examine n-prime radicals $\beta^n(\mathbb{Z}_{36})$.

I_i ideal	$\mathcal{L}_{\mathbb{Z}_{36}}(I_i)$
$I_0 = (0), I_6 = (6), I_{12} = (12), I_{18} = (18)$	$\mathcal{L}_{\mathbb{Z}_{36}}(I_0) = \mathcal{L}_{\mathbb{Z}_{36}}(I_6) = \mathcal{L}_{\mathbb{Z}_{36}}(I_{12}) = \mathcal{L}_{\mathbb{Z}_{36}}(I_{18}) = (6)$
$I_1 = (1) = \mathbb{Z}_{36}$	$\mathcal{L}_{\mathbb{Z}_{36}}(I_1) = \mathbb{Z}_{36}$
$I_2 = (2), I_4 = (4)$	$\mathcal{L}_{\mathbb{Z}_{36}}(I_2) = \mathcal{L}_{\mathbb{Z}_{36}}(I_4) = (2)$
$I_3 = (3), I_9 = (9)$	$\mathcal{L}_{\mathbb{Z}_{36}}(I_3) = \mathcal{L}_{\mathbb{Z}_{36}}(I_9) = (3)$
$I_8 = (8)$	$\mathcal{L}_{\mathbb{Z}_{36}}(I_8) = (4)$
$I_{16} = (16)$	$\mathcal{L}_{\mathbb{Z}_{36}}(I_{16}) = (8)$

Since $I_{12} \cap I_{18} = (0)$,

$$\mathcal{L}_{\mathbb{Z}_{36}}(I_{12} \cap I_{18}) = \mathcal{L}_{\mathbb{Z}_{36}}(0) = I_6.$$

In this case, $\beta(\mathbb{Z}_{36}) = \sqrt{(0)} = I_2 \cap I_3 = I_6$. Furthermore,

$$\mathcal{L}_{\mathbb{Z}_{36}}^{-1}(\sqrt{(0)}) = I_0 \cap I_6 \cap I_{12} \cap I_{18} = (0)$$

and

$$\beta^2(\mathbb{Z}_{36}) = \sqrt[3]{(0)} = \sqrt[1+2]{(0)} = L_{\mathbb{Z}_{36}}^{-1}(\sqrt{(0)}) = \mathcal{L}_{\mathbb{Z}_{36}}^{-1}((6)) = (0)$$

and $\mathbb{Z}_{36} \,/\, \beta^2(\mathbb{Z}_{36}) \cong \mathbb{Z}_{36}$. Therefore,

$$\beta^2(\mathbb{Z}_{36} \,/\, \beta^2(\mathbb{Z}_{36})) = (0).$$

Example 4. Let $(F, +, .)$ be a field and $R = \left\{\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, d, e \in F\right\}$ be a ring. Then,

$$\beta(R) = \sqrt{(0)} = \left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in F\right\},$$

$$\mathcal{L}_R^{-1}(\sqrt{(0)}) = \left\{\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in F\right\}$$

and

$$\beta^2(R) = \sqrt[3]{(0)} = \sqrt[2+1]{(0)} = \mathcal{L}_R^{-1}(\sqrt{(0)}) = \left\{\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in F\right\}.$$

Quotient ring as

$$\overline{R} = \left\{\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} + \beta^2(R) : a, b, c, d, e \in F\right\} = \left\{\begin{pmatrix} a & b & 0 \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : a, b, d, e \in F\right\}$$

where

$$\begin{pmatrix} a & b & 0 \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} x & y & 0 \\ 0 & z & t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a+x & b+y & 0 \\ 0 & c+z & d+t \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b & 0 \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x & y & 0 \\ 0 & z & t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} ax & ay+bz & bt \\ 0 & cz & ct \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $I_1 = \left\{\begin{pmatrix} a & b & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} : a, b, e \in F\right\}$, $I_2 = \left\{\begin{pmatrix} 0 & b & 0 \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : b, d, e \in F\right\}$,

$I_3 = \left\{\begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a, b \in F\right\}$, $I_4 = \left\{\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} : b, e \in F\right\}$,

$I_5 = \left\{\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in F\right\}$, $I_6 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} : e \in F\right\}$ are semiprime ideals of $\overline{R}$. Then, $\mathcal{L}_{\overline{R}}(I_1) = I_1$, $\mathcal{L}_{\overline{R}}(I_2) = I_4$, $\mathcal{L}_{\overline{R}}(I_3) = I_1$, $\mathcal{L}_{\overline{R}}(I_4) = I_4$, $\mathcal{L}_{\overline{R}}(I_5) = I_4$, $\mathcal{L}_{\overline{R}}(I_6) = I_4$, and $\mathcal{L}_{\overline{R}}(0) = I_4$. Herefrom,

$$I_2 = \left\{\begin{pmatrix} 0 & b & 0 \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : b, d, e \in F\right\} \text{and} I_1 = \left\{\begin{pmatrix} a & b & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} : a, b, e \in F\right\}$$

are prime ideal. Moreover, $I_1 \cap I_2 = I_4$ is a semiprime ideal. $A(I_2) = \{I_2\}$ and $A(I_1) = \{I_1, I_3\}$ are obtained. It is observed that

$$\beta(\overline{R}) = \sqrt{(0)} = I_4$$

Since $\mathcal{L}_{\overline{R}}^{-1}\left(\sqrt{(0)}\right) = (0)$, $\beta^2(\overline{R}) = (0)$. Thus,

$$\beta^2\left(\overline{R} \,/\, \beta^2(\overline{R})\right) = \beta^2\left(\overline{R}\right) = (0).$$

Theorem 3.15. If I is an ideal of R, then $\mathcal{L}_R(I) \subset \sqrt{I}$.

Proof. For all $a \in \mathcal{L}_R(I)$, $aRa \subset I \subset \sqrt{I}$. Since $\sqrt{I}$ is a semiprime ideal, $a \in \sqrt{I}$. Hence, $\mathcal{L}_R(I) \subset \sqrt{I}$.

Theorem 3.16. If I is an ideal of R, then $\mathcal{L}_R^n(I) \subset \sqrt{I}$, for all $n \in \mathbb{N}$.

Proof. From Theorem 6, $\mathcal{L}_R(I) \subset \sqrt{I}$, for $n = 1$. Let

$$\mathcal{L}_R^n(I) \subset \sqrt{I}$$

for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{L}_R(\mathcal{L}_R^n(I)) = \mathcal{L}_R^{n+1}(I) \subset \mathcal{L}_R(\sqrt{I}) = \sqrt{I}.$$

Theorem 3.17: Let R be a ring and let $\{H_i\}_{i\in\Lambda}$ be a n-prime ring family. If R is isomorphic to a subdirect sum of ring H_i, then $\beta(R) = \mathcal{L}_R^n(0)$.

Proof. Since R is isomorphic to a subdirect sum of ring H_i, there is an ideal K_i of R such that $R \,/\, K_i \simeq H_i$ and $\bigcap_{i\in\Lambda} K_i = (0)$.

On the other hand, from [[3], Theorem 4], $\pi^{-1}\left(\mathcal{L}_{H_i}^n(0)\right) = \mathcal{L}_R^n(K_i)$ is a prime ideal. Therefore,

$$\beta(R) = \bigcap_{j\in\Lambda} P_j \subset \bigcap_{i\in\Lambda} \mathcal{L}_R^n(K_i) = \mathcal{L}_R^n\left(\bigcap_{i\in\Lambda} K_i\right) = \mathcal{L}_R^n(0).$$

Hence we get $\beta(R) = \mathcal{L}_R^n(0)$.

Theorem 3.18. Let R be a commutative ring with identity. Suppose that $(\mathcal{L}_R^n(0))^2 = (0)$ and each ideal of ring R has pairwise comaximal ideals in R. If $\beta(R) = \mathcal{L}_R^n(0)$, then R is isomorphic to a subdirect sum of ring H_i where H_i are n-prime rings.

Proof. Suppose that $\mathcal{L}_R^n(0) = \beta(R)$. Hence,

$$\mathcal{L}_R^n(0) = \bigcap_{i\in\Lambda} P_i$$

where P_i is a prime ideal, for all $i \in \Lambda$. Since P_i is a pairwise comaximal ideals in R,

$$\bigcap_{i\in\Lambda} P_i^2 = \left(\bigcap_{i\in\Lambda} P_i\right)^2 = (\mathcal{L}_R^n(0))^2 = (0)$$

Thus, $\bigcap_i K_i = 0$ where $P_i^2 = K_i$, for all $i \in \Lambda$. On the other hand, since $\mathcal{L}_R^n(K_i) = \mathcal{L}_R^n(P_i^2) = P_i$, $\mathcal{L}_R^n(K_i)$ is a prime ideal and the set $R / K_i \simeq H_i$ is a n-prime ring. Let $\pi_i: R \to R / K_i$ be a natural epimorphism with $\pi_i(r) = 0$, for all $i \in \Lambda$ and $0 \neq r$. As a consequence, $r \in \bigcap_{i\in\Lambda} K_i = (0)$. This is a contradiction.

Remark 1. If $\beta(R) = \mathcal{L}_R^n(0)$, then $\beta^n\big(R/\beta^n(R)\big) = (0)$.

Proof. If $\mathcal{L}_R^n(0) = \mathcal{L}_R(\mathcal{L}_R^{n-1}(0)) = \beta(R)$, then $(0) \in A^{n-1}\big(\beta(R)\big)$. Therefore,

$$\beta^n(R) = \sqrt[n+1]{(0)} = \mathcal{L}_R^{-(n-1)}(0) = \bigcap_{J\in A^{n-1}(\beta(R))} J = (0)$$

and

$$\beta^n\big(R/\beta^n(R)\big) = (0).$$

4. CONCLUSION

This article attempts to generalize the prime radical in a promising way. It also investigates the properties of the basic notions essential for this generalization specifically $A^n(I)$ and $\mathcal{L}_{\mathcal{R}}{}^{-n}(I)$. The paper introduces the definitions of n-minimal semigroup semiprime ideal, n-prime radical of ideal I , and n-prime radical of ring R. Future research could extend these results to different rings, utilizing the generalization of prime radicals, thereby contributing significantly to ring theory. Additionally, the paper highlights open problems that may guide future studies.

5. OPEN PROBLEMS

1. Minimal $(n-1)-$ prime radical is a subideal of intersection of $(n-1)-$ prime ideals.

2. Let R be a ring. Then, $\beta^n\big(R/\beta^n(R)\big) = (0)$.

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