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Existence and Uniqueness Solution for a Mathematical Model with Mittag-Leffler Kernel

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Abstract: In this work, we analyse the fractional order West Nile Virus model involving the Atangana-Baleanu derivatives. Existence and uniqueness solutions were obtained by the fixed-point theorem. Another impressive aspect of the work is illustrated by simulations of different fractional orders by calculating the numerical solutions of the mathematical model.

Mittag-Leffler Çekirdeği ile bir Matematiksel Modelin Varlığı ve Tekliğinin Çözümü

Anahtar Kelimeler
Virüs, Matematiksel modelleme, Kesirli türev ve integraller, Nümerik çözüm

Öz: Bu çalışmada Atangana-Baleanu türevlerini içeren kesirli dereceli Batı Nil Virüsü modelini analiz ediyoruz. Varlık ve teklik çözümleri sabit nokta teoremi ile elde edildi. Çalışmanın bir diğer etkileyici yanı ise matematiksel modelin sayısal çözümlerinin hesaplanarak farklı kesirli derecelerin simülasyonları ile ortaya konulmasıdır.

1. INTRODUCTION

Perhaps one of the most prevalent ideas in applied mathematics is the notion of a derivative. This idea was first developed to explain how quickly a particular function changes, and it was later applied to create mathematical equations that explain how situations in the actual world behave. However, the idea was updated to the idea of fractional derivatives due to the complexity of the situations in the actual world. One of the most comprehensive books on the fractional derivative was handled by I. Podlubny. Here, all fractional derivative and integral operators that have contributed to the literature are discussed in detail with all their properties [17]. Kilbas et al. have discussed the fractional derivative in a comprehensive way and the applications of the fractional derivative are also shown [11]. It quickly became apparent that the fractional derivative notion was better suited than the local derivative for simulating real-world problems. It makes sense that many scholars have focused on creating a new definition of fractional derivative. Fractional derivative and integral have been used in many disciplines, including engineering, chemistry, physics, and others, as a result of this significant advancement [3, 4, 13]. Numerous applications made use of the Caputo fractional derivative [7]. However, due to kernel singularity, this concept has a significant flaw. We can see that this problem has been addressed by the work of Atangana and Baleanu [1], which changed the kernel. In addition, there are very important studies on the fractional derivative. For example, the stability of fractional differential equations (FDEs) whose parameters are unknown has been studied [14]. The authors [15] investigated the time and frequency domain characteristics of the circuit.
Bacteria and viruses are common ways for many diseases to spread from animals to people. Carriers have

the ability to spread bacteria or viruses to others, which has the potential to cause a pandemic. The measles, sometimes called as the "Flower" pandemic, is a term used to describe the epidemic that died an estimated 5 million people between 165 and 180 AD. Between 30 and 50 million people died during the Justinian Plague (1st Plague Outbreak) in the middle of the fifth century, which was brought on by a strain of the bacterium Yersinia pestis. Throughout history, there have been numerous more outbreaks that are comparable. The Spanish flu (1918-1919), HIV/AIDS (1981present), and the yellow fever outbreak (late 1800s) are the most common causes of death among them. The number of pandemic diseases has dramatically increased in the twenty-first century. Since the early 2000s, nations, continents, and possibly the entire planet have been in danger from virus-borne epidemics including SARS, Swine flu, Ebola, and MERS. One of the most significant recent instances is the coronavirus, also known as COVID-19, which is still active today and has already claimed many lives. Throughout human history, diseases have always existed and have caused the death of people. The analysis of mathematical modeling of epidemics by fractional derivative operators has been the subject of many studies. For example, the garden equation is analyzed with both Caputo and Caputo-Fabrizio fractional derivative operators [8]. The mathematical model of the virus, named COVID-19, which has recently affected the whole world, has been analyzed in detail with the Caputo-Fabrizio fractional derivative operator, and the existence and uniqueness of its solution has been examined. Numerical solutions are also included in the study [9]. The SIQR model is solved numerically with the help of Caputo Fractional derivative operator [12].

In this study, we analyzed West Nile (WN) Virus model. Human, equine, and avian neuropathogens include the flavivirus West Nile that is spread by mosquitoes. The virus is native to Asia, Africa, Australia and Europe. Recently, it has produced significant epidemics in Israel, Romania, and Russia. The WN virus was very recently discovered in North America after being discovered there in 1999 during a meningoencephalitis pandemic in New York City. The majority of WN virus infections in humans are asymptomatic, the risk of developing severe neuroinvasive disease and dying rises with age [6].

New information regarding the dynamics and epidemiology of WNV transmission was mentioned in Hayes et al [10]. They gave the chance to do this subject investigation as well. A reaction-diffusion model was created and examined by Lewis et al [16]. for the spatial distribution of the West Nile virus. Wonham et al. [19] offered a straightforward new analytical and graphical method for calculating the essential mosquito control levels from common public health indicators. A free boundary problem with a coupled system was taken into consideration by Tarboush et al. [18] in their study of the PDE and ODE models that represent the movement of birds and mosquitoes, respectively.

2. PRELIMINARIES

Fundamental definitions relating to fractional derivatives and integral operators are to be presented in this part [1, 7, 17].

Definition 2.1 *Following is a definition of the well-known fractional order Caputo derivative [7],*

$$ {}_a^C D_t^{\varepsilon}\phi(t) = \frac{1}{\Gamma(m-\varepsilon)}\int_a^t \frac{\phi^{(m)}(\rho)}{(t-\rho)^{\varepsilon+1-m}}d\rho, \quad (1)$$

$m-1<\varepsilon<m\in N$ with $\emptyset \in H^1(a,b), b>a$.

Definition 2.2 *The Riemann-Liouville (RL) fractional integral is defined as [17]:*

$$ J^{\varepsilon}\phi(t) = \frac{1}{\Gamma(\varepsilon)}\int_a^t \phi(\rho)(t-\rho)^{\varepsilon-1}d\rho. \quad (2)$$

Definition 2.3 *The definition of the Sobolev space of order 1 in (a,b) is [7]:*

$$ H^1(a,b) = \{u \in L^2(a,b) : u' \in L^2(a,b)\}. \quad (3)$$

Definition 2.4 *Let a function $g \in H^1(a,b)$ and $\varepsilon \in (0,1)$. The following is the definition of the AB fractional derivative in the Caputo and RL sense of order ε of g with a basis point a [1]:*

$$ {}_a^{ABC} D_t^{\varepsilon} g(t) = \frac{B(\varepsilon)}{1-\varepsilon}\int_a^t g'(s) * E_{\varepsilon}[\frac{\varepsilon}{1-\varepsilon}(\rho-t)^{\varepsilon}]d\rho, \quad (4)$$

and

$$ {}_a^{ABR} D_t^{\varepsilon} g(t) = \frac{B(\varepsilon)}{1-\varepsilon}\frac{d}{dt}\int_a^t g(\rho) * E_{\varepsilon}[\frac{\varepsilon}{1-\varepsilon}(\rho-t)^{\varepsilon}]d\rho, \quad (5)$$

where

$$ B(\varepsilon) = 1-\varepsilon+\frac{\varepsilon}{\Gamma(\varepsilon)},$$

Definition 2.5 *With base point a, the Atangana-Baleanu fractional integral of order ε is defined as [1]:*

$$ {}^{AB}I_t^{\varepsilon} g(t) = \frac{1-\varepsilon}{B(\varepsilon)}g(t) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_a^t g(\rho)(t-\rho)^{\varepsilon-1}d\rho.$$

3. THE MATHEMATICAL MODEL AND ITS DERIVATION

3.1. Classical Model

Bowman et al. [5] provided a mathematical model of the West Nile virus (WNV) in 2015. Given below is a mathematical model of how the female mosquitoes that feed on birds as intermediate hosts and disseminate the West Nile virus between humans and domestic animals.

$$\begin{aligned}
\frac{dM_S(t)}{dt} &= \Lambda_M - \frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)} - \eta_M M_S(t),\\
\frac{dM_i(t)}{dt} &= \frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)} - \eta_M M_i(t),\\
\frac{dB_S(t)}{dt} &= \Lambda_B - \frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)} - (\phi_B+\eta_B)B_S(t),\\
\frac{dB_i(t)}{dt} &= \frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)} - (\nu_B+\phi_B+\eta_B)B_i(t),\\
\frac{dS(t)}{dt} &= \Lambda_H - \frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)} - \eta_H S(t),\\
\frac{dE(t)}{dt} &= \frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)} - (\gamma+\eta_H)E(t),\\
\frac{dI(t)}{dt} &= \gamma E(t) - (\alpha+\nu_I - r+\eta_H)I(t),\\
\frac{dH(t)}{dt} &= \alpha I(t) - (\nu_H+\mu+\eta_H)H(t),\\
\frac{dR(t)}{dt} &= \mu H(t) + rI(t) - \eta_H R(t).
\end{aligned}$$

They fall under the first group's categories of susceptibility $S(t)$, exposure $E(t)$, infectiousness $I(t)$, hospitalization $H(t)$, and recovery $R(t)$. Alternatively, when formulated mathematically,

$$N_M(t) = M_S(t) + M_i(t), \quad N_B(t) = B_S(t) + B_i(t),$$
$$N_H(t) = S(t) + E(t) + I(t) + H(t) + R(t).$$

The recruitment rates of mosquitoes (assumed susceptible), birds (assumed susceptible), and insects (assumed susceptible) are described by the variables Λ_M, Λ_B, and Λ_H in the equation system above, respectively. The per capita rate of mosquito bites on the primary host (birds) and the per capita rate of mosquito bites on the human host are then described in b_1 and b_2, respectively. Additionally, ζ_1, ζ_2 and ζ_3 represent the likelihood that WNV will spread from an infected bird to a mosquito that is susceptible to the virus, the likelihood that WNV will spread from an infected mosquito to a bird that is susceptible to the virus, and the likelihood that WNV will spread from mosquitoes to humans, respectively. The natural death rates for humans η_H and for animals η_M are denoted by the symbols η_M and η_H, respectively. The rate of bird migration is ϕ_B. The percentage of birds dying because of WNV is ν_B. The pace at which WNV's clinical symptoms appear is γ. The death rates of people hospitalized and those caused by the WNV are denoted by the variables ν_I and ν_H, respectively. The natural recovery rate is r, the treatment-induced recovery rate is μ, and the hospitalization rate for infectious people is α [5].

Table 1. Values of the parameters of the system (12)

Parameter	Value
γ	[0,0.05]
θ_2	0.03
ζ_1	0.1245
η_1	2×10^7
μ_2	0.18
β	1×10^{-9}
α	1
η_3	10
ζ_2	5
θ_3	1×10^3

3.2. Existence Solution

Using a fixed-point technique, the existence of a solution for a fractional WNV mathematical model will be investigated. The following form results when the system (7) is expressed using the ABC fractional operator:

$$\begin{aligned}
{}_0^{ABC}D_t^{\varepsilon} M_S(t) &= \Lambda_M - \frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)} - \eta_M M_S(t),\\
{}_0^{ABC}D_t^{\varepsilon} M_i(t) &= \frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)} - \eta_M M_i(t),\\
{}_0^{ABC}D_t^{\varepsilon} B_S(t) &= \Lambda_B - \frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)} - (\phi_B+\eta_B)B_S(t),\\
{}_0^{ABC}D_t^{\varepsilon} B_i(t) &= \frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)} - (\nu_B+\phi_B+\eta_B)B_i(t),\\
{}_0^{ABC}D_t^{\varepsilon} S(t) &= \Lambda_H - \frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)} - \eta_H S(t),\\
{}_0^{ABC}D_t^{\varepsilon} E(t) &= \frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)} - (\gamma+\eta_H)E(t),\\
{}_0^{ABC}D_t^{\varepsilon} I(t) &= \gamma E(t) - (\alpha+\nu_I - r+\eta_H)I(t),\\
{}_0^{ABC}D_t^{\varepsilon} H(t) &= \alpha I(t) - (\nu_H+\mu+\eta_H)H(t),\\
{}_0^{ABC}D_t^{\varepsilon} R(t) &= \mu H(t) + rI(t) - \eta_H R(t).
\end{aligned}$$

under initial conditions that are not negative

$$M_{S_0}(0)=M_S(0), M_{i_0}(0)=M_i(0),$$
$$B_{i_0}(0)=B_i(0), E_0(0)=E(0), S_0(0)=S(0),$$
$$H_0(0)=H(0), I_0(0)=I(0), R_0(0)=R(0).$$

Using the definition (2.6), the system above can be expressed as follows:

$$\begin{aligned}
M_S(t)-M_S(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\Big(\Lambda_M-\frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)}-\eta_M M_S(t)\Big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\Big(\Lambda_M-\frac{b_1\zeta_1 M_S(y)B_i(y)}{N_B(y)}-\eta_M M_S(y)\Big)dy,\\
M_i(t)-M_i(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\Big(\frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)}-\eta_M M_i(t),\Big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\Big(\frac{b_1\zeta_1 M_S(y)B_i(y)}{N_B(y)}-\eta_M M_i(y),\Big)dy,\\
B_S(t)-B_S(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\Big(\Lambda_B-\frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)}-(\phi_B+\eta_B)B_S(t)\Big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\Big(\Lambda_B-\frac{b_1\zeta_2 M_i(y)B_s(y)}{N_B(y)}-(\phi_B+\eta_B)B_S(y)\Big)dy,\\
B_i(t)-B_i(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\Big(\frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)}-(\nu_B+\phi_B+\eta_B)B_i(t)\Big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\Big(\frac{b_1\zeta_2 M_i(y)B_s(y)}{N_B(y)}-(\nu_B+\phi_B+\eta_B)B_i(y)\Big)dy,\\
S(t)-S(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\Big(\Lambda_H-\frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)}-\eta_H S(t)\Big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\Big(\Lambda_H-\frac{b_2\zeta_3 M_i(y)S(y)}{N_H(y)}-\eta_H S(y)\Big)dy,\\
E(t)-E(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\Big(\frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)}-(\gamma+\eta_H)E(t)\Big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\Big(\frac{b_2\zeta_3 M_i(y)S(y)}{N_H(y)}-(\gamma+\eta_H)E(y)\Big)dy,\\
I(t)-I(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\big(\gamma E(t)-(\alpha+\nu_I-r+\eta_H)I(t)\big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\big(\gamma E(y)-(\alpha+\nu_I-r+\eta_H)I(y)\big)dy,\\
H(t)-H(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\big(\alpha I(t)-(\nu_H+\mu+\eta_H)H(t)\big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\big(\alpha I(y)-(\nu_H+\mu+\eta_H)H(y)\big)dy,\\
R(t)-R(0) &= \frac{1-\varepsilon}{B(\varepsilon)}\big(\mu H(t)+rI(t)-\eta_H R(t)\big)\\
&\quad+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)}\big(\mu H(y)+rI(y)-\eta_H R(y))\big)dy,
\end{aligned}$$

To keep the kernels simple, one can write as follows:

$$M_1(t,M_S)=\Lambda_M-\frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)}-\eta_M M_S(t),$$
$$M_2(t,M_i)=\frac{b_1\zeta_1 M_S(t)B_i(t)}{N_B(t)}-\eta_M M_i(t),$$
$$M_3(t,B_S)=\Lambda_B-\frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)}$$
$$-(\phi_B+\eta_B)B_S(t),$$
$$M_4(t,B_i)=\frac{b_1\zeta_2 M_i(t)B_s(t)}{N_B(t)}$$
$$-(\nu_B+\phi_B+\eta_B)B_i(t),$$
$$M_5(t,S)=\Lambda_H-\frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)}-\eta_H S(t),$$
$$M_6(t,E)=\frac{b_2\zeta_3 M_i(t)S(t)}{N_H(t)}-(\gamma+\eta_H)E(t),$$
$$M_7(t,I)=\gamma E(t)-(\alpha+\nu_I-r+\eta_H)I(t),$$
$$M_8(t,H)=\alpha I(t)-(\nu_H+\mu+\eta_H)H(t),$$
$$M_9(t,R)=\mu H(t)+rI(t)-\eta_H R(t).$$

and

$$\Psi_1=b_1\zeta_1\frac{L_4}{L_{10}}+\eta_M$$
$$\Psi_2=\eta_M$$
$$\Psi_3=b_1\zeta_2\frac{L_2}{L_{10}}+\phi_B+\eta_B$$
$$\Psi_4=\nu_B+\phi_B+\eta_B$$
$$\Psi_5=b_2\zeta_3\frac{L_2}{L_{12}}+\eta_H$$
$$\Psi_6=\gamma+\eta_H$$
$$\Psi_7=\alpha+\nu_I-r+\eta_H$$
$$\Psi_8=\nu_H+\mu+\eta_H$$
$$\Psi_9=\eta_H$$

It is assumed that C: $M_S(t), M_i(t), B_S(t), B_i(t), S(t), E(t), I(t), H(t), R(t), M_{S1}(t), M_{i1}(t), B_{S1}(t), B_{i1}(t), S_1(t), E_1(t), I_1(t), H_1(t), R_1(t) \in L[0,1]$, are continuous functions, so that $||M_S(t)|| \le L_1$, $||M_i(t)|| \le L_2$, $||B_S(t)|| \le L_3$, $||B_i(t)|| \le L_4$, $||S(t)|| \le L_5$, $||E(t)|| \le L_6$, $||I(t)|| \le L_7$, $||H(t)|| \le L_8$, $||R(t)|| \le L_9$ respectively. Also,

$$N_B(t)=B_i(t)+B_S(t)\le L_3+L_4=L_{10}$$
$$N_M(t)=M_i(t)+M_S(t)\le L_1+L_2=L_{11}.$$

Theorem 3.1 *If the assumption C is true, the kernels $M_i, i=1,2,3,\dots,9$ satisfythe Lipschitz condition and are contraction s provided that* $\Psi_i<1$ *for* $(\forall \in i=1\dots 9)$

Proof 3.1 *We now demonstrate that the Lipschitz condition is satisfied by $M_1(t,M_S)$. Let $M_S(t)$ and $M_{S_1}(t)$ be two different functions.*

$$\|M_1(t,M_S)-M_1(t,M_{S_1})\| =\|(\Lambda_M-\frac{b_1\zeta_1 M_S B_i}{N_B}-\eta_M M_S) -(\Lambda_M-\frac{b_1\zeta_1 M_{S_1} B_i}{N_B}-\eta_M M_{S_1})\| \le (b_1\zeta_1\frac{L_4}{L_{10}}+\eta_M)\|M_S-M_{S_1}\| \le \Psi_1\|M_S-M_{S_1}\|.$$

We demonstrate that the Lipschitz condition is satisfied by $M_2(t,M_i)$. Let's assume there are two functions, $M_i(t)$. and $M_{i_1}(t)$.

$$\|M_2(t,M_i)-M_2(t,M_{i_1})\| =\left\|\left(\frac{b_1\zeta_1 M_S B_i}{N_B}-\eta_M M_i\right)-\left(\frac{b_1\zeta_1 M_S B_i}{N_B}-\eta_M M_{i_1}\right)\right\| \le \eta_M\|M_i-M_{i_1}\| \le \Psi_2\|M_i-M_{i_1}\|.$$

We demonstrate that the Lipschitz condition is satisfied by $M_3(t,B_S)$. Let's asuume there are two functions, $B_S(t)$ and $B_{S_1}(t)$.

$$\|M_3(t,B_S)-M_3(t,B_{S_1})\| =\|(\Lambda_B-\frac{b_1\zeta_2 M_i B_S}{N_B}-(\phi_B+\eta_B)B_S) -(\Lambda_B-\frac{b_1\zeta_2 M_i B_{S_1}}{N_B}-(\phi_B+\eta_B)B_{S_1})\| \le (b_1\zeta_2\frac{L_2}{L_{10}}+\phi_B+\eta_B)\|B_S-B_{S_1}\| \le \Psi_3\|B_S-B_{S_1}\|.$$

We demonstrate that the Lipschitz condition is satisfied by $M_4(t,B_i)$. Let's asuume there are two functions, $B_i(t)$ and$B_{i_1}(t)$.

$$\|M_4(t,B_i)-M_4(t,B_{i_1})\| =\left\|\begin{pmatrix}\frac{b_1\zeta_2 M_i B_S}{N_B}-\\ (\nu_B+\phi_B+\eta_B)B_i\end{pmatrix}-\begin{pmatrix}\frac{b_1\zeta_2 M_i B_S}{N_B}-\\ (\nu_B+\phi_B+\eta_B)B_{i_1}\end{pmatrix}\right\| \le (\nu_B+\phi_B+\eta_B)\|B_i-Bi_1\| \le \Psi_4\|B_i-Bi_1\|.$$

We demonstrate that the Lipschitz condition is satisfied by $M_5(t,S)$. Let's asuume there are two functions, $S(t)$ and $S_1(t)$.

$$\| \mathrm{M}_5(t,S)-\mathrm{M}_5(t,S_1)\| \begin{aligned} &=\| (\Lambda_H-\frac{b_2\zeta_3 M_i S}{N_H}-\eta_H S) \\ &-(\Lambda_H-\frac{b_2\zeta_3 M_i S_1}{N_H}-\eta_H S_1)\| \\ &\le (b_2\zeta_3\frac{\mathrm{L}_2}{\mathrm{L}_{12}}+\eta_H)\| S-S_1\| \\ &\le \Psi_5 \| S-S_1\| . \end{aligned}$$

We demonstrate that the Lipschitz condition is satisfied by $M_6(t,E)$. Let's asuume there are two functions, $E(t)$ and $E_1(t)$.

$$\| \mathrm{M}_6(t,E)-\mathrm{M}_6(t,E_1)\| \begin{aligned} &=\left\| \left(\frac{b_2\zeta_3 M_i S}{N_H}-(\gamma+\eta_H)E\right)\right. \\ &\left.-\left(\frac{b_2\zeta_3 M_i S}{N_H}-(\gamma+\eta_H)E_1\right)\right\| \\ &\le (\gamma+\eta_H)\| E-E_1\| \\ &\le \Psi_6 \| E-E_1\| . \end{aligned}$$

We demonstrate that the Lipschitz condition is satisfied by $M_7(t,I)$. Let's asuume there are two functions, $I(t)$ and $I_1(t)$.

$$\| \mathrm{M}_7(t,I)-\mathrm{M}_7(t,I_1)\| \begin{aligned} &=\| (\gamma E-(\alpha+v_I-r+\eta_H)I) \\ &-(\gamma E-(\alpha+v_I-r+\eta_H)I_1)\| \\ &\le (\alpha+v_I-r+\eta_H)\| I-I_1\| \\ &\le \Psi_7 \| I-I_1\| . \end{aligned}$$

We demonstrate that the Lipschitz condition is satisfied by $M_8(t,H)$. Let's asuume there are two functions, $H(t)$ and $H_1(t)$.

$$\| \mathrm{M}_8(t,H)-\mathrm{M}_8(t,H_1)\| \begin{aligned} &=\| (\alpha I-(v_H+\mu+\eta_H)H) \\ &-(\alpha I-(v_H+\mu+\eta_H)H_1)\| \\ &\le (v_H+\mu+\eta_H))\| H-H_1\| \\ &\le \Psi_8 \| H-H_1\| . \end{aligned}$$

We demonstrate that the Lipschitz condition is satisfied by $M_9(t,R)$. Let's asuume there are two functions, $R(t)$ and $R_1(t)$.

$$\| \mathrm{M}_9(t,R)-\mathrm{M}_9(t,R_1)\| \begin{aligned} &=\| (\mu H+rI-\eta_H R) \\ &-(\mu H+rI-\eta_H R_1)\| \\ &\le (\eta_H)\| R-R_1\| \\ &\le \Psi_9 \| R-R_1\| . \end{aligned}$$

The kernels $M_i, i=1,2,3,\ldots,9$ are contractions with $\Psi_i<1$ $I\backslash in\ 1,\ldots 9$, and they satisfy the Lipschitz conditions. The proof is complete.

We rewrite the system given as follows, using the kernels $M_i, i=1,2,3,\ldots,9$ and all initial conditions being zero:

$$\begin{aligned}
M_S(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_1(t,M_S(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_1(y,M_S(y))dy, \\
M_i(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_2(t,M_i(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_2(y,M_i(y))dy, \\
B_S(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_3(t,B_S(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_3(y,B_S(y))dy, \\
B_i(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_4(t,B_i(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_4(y,B_i(y))dy, \\
S(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_5(t,S(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_5(y,S(y))dy, \\
E(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_6(t,E(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_6(y,E(y))dy, \\
I(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_7(t,I(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_7(y,I(y))dy, \\
H(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_8(t,H(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_8(y,H(y))dy, \\
R(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_9(t,R(t))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_9(y,R(y))dy.
\end{aligned} \tag{10}$$

Next, we have the following system of equations defined via recursive formulas:

$$
\begin{aligned}
M_{S_n}(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_1(t, M_{S_{n-1}}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_1(y, M_{S_{n-1}}(y))dy,\\
M_{i_n}(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_2(t, M_{i_{n-1}}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_2(y, M_{i_{n-1}}(y))dy,\\
B_{S_n}(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_3(t, B_{S_{n-1}}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_3(y, B_{S_{n-1}}(y))dy,\\
B_{i_n}(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_4(t, B_{i_{n-1}}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_4(y, B_{i_{n-1}}(y))dy,\\
S_n(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_5(t, S_{n-1}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_5(y, S_{n-1}(y))dy,\\
E_n(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_6(t, E_{n-1}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_6(y, E_{n-1}(y))dy,\\
I_n(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_7(t, I_{n-1}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_7(y, I_{n-1}(y))dy,\\
H_n(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_8(t, H_{n-1}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_8(y, H_{n-1}(y))dy,\\
R_n(t) &= \frac{1-\varepsilon}{B(\varepsilon)}\mathrm{M}_9(t, R_{n-1}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \mathrm{M}_9(y, R_{n-1}(y))dy.
\end{aligned} \tag{11}
$$

Additionally, each equation's difference can be expressed as follows:

$$
(M_{S_{n+1}} - M_{S_n})(t) = \frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_1(t, M_{S_n}(t)) - \mathrm{M}_1(t, M_{S_{n-1}}(t))) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} (\mathrm{M}_1(y, M_{S_n}(y)) - \mathrm{M}_1(y, M_{S_{n-1}}(y)))dy,
$$

$$
(M_{i_{n+1}} - M_{i_n})(t) = \frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_2(t, M_{i_n}(t)) - \mathrm{M}_2(t, M_{i_{n-1}}(t))) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} (\mathrm{M}_2(t, M_{i_n}(y)) - \mathrm{M}_2(t, M_{i_{n-1}}(y)))dy,
$$

$$
(B_{S_{n+1}} - B_{S_n})(t) = \frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_3(t, B_{S_n}(t)) - \mathrm{M}_3(t, B_{S_{n-1}}(t))) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} (\mathrm{M}_3(t, B_{S_n}(y)) - \mathrm{M}_3(t, B_{S_{n-1}}(y)))dy,
$$

$$
(B_{i_{n+1}} - B_{i_n})(t) = \frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_4(t, B_{i_n}(t)) - \mathrm{M}_4(t, B_{i_{n-1}}(t))) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} (\mathrm{M}_4(y, B_{i_n}(y)) - \mathrm{M}_4(y, B_{i_{n-1}}(y)))dy,
$$

$$
(S_{n+1} - S_n)(t) = \frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_5(t, S_n(t)) - \mathrm{M}_5(t, S_{n-1}(t))) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} (\mathrm{M}_5(t, S_n(y)) - \mathrm{M}_5(t, S_{n-1}(y)))dy,
$$

$$
(E_{n+1} - E_n)(t) = \frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_6(t, E_n(t)) - \mathrm{M}_6(t, E_{n-1}(t))) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} (\mathrm{M}_6(y, E_n(y)) - \mathrm{M}_6(y, E_{n-1}(y)))dy,
$$

$$(I_{n+1}-I_n)(t)=\frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_7(t,I_n(t))-\mathrm{M}_7(t,I_{n-1}(t)))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}(\mathrm{M}_7(t,I_n(y))-\mathrm{M}_7(t,I_{n-1}(y)))dy,$$

$$(H_{n+1}-H_n)(t)=\frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_8(t,H_n(t))-\mathrm{M}_8(t,H_{n-1}(t)))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}(\mathrm{M}_8(t,H_n(y))-\mathrm{M}_8(t,H_{n-1}(y)))dy,$$

$$(R_{n+1}-R_n)(t)=\frac{1-\varepsilon}{B(\varepsilon)}(\mathrm{M}_9(t,R_n(t))-\mathrm{M}_9(t,R_{n-1}(t)))+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}(\mathrm{M}_9(t,R_n(y))-\mathrm{M}_9(t,R_{n-1}(y)))dy.$$

The norm of the two sides of the aforementioned equations, when taken,

$$\|(M_{S_{n+1}}-M_{S_n})(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_1(t,M_{S_n}(t))-\mathrm{M}_1(t,M_{S_{n-1}}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_1(y,M_{S_n}(y))-\mathrm{M}_1(y,M_{S_{n-1}}(y)))\|dy,$$

$$\|(M_{i_{n+1}}-M_{i_n})(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_2(t,M_{i_n}(t))-\mathrm{M}_2(t,M_{i_{n-1}}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_2(t,M_{i_n}(y))-\mathrm{M}_2(t,M_{i_{n-1}}(y)))\|dy,$$

$$\|(B_{S_{n+1}}-B_{S_n})(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_3(t,B_{S_n}(t))-\mathrm{M}_3(t,B_{S_{n-1}}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_3(t,B_{S_n}(y))-\mathrm{M}_3(t,B_{S_{n-1}}(y)))\|dy,$$

$$\|(B_{i_{n+1}}-B_{i_n})(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_4(t,B_{i_n}(t))-\mathrm{M}_4(t,B_{i_{n-1}}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_4(y,B_{i_n}(y))-\mathrm{M}_4(y,B_{i_{n-1}}(y)))\|dy,$$

$$\|(S_{n+1}-S_n)(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_5(t,S_n(t))-\mathrm{M}_5(t,S_{n-1}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_5(t,S_n(y))-\mathrm{M}_5(t,S_{n-1}(y)))\|dy,$$

$$\|(E_{n+1}-E_n)(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_6(t,E_n(t))-\mathrm{M}_6(t,E_{n-1}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_6(y,E_n(y))-\mathrm{M}_6(y,E_{n-1}(y)))\|dy,$$

$$\|(I_{n+1}-I_n)(t)\|=\frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_7(t,I_n(t))-\mathrm{M}_7(t,I_{n-1}(t)))\|+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t(t-y)^{\varepsilon-1}\|(\mathrm{M}_7(t,I_n(y))-\mathrm{M}_7(t,I_{n-1}(y)))\|dy,$$

$$\|(H_{n+1}-H_n)(t)\| = \frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_8(t,H_n(t))-\mathrm{M}_8(t,H_{n-1}(t)))\| + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1}\|(\mathrm{M}_8(t,H_n(y))-\mathrm{M}_8(t,H_{n-1}(y)))\|\,dy,$$

$$\|(R_{n+1}-R_n)(t)\| = \frac{1-\varepsilon}{B(\varepsilon)}\|(\mathrm{M}_9(t,R_n(t))-\mathrm{M}_9(t,R_{n-1}(t)))\| + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1}\|(\mathrm{M}_9(t,R_n(y))-\mathrm{M}_9(t,R_{n-1}(y)))\|\,dy.$$

Theorem 3.2 *If the following inequality is reached, a solution to the West Nile Virus mathematical model (9) can be found:*

$$\Upsilon = max\{\Psi_i\} < 1, \quad i = 1,2,\dots 9. \tag{12}$$

Proof 3.2 *Let us consider the following equations,*

$$\mathrm{K}_{1n}(t) = M_{S_{n+1}}(t) - M_S(t), \qquad \mathrm{K}_{2n}(t) = M_{i_{n+1}}(t) - M_i(t),$$

$$\mathrm{K}_{3n}(t) = B_{S_{n+1}}(t) - B_S(t), \qquad \mathrm{K}_{4n}(t) = B_{i_{n+1}}(t) - B_i(t),$$

$$\mathrm{K}_{5n}(t) = S_{n+1}(t) - S(t), \qquad \mathrm{K}_{6n}(t) = E_{n+1}(t) - E(t),$$

$$\mathrm{K}_{7n}(t) = I_{n+1}(t) - I(t), \qquad \mathrm{K}_{8n}(t) = H_{n+1}(t) - H(t),$$

$$\mathrm{K}_{9n}(t) = R_{n+1}(t) - R(t).$$

We start with $\mathrm{K}_{1n}(t)$,

$$\begin{aligned}\|\mathrm{K}_{1n}(t)\| &\le \frac{1-\varepsilon}{B(\varepsilon)}\|\mathrm{M}_1(t,M_{S_n}(t)) - \mathrm{M}_1(t,M_S(t))\| \\ &+ \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1}\|\mathrm{M}_1(y,M_{S_n}(y)) - \mathrm{M}_1(y,M_S(y))\|\,dy \\ &\le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)\Psi_1\|M_{S_n} - M_S\| \\ &\le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|M_S - M_{S_1}\|.\end{aligned} \tag{13}$$

Similarly,

$$\|\mathrm{K}_{2n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|M_i - M_{i_1}\|,$$

$$\|\mathrm{K}_{3n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|B_S - B_{S_1}\|,$$

$$\|\mathrm{K}_{4n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|B_i - B_{i_1}\|,$$

$$\|\mathrm{K}_{5n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|S - S_1\|,$$

$$\|\mathrm{K}_{6n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|E - E_1\|,$$

$$\|\mathrm{K}_{7n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|I - I_1\|,$$

$$\|\mathrm{K}_{8n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|H - H_1\|,$$

$$\|\mathrm{K}_{9n}(t)\| \le \left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)^n \Upsilon^n \|R - R_1\|.$$

It can be found that $\mathrm{K}_{in}(t) \to 0, i = 1,2,\dots 9,$ as $n \to \infty$ for $\Upsilon < 1$. This completes the proof.

3.3. Uniqueness Solution

In this section, we will demonstrate the mathematical model for the West Nile Virus's uniqueness of solution.

Theorem 3.3 *In the situation where the following inequality holds true, the WNV model (9) has an unique solution:*

$$\left(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\right)\Psi_i \le 1, \quad i = 1,2,\dots 9.$$

Proof 3.3 *Let us assume that the system (9) has solutions* $M_S(t), M_\mathrm{i}(t), B_S(t), B_\mathrm{i}(t), S(t), E(t), I(t), H(t), R(t)$, *as well as* $\widetilde{M_S}(t), \widetilde{M_\iota}(t), \widetilde{B_S}(t), \widetilde{B_\iota}(t), \tilde{S}(t), \tilde{E}(t), \tilde{I}(t), \tilde{H}(t), \tilde{R}(t)$. *Then, the system can also be written as,*

$$\tilde{M}_S(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_1(t,\tilde{M}_S(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_1(y,\tilde{M}_S(y))dy,$$

$$\tilde{M}_i(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_2(t,\tilde{M}_i(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_2(y,\tilde{M}_i(y))dy,$$

$$\tilde{B}_S(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_3(t,\tilde{B}_S(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_3(y,\tilde{B}_S(y))dy,$$

$$\tilde{B}_i(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_4(t,\tilde{B}_i(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_4(y,\tilde{B}_i(y))dy,$$

$$\tilde{S}(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_5(t,\tilde{S}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_5(y,\tilde{S}(y))dy,$$

$$\tilde{E}(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_6(t,\tilde{E}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_6(y,\tilde{E}(y))dy,$$

$$\tilde{I}(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_7(t,\tilde{I}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_7(y,\tilde{I}(y))dy,$$

$$\tilde{H}(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_8(t,\tilde{H}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_8(y,\tilde{H}(y))dy,$$

$$\tilde{R}(t) = \frac{1-\varepsilon}{B(\varepsilon)}M_9(t,\tilde{R}(t)) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{(\varepsilon-1)} M_9(y,\tilde{R}(y))dy.$$

When the norm is determined for the two above systems of equations, first

$$\begin{aligned}\| M_S(t)-\tilde{M}_S(t)\| &\le \frac{1-\varepsilon}{B(\varepsilon)}\| M_1(t,M_S(t)) - M_1(t,\tilde{M}_S(t))\| \\ &+ \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-y)^{\varepsilon-1} \| M_1(y,M_S(y)) - M_1(y,\tilde{M}_S(y))\| dy \\ &\le \frac{1-\varepsilon}{B(\varepsilon)}\Psi_1 \| M_S - \tilde{M}_S \| \\ &+ \frac{\Psi_1}{B(\varepsilon)\Gamma(\varepsilon)}\| M_S - \tilde{M}_S \| .\end{aligned} \quad (15)$$

The following inequality can be written,

$$\left(\frac{1-\varepsilon}{B(\varepsilon)}\Psi_1 + \frac{\Psi_1}{B(\varepsilon)\Gamma(\varepsilon)} - 1\right)\| M_S - \tilde{M}_S \| \ge 0.$$

Thus, $\| M_S - \tilde{M}_S \| = 0, ise \quad M_S(t) = \tilde{M}_S(t)$ This implies

$M_i(t) = \tilde{M}_i(t), B_S(t) = \tilde{B}_S(t), B_i(t) = \tilde{B}_i(t), S(t) = \tilde{S}(t),$
$E(t) = \tilde{E}(t), I(t) = \tilde{I}(t), H(t) = \tilde{H}(t), R(t) = \tilde{R}(t)$

Thus, the model has a unique solution.

4. NUMERICAL SIMULATIONS

The Atangana-Baleanu fractional derivative [2] will be discretized using the approach for fractional differential equations in this section. The following differential equation of fractional order is first taken into consideration.

$${}_0^{ABC}D_t^{\varepsilon} x(t) = (f(t,x(t))),$$

The following equation is created when the equation (16) is arranged.

$$x(t)-x(0) = \frac{1-\varepsilon}{ABC(\varepsilon)} f(t,x(t)) + \frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-\tau)^{\varepsilon-1} f(\tau,x(\tau))d\tau,$$

consequently,

$$x(t_{n+1})-x(0) = \frac{1-\varepsilon}{ABC(\varepsilon)} f(t_n,x_n) + \frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)}\int_0^{t_{n+1}} (t_{n+1}-t)^{\varepsilon-1} f(t,x(t))dt,$$

and at t_n

$$x(t_n)-x(0)=\frac{1-\varepsilon}{ABC(\varepsilon)}f(t_{n-1},x_{n-1})+ \frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)}\int_0^{t_n}(t_n-t)^{\varepsilon-1}f(t,x(t))dt, \quad (19)$$

The following equation can be discovered by combining equations (18) and (19).

$$x(t_{n+1})-x(t_n) = \frac{1-\varepsilon}{ABC(\varepsilon)}\{f(t_n,x_n)-f(t_{n-1},x_{n-1})\} + \frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)} \times\int_0^{t_{n+1}}(t_{n+1}-t)^{\varepsilon-1}f(t,x(t))dt - \frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)} \int_0^{t_n}(t_n-t)^{\varepsilon-1}f(t,x(t))dt$$

Thus,

$$x(t_{n+1})-x(t_n)=\frac{1-\varepsilon}{ABC(\varepsilon)}\{f(t_n,x_n)- f(t_{n-1},x_{n-1})\}+A_{\varepsilon,1}-A_{\varepsilon,2}$$

Without loss of generality,

$$A_{\varepsilon,1}=\frac{\varepsilon f(t_n,x_n)}{ABC(\varepsilon)\Gamma(\varepsilon)h}\{\frac{2ht_{n+1}^{\varepsilon}}{\varepsilon}-\frac{t_{n+1}^{\varepsilon+1}}{\varepsilon+1}\}- \frac{\varepsilon f(t_{n-1},x_{n-1})}{ABC(\varepsilon)\Gamma(\varepsilon)h}\{\frac{ht_{n+1}^{\varepsilon}}{\varepsilon}-\frac{t_{n+1}^{\varepsilon+1}}{\varepsilon+1}\}$$

and similiarly,

$$A_{\varepsilon,2}=\frac{\varepsilon f(t_n,x_n)}{ABC(\varepsilon)\Gamma(\varepsilon)h}\{\frac{ht_n^{\varepsilon}}{\varepsilon}-\frac{t_n^{\varepsilon+1}}{\varepsilon+1}\}- \frac{f(t_{n-1},x_{n-1})}{ABC(\varepsilon)\Gamma(\varepsilon)h}$$

As a result,

$$x_{n+1} = x_n+f(t_n,x_n)\{\frac{1-\varepsilon}{ABC(\varepsilon)}+\frac{\varepsilon}{ABC(\varepsilon)h} [\frac{2ht_{n+1}^{\varepsilon}}{\varepsilon}-\frac{t_{n+1}^{\varepsilon+1}}{\varepsilon+1}] -\frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)h}[\frac{ht_n^{\varepsilon}}{\varepsilon}-\frac{t_n^{\varepsilon+1}}{\varepsilon+1}]\}+f(t_{n-1},x_{n-1}) \times\{\frac{\varepsilon-1}{ABC(\varepsilon)}-\frac{\varepsilon}{hABC(\varepsilon)\Gamma(\varepsilon)} [\frac{ht_{n+1}^{\varepsilon}}{\varepsilon}-\frac{t_{n+1}^{\varepsilon+1}}{\varepsilon+1}+\frac{t^{\varepsilon+1}}{hABC(\varepsilon)\Gamma(\varepsilon)}]\}$$

Theorem 4.1 *Let* $x(t)$ *be a soluiton of*

$${}_0^{ABC}D_t^{\varepsilon}x(t)=f(t,x(t)),$$

with f being continuous and bounded, the numerical solution of $x(t)$ is given as

$$x_{n+1} = x_n+f(t_n,x_n)\{\frac{1-\varepsilon}{ABC(\varepsilon)}+\frac{\varepsilon}{ABC(\varepsilon)h} [\frac{2ht_{n+1}^{\varepsilon}}{\varepsilon}-\frac{t_{n+1}^{\varepsilon+1}}{\varepsilon+1}] -\frac{\varepsilon}{ABC(\varepsilon)\Gamma(\varepsilon)h}[\frac{ht_n^{\varepsilon}}{\varepsilon}-\frac{t_n^{\varepsilon+1}}{\varepsilon+1}]\}+f(t_{n-1},x_{n-1}) \times\{\frac{\varepsilon-1}{ABC(\varepsilon)}-\frac{\varepsilon}{hABC(\varepsilon)\Gamma(\varepsilon)} [\frac{ht_{n+1}^{\varepsilon}}{\varepsilon}-\frac{t_{n+1}^{\varepsilon+1}}{\varepsilon+1}+\frac{t^{\varepsilon+1}}{hABC(\varepsilon)\Gamma(\varepsilon)}]\}+R_{\varepsilon}$$

where $\|R_{\varepsilon}\|_{\infty}<M$.

4.1. Numerical simulations for the model

This section presents the model's simulations. Using the method outlined in the preceding section, numerical results were obtained and graphically presented [2]. For various values of the fractional derivative, numerical results are displayed. The simulation of $M_S(t), M_{\text{i}}(t), B_S(t), B_{\text{i}}(t), S(t)$, $E(t), I(t), H(t), R(t)$,for diffeent value of ε.

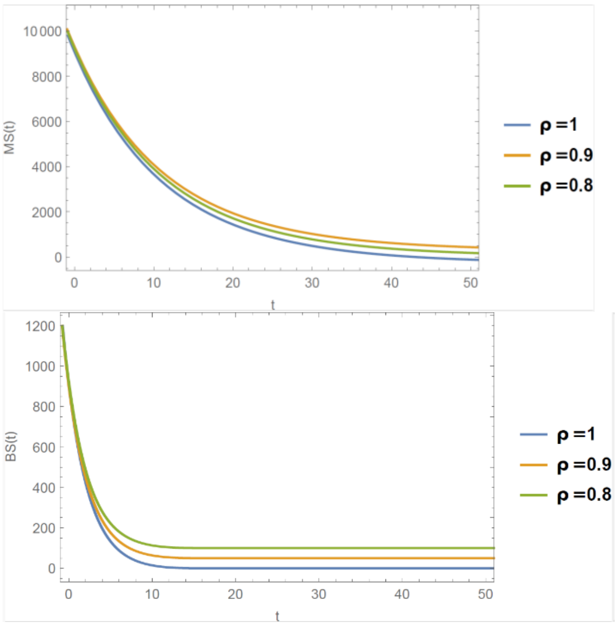

Figure 1. Simulations of the M_S and B_S function for different ε values.

In Fig.1, M_S and B_S functions are simulated. As is known, in the mathematical model, M_S and B_S represent susceptible mosquito and bird populations, respectively. It is seen that both populations decrease with time. This indicates that the susceptible population becomes infected over time. As can be expected, the suspected population in virus spreads is expected to decrease over time. The first of the important reasons for this is that a large part of the population is infected.

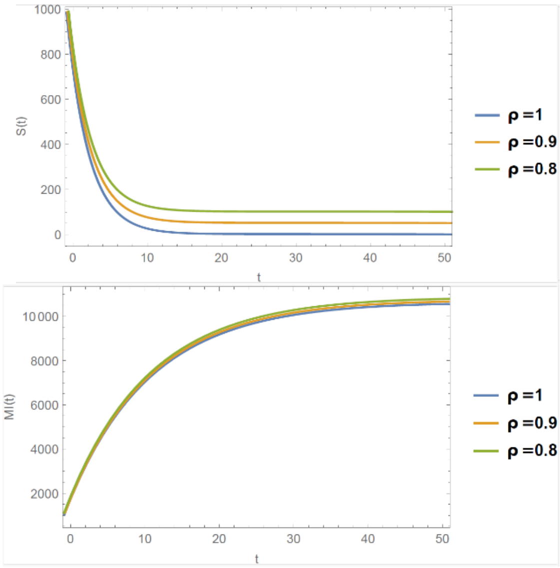

Figure 2. Simulations of the S and M_I function for different ε values.

In Fig.2, S and M_I functions are simulated. S and M_I represent susceptible human and infected mosquito populations, respectively. While the susceptible human population is decreasing over time, the infected mosquito population is increasing. The infected mosquito population has increased over time because it has passed on the virus from an infected human or bird population. The suspected human population is also naturally declining. One of the main reasons is that they are infected.

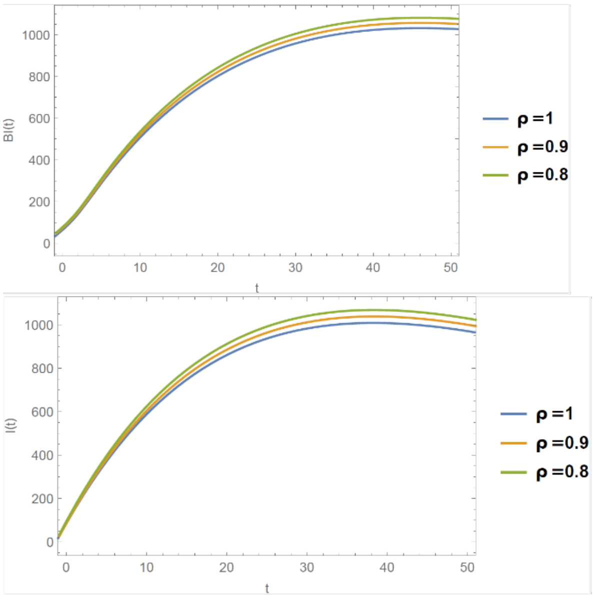

Figure 3. Simulations of the B_I and I function for different ε values.

In Fig.3, B_I and I functions are simulated. B_I and I represent susceptible infected bird populations and infectious human populations, respectively. According to the graph, it is seen that both populations have increased. Considering that the infected bird population and the virus are transmitted to humans, it is a normal situation.

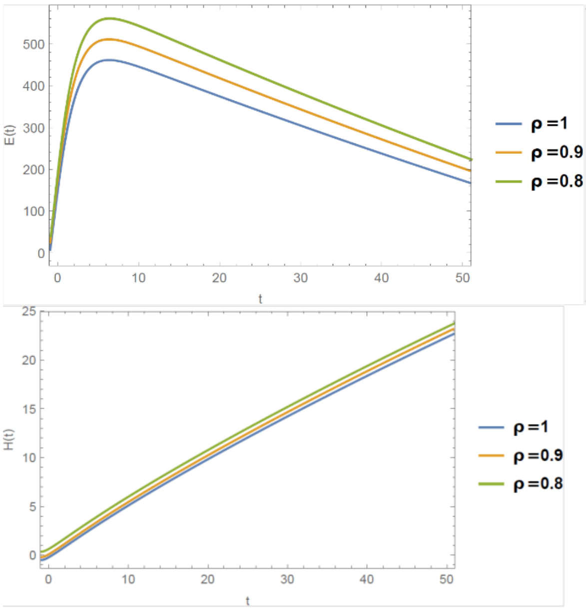

Figure 4. Simulations of the E and H function for different ε values.

In Fig.4, E and H functions are simulated. E and H represent the infected human population and the hospitalized human population, respectively. The infected human population has increased very rapidly for a certain period of time and then declined after a certain time. Among the reasons for this, it can be said that the development of treatment against the virus and the discovery of a vaccine. It has been observed that the population of people treated in the hospital is constantly increasing. This clearly demonstrates the rapid spread of the virus.

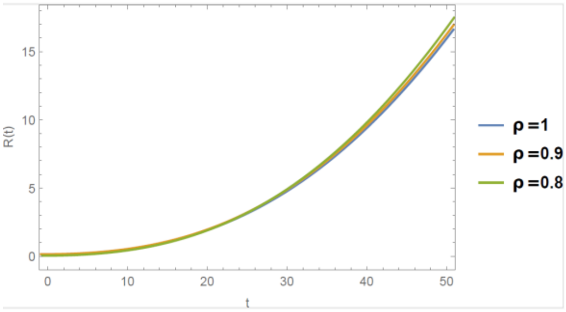

Figure 5. Simulations of the R function for different ε values.

In Fig.5, R function are simulated. R represents the recovered human population. Naturally, over time, the population of people recovering from infection is constantly increasing. This clearly shows that there is an effective treatment method against the virus

5. CONCLUSION

In this study, the West Nile Virus mathematical model was analyzed. This model is extended with the help of the Atangana-Baleanu fractional derivative operator and the existence, uniqueness and stability of its solution are analyzed. In addition, the mathematical model is solved with the Adam-Bashford numerical approach. As a result, the graphics of the mathematical model called West Nile Virus were analyzed. Three types of populations were studied in this mathematical model. These are mosquito, bird and human populations, respectively. It is easy to see how the virus affects these three populations with the help of graphs. On the other hand, the model, which was analyzed by expanding it to the Atangana-Baleanu fractional derivative, was examined for fractional derivative values of different orders. When the graphs are examined, accurate and logical results have been obtained for the fractional derivative values of different orders.

The effect of the West Nile virus affects many countries in certain periods. Even in Greece in 2022, many people were affected by this virus. If timely measures are not taken, it can cause significant damage to human metabolism and even result in death. Therefore, it is important to analyze this virus using control theory, different fractional derivative operators, and examining it with different numerical methods.

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