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Coloring sums of extensions of certain graphs

Research Article

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Abstract: We recall that the minimum number of colors that allow a proper coloring of graph G is called the chromatic number of G and denoted $\chi(G)$. Motivated by the introduction of the concept of the b-chromatic sum of a graph the concept of χ'-chromatic sum and χ^+-chromatic sum are introduced in this paper. The extended graph G^x of a graph G was recently introduced for certain regular graphs. This paper furthers the concepts of χ'-chromatic sum and χ^+-chromatic sum to extended paths and cycles. Bipartite graphs also receive some attention. The paper concludes with patterned structured graphs. These last said graphs are typically found in chemical and biological structures.

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1. Introduction

For general notation and concepts in graph and digraph theory, we refer to [2, 4, 5]. Unless mentioned otherwise, all graphs mentioned in this paper are simple, connected, finite and undirected graphs of order $n \geq 2$. We recall that the minimum number of colors that allow a proper coloring of graph G is called the chromatic number of G and denoted $\chi(G)$. Since coloring may be effected from any alphabet like in an application of cryptographical coding of vertices, we use a preferred definition slightly different from the contemporary definition found in the literature. Consider a *proper k-coloring* of a graph G and denote the set of k colors, $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_k\}$. Also consider the disjoint subsets of $V(G)$ i.e. $V_{c_i} = \{v_j : v_j \to c_i, v_j \in V(G), c_i \in \mathcal{C}\}$, $1 \leq i \leq k$. Clearly, $V(G) = \bigcup\limits_{i=1}^{k} V_{c_i}$. If for largest $k \in \mathbb{N}$, a proper k-coloring is found such that, for all color class $V_{\mathcal{C}_i}$; $1 \leq i \leq k$, there exists at least one $v_s \in V_{\mathcal{C}_i}$ such that $v_s v_t \in E(G)$, for at least one $v_t \in V_{\mathcal{C}_j}$; $1 \leq j \leq k$, $i \neq j$, then the b-chromatic number of G is defined to be $\varphi(G) = k$. Such a coloring is called a b-coloring of G.

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The concept of extending regular graphs has been introduced in [1]. We shall use the idea of extending a graph (also called an extended graph) in general and study how certain interesting invariants change in the extended graph.

2. On extensions of graphs

We formally define the notion of a *degree-extension* of a graph G.

Definition 2.1. *(i) Let G be a graph of even order $n \geq 2$. If the complement graph G^c has a perfect matching say M, then $G^x_c = G + M$ is called a complete degree-extension of G, also called an c-extended G.*
(ii) Let G be a graph of odd order $n \geq 3$ and let $u \in V(G)$ such that $d_G(u) = \Delta(G)$. If the complement graph $(G-u)^c$ has a perfect matching say M, then $G^x_{ic} = G+M$ is called an incomplete degree-extension of G, also called an ic-extended G.

The choices of vertex u, $d_G(u) = \Delta(G)$ in Definition 2.1(ii) is by default. Other criteria can apply of which u, $d_G(u) = \delta(G)$ is an obvious other criteria. An immediate and important result following from Definition 2.1 is given by the next corollary.

Corollary 2.2. *(i) In G^x_c, $d_{G^x_c}(v) = d_G(v) + 1$, $\forall v \in V(G)$.*
(ii) In G^x_{ic}, $d_{G^x_{ic}}(v) = d_G(v) + 1$, $\forall v \in V(G)$, $v \neq u$ and $\Delta(G^x_{ic}) \in \{\Delta(G), \Delta(G) + 1\}$.

In [1] it is shown that an r-regular graph G of even order n with $r < \frac{n}{2}$ always has a degree-extension G^x_c. It means that G can always be extended to an $(r+1)$-regular graph G^x_c by adding edges to G. First, we generalise the result to any simple connected graph G for which $\Delta(G) < \frac{n}{2}$.

Theorem 2.3. *Let G be a graph of order n, n even with $\Delta(G) < \frac{n}{2}$, then graph G has a complete degree-extension G^x_c. Also, if n is odd and a unique vertex u exists such that $d_G(u) = \Delta(G)$ then G has a complete degree-extension.*

Proof. Following from Definition 2.1 we consider two cases.

Case (i): See [1]. Assume that n is even. Since $\Delta(G) < \frac{n}{2}$ the complement graph G^c has $\delta(G^c) \geq \frac{n}{2}$. Recalling Dirac's Theorem it follows that G^c is Hamiltonian. So for each Hamilton cycle in G^c exactly two perfect matchings exist in G^c. Therefore, G has at least two complete degree-extensions, G^x_c.

Case (ii): Assume that n is odd and let here exists a unique vertex $u \in V(G)$ such that $d_G(u) = \Delta(G)$. Consider the graph $G-u$ then certainly $\Delta(G-u) < \frac{n-1}{2}$. Following from Case (i) we have that $G-u$ has a degree extension $(G-u)^x_c$. From Definition 2.1(ii) it follows that $(G-u)^x + u$ is an incomplete degree-extension G^x_{ic} of G, with $\Delta(G^x_{ic}) \in \{\Delta(G), \Delta(G) + 1\}$. □

It is observed that $\lfloor \frac{n}{2} \rfloor$ edges must be added to obtain an incomplete degree-extension G^x_{ic}. Henceforth we will only consider a graph G of even order n.

If n is even and if k distinct Hamilton cycles exist in G^c then $2k$ complete extended graphs exist. For graphs on even number of vertices we have the next result.

Theorem 2.4. *Let G be a graph of even order n. If the complement graph G^c has a path of length $n-1$ (spanning path), then the graph G has a complete degree-extension G^x_c.*

Proof. If the complement graph G^c has a spanning path of length $n-1$ exactly one perfect matching corresponding to that path in G^c exists. The result follows immediately. □

Corollary 2.5. *Let G be a graph of even order n. If $V(G)$ can be partitioned into a number of subsets $V_1(G), V_2(G), \ldots, V_\ell(G)$, each containing even number of vertices such that the complement of each induced subgraph $\langle V_i(G) \rangle^c$, $1 \leq i \leq \ell$ has a spanning path, then G has a complete degree-extension G^x_c.*

Proof. Denote the spanning path in $\langle V_i(G)\rangle^c$ by$P^{(i)}$, $1 \leq i \leq \ell$. From each $P^{(i)}$ we select the unique perfect matching $S^{(i)}$ in $\langle V_i(G)\rangle^c$. Clearly, $\bigcup\limits_{i=1}^{\ell} S^{(i)}$ is a perfect matching in G^c. □

Clearly, the smallest path for which Theorem 2.2 finds application is P_4. Let the vertices of P_4 consecutively be labelled v_1, v_2, v_3, v_4. Then, the unique $P^x_{4,c} = P_4 + (v_1v_3, v_2v_4)$. For P_6 we find the application of Theorem 2.2 allows the complete degree-extensions, $P^{x_1}_{6,c} = P_6 + (v_1v_6, v_2v_4, v_3v_5)$ or $P^{x_2}_{6,c} = P_6 + (v_1v_3, v_2v_5, v_4v_6)$ or $P^{x_3}_{6,c} = P_6 + (v_1v_4, v_2v_5, v_3v_6)$ or $P^{x_4}_{6,c} = P_6 + (v_1v_5, v_2v_4, v_3v_6)$ or $P^{x_5}_{6,c} = P_6 + (v_1v_4, v_2v_6, v_3v_5)$.

3. χ'-chromatic sum and χ^+-chromatic sum of certain graphs

We recall that a vertex coloring of a graph G such that adjacent vertices are not allocated the same color is called a proper coloring of G. The minimum number of colors in a proper coloring of G is called the chromatic number, $\chi(G)$.

3.1. The χ'-chromatic sum and χ^+-chromatic sum of extended paths and extended cycles

A new concept called the b-chromatic sum of a graph was introduced in [10]. Analogous to this the notion of general color sum of graphs has been defined in [7] as:

Let $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_k\}$ allow a b-coloring $\mathcal{S}$ of G. As stated in [7] there are $k!$ ways of allocating the colors to the vertices of G. Let the *color weight* $\theta(c_i)$ be the number of times a color c_i is allocated to vertices. In general we refer to the *color sum* of a coloring $\mathcal{S}$ and define it as, $\omega(\mathcal{S}) = \sum\limits_{i=1}^{k} i \cdot \theta(c_i)$. The b-chromatic sum is given by $\varphi'(G) = min\{\sum\limits_{i=1}^{k} i \cdot \theta(c_i) : \forall\ b\text{-colorings of } G\}$.

The general color sums of certain cycle related graphs has been studied in [11]. This interesting new invariant motivated similar concepts in graph coloring.

Definition 3.1. [7] For a graph G the χ'-chromatic sum is defined to be:
$\chi'(G) = min\{\sum\limits_{i=1}^{k} i \cdot \theta(c_i) : \forall \text{ minimum proper colorings of } G\}$.

Definition 3.2. citeKS1 For a graph G the χ^+-chromatic sum is defined to be:
$\chi^+(G) = max\{\sum\limits_{i=1}^{k} i \cdot \theta(c_i) : \forall \text{ minimum proper colorings of } G\}$.

Further motivation for these new invariants is as follows. If the colors represent different technology types and the configuration requirement is that at least one unit per technology type must be placed at a point in a network without similar technology types being adjacent, two further considerations come into play. Firstly, if the higher indexed colors represent technology types with higher failure rate (risk) then the placement of the maximal number of higher indexed units is the solution to ensure a functional network. On the other hand, if the lower indexed colors represent a less costly (procurement, installation, commissioning and maintenance) technology type, and minimising total cost is the priority, then the placement of maximal number of lower indexed units is the desired solution. We recall two important results from [7].

Theorem 3.3. [7] *For a path P_n, $n \geq 1$ the χ'-chromatic sum and χ^+-chromatic sum are given by:*

(i)

$$\chi'(P_n)) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{3n}{2}, & \text{if } n \text{ is even}, \\ 3 \cdot \lfloor \frac{n}{2} \rfloor + 1, & \text{if } n \text{ is odd}. \end{cases}$$

(ii)

$$\chi^+(P_n)) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{3n}{2}, & \text{if } n \text{ is even}, \\ 3 \cdot \lfloor \frac{n}{2} \rfloor + 2, & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 3.4. [7] *For a cycle C_n the χ'-chromatic sum and χ^+-chromatic sum are given by:*

(i)

$$\chi'(C_n)) = 3 \cdot \lceil \frac{n}{2} \rceil.$$

(ii)

$$\chi^+(C_n)) = \begin{cases} \frac{3n}{2}, & \text{if } n \text{ is even}, \\ 5 \cdot \lfloor \frac{n}{2} \rfloor + 1, & \text{if } n \text{ is odd}. \end{cases}$$

From $P^{x_i}_{6,c}$, $1 \leq i \leq 5$ we note that $P^{x_3}_{6,c}$ allows a minimum proper coloring $v_1 \to c_1, v_2 \to c_2, v_3 \to c_1, v_4 \to c_2, v_5 \to c_1, v_6 \to c_2$, such that $\chi'(P^{x_3}_{6,c}) = 9 = min\{\chi'(P^{x_i}_6) : 1 \leq i \leq 5\}$. Also we note that $P^{x_4}_6$, (by symmetry $P^{x_5}_6$ as well) allows a minimum proper coloring $v_1 \to c_3, v_2 \to c_2, v_3 \to c_1, v_4 \to c_3, v_5 \to c_2, v_6 \to c_3$, such that $\chi^+(P^{x_4}_{6,c}) = 14 = max\{\chi^+(P^{x_i}_{6,c}) : 1 \leq i \leq 5\}$. Up to isomorphism P_6 has 4 distinct complete extensions. These observations lead to the next results which indeed follow directly from Definitions 3.1 and 3.2.

Proposition 3.5. *(i) For a graph G of even order n, such that the complement graph G^c has at least one perfect matching then, if up to isomorphism the graph G has $G^{x_i}_c$, $1 \leq i \leq \ell$ complete degree-extensions then, $\chi'(G^x_c) = min\{\chi'(G^{x_i}_c) : 1 \leq i \leq \ell\}$.*

(ii) For a graph G of even order n, such that the complement graph G^c has at least one perfect matching then, if up to isomorphism the graph G has $G^{x_i}_c$, $1 \leq i \leq \ell$ complete degree-extensions then, $\chi^+(G^x_c) = max\{\chi^+(G^{x_i}_c) : 1 \leq i \leq \ell\}$.

Theorem 3.6. *For a path P_n, $n \geq 4$ and even, we have*

(i) $\chi'(P^x_{4,c}) = 7$ *and* $\chi^+(P^x_{4,c}) = 9$.

(ii) $\chi'(P^x_{n,c}) = \frac{3n}{2}$ *and* $\chi^+(P^x_{n,c}) = \frac{5n}{2} - 1$, *for* $n \geq 6$.

Proof. Case (i): Consider the unique $P^x_{4,c} = P_4 + (v_1v_3, v_2v_4)$ and allocate the minimum proper coloring $v_1 \to c_1, v_2 \to c_2, v_3 \to c_3, v_4 \to c_1$. It follows that, $\chi'(P^x_{4,c}) = 7$. Now interchange the colors c_1 and c_3. It follows that, $\chi^+(P^x_{4,c}) = 9$.

Case (ii)(a): Consider P_n, $n \geq 6$.

Subcase (ii)(a)(1): If $\frac{n}{2}$ is odd, consider $P^x_{n,c} = P_n + (v_1v_{\frac{n}{2}+1}, v_{\frac{n}{2}}v_n, v_{1+i}v_{n-i})$, $1 \leq i \leq \frac{n}{2} - 1$. Allocate the minimum proper coloring $v_1 \to c_1, v_2 \to c_2, v_3 \to c_1, \dots, v_n \to c_2$. So, $\theta(c_1) = \frac{n}{2}$ and $\theta(c_2) = \frac{n}{2}$. Hence, $\chi'(P^x_{n,c}) = 1 \cdot \theta(c_1) + 2 \cdot \theta(c_2) = \frac{n}{2} + 2 \cdot \frac{n}{2} = \frac{3n}{2} = min\{\chi'(P^{x_i}_{n,c}) : \forall P^{x_i}_{n,c}\}$.

Subcase (ii)(a)(2): If $\frac{n}{2}$ is even, consider $P^x_{n,c} = P_n + (v_1v_{\frac{n}{2}}, v_{\frac{n}{2}+1}v_n, v_{1+i}v_{n-i})$, $1 \leq i \leq \frac{n}{2} - 1$. The result now follows like in Subcase (ii)(a)(1).

Case (ii)(b): Consider P_n, $n \geq 6$.

Subcase (ii)(b)(1): If $\frac{n}{2}$ is odd let, $t = \frac{n}{2}$. Construct the graph $P_n^* = P_n + (v_1v_{t+1}, v_{t-1}v_{t+3}, v_tv_{t+2})$. Now, allocate the proper coloring $v_1 \to c_3, v_2 \to c_2, v_3 \to c_3, \dots, v_{t-1} \to c_2, v_t \to c_3, v_{t+1} \to c_1, v_{t+2} \to c_2, v_{t+3} \to c_3, \dots, v_{n-1} \to c_2, v_n \to c_3$. Since equal number of vertices are left with degree 2 and colored c_3, c_2 respectively, it is always possible to complete the extension to obtain $P^x_{n,c}$ having the allocated minimum proper coloring. Clearly, $\theta(c_1) = 1$, $\theta(c_2) = t-1$ and $\theta(c_3) = t$. Hence, $\chi^+(P^x_{n,c}) = 3 \cdot \frac{n}{2} + 2 \cdot (\frac{n}{2} - 1) + 1 = \frac{5n}{2} - 1 = max\{\chi^+(P^{x_i}_{n,c}) : \forall P^{x_i}_{n,c}\}$.

Subcase (ii)(b)(2): If $\frac{n}{2}$ is even let, $t = \frac{n}{2}$. Construct the graph $P_n^* = P_n + (v_1v_{t+1}, v_{t-1}v_{t+3}, v_tv_{t+2})$. Now, allocate the proper coloring $v_1 \to c_3, v_2 \to c_2, v_3 \to c_3, \dots, v_{t-1} \to c_2, v_t \to c_3, v_{t+1} \to c_1, v_{t+2} \to c_2, v_{t+3} \to c_3, \dots, v_{n-1} \to c_2, v_n \to c_3$. Since equal number of vertices are left with degree 2 and colored c_3, c_2 respectively, it is always possible to complete the extension to obtain $P^x_{n,c}$ having the allocated minimum proper coloring. Clearly, $\theta(c_1) = 1$, $\theta(c_2) = t-1$ and $\theta(c_3) = t$. Hence, $\chi^+(P^x_{n,c}) = 3 \cdot \frac{n}{2} + 2 \cdot (\frac{n}{2} - 1) + 1 = \frac{5n}{2} - 1 = max\{\chi^+(P^{x_i}_{n,c}) : \forall P^{x_i}_{n,c}\}$. □

Theorem 3.7. *For a cycle C_n, $n \geq 4$, we have:*

(i) $\chi'(C^x_{4,c}) = \chi^+(C^x_{4,c}) = 10$.

(ii) $\chi'(C^x_{n,c}) = \frac{3n}{2}$ *and:*

$$\chi^+(C^x_{n,c})) = \begin{cases} \frac{5n}{2} - 1, & \text{if } \frac{n}{2} \text{ is even, } n \geq 6, \\ \frac{5n}{2} - 3, & \text{if } \frac{n}{2} \text{ is odd, } n \geq 6. \end{cases}$$

Proof. Case (i): Consider the unique $C^x_{4,c} = C_4 + (v_1v_3, v_2v_4) = K_4$. Hence, $\chi^+(C^x_{4,c}) = 10$.

Case (ii)(a): Consider C_n, $n \geq 6$.

Subcase (ii)(a)(1): If $\frac{n}{2}$ is odd, consider $C^x_{n,c} = C_n + (v_1v_{\frac{n}{2}+1}, v_{\frac{n}{2}}v_n, v_{1+i}v_{n-i})$, $1 \leq i \leq \frac{n}{2} - 1$. Allocate the minimum proper coloring $v_1 \to c_1, v_2 \to c_2, v_3 \to c_1, \dots, v_n \to c_2$. So, $\theta(c_1) = \frac{n}{2}$ and $\theta(c_2) = \frac{n}{2}$. Hence, $\chi'(C^x_{n,c}) = 1 \cdot \theta(c_1) + 2 \cdot \theta(c_2) = \frac{n}{2} + 2 \cdot \frac{n}{2} = \frac{3n}{2} = min\{\chi'(C^{x_i}_{n,c}) : \forall C^{x_i}_{n,c}\}$.

Subcase (ii)(a)(2): If $\frac{n}{2}$ is even, consider $C^x_{n,c} = C_n + (v_1v_{\frac{n}{2}}, v_{\frac{n}{2}+1}v_n, v_{1+i}v_{n-i})$, $1 \leq i \leq \frac{n}{2} - 1$. The result now follows like in Subcase (ii)(a)(1).

Case (ii)(b): Consider C_n, $n \geq 6$.

Subcase (ii)(b)(1): If $\frac{n}{2}$ is even let $t = \frac{n}{2}$. Construct the graph $C_n^* = C_n + (v_1v_{t+1}, v_{t-1}v_{t+3}, v_tv_{t+2})$. Now, allocate the proper coloring $v_1 \to c_3, v_2 \to c_2, v_3 \to c_3, \dots, v_{t-1} \to c_2, v_t \to c_3, v_{t+1} \to c_1, v_{t+2} \to c_2, v_{t+3} \to c_3, \dots, v_{n-1} \to c_2, v_n \to c_1$. Since equal number of non-adjacent vertices are left with degree 2 and half colored c_2 and other half colored c_3 respectively, it is always possible to complete the extension to obtain $C^x_{n,c}$ having the allocated minimum proper coloring. Clearly, $\theta(c_1) = 1$, $\theta(c_2) = t-1$ and $\theta(c_3) = t$. Hence, $\chi^+(C^x_{n,c}) = 3 \cdot \frac{n}{2} + 2 \cdot (\frac{n}{2} - 1) + 1 = \frac{5n}{2} - 1 = max\{\chi^+(C^{x_i}_{n,c}) : \forall C^{x_i}_{n,c}\}$.

Subcase (ii)(b)(2): If $\frac{n}{2}$ is odd let $t = \frac{n}{2}$. Construct the graph $C_n^* = C_n + (v_1v_{t+1}, v_{t-1}v_{t+3}, v_tv_{t+2})$. Now, allocate the proper coloring $v_1 \to c_3, v_2 \to c_2, v_3 \to c_3, \dots, v_{t-1} \to c_2, v_t \to c_3, v_{t+1} \to c_1, v_{t+2} \to c_2, v_{t+3} \to c_3, \dots, v_{n-1} \to c_1, v_n \to c_3$. Since equal number of non-adjacent vertices are left with degree 2 and half colored c_2 and other half colored c_3 respectively, it is always possible to complete the extension to obtain $C^x_{n,c}$ having the allocated minimum proper coloring. Clearly, $\theta(c_1) = 2$, $\theta(c_2) = t-1$ and $\theta(c_3) = t-1$. Hence, $\chi^+(C^x_{n,c}) = 3 \cdot (\frac{n}{2} - 1) + 2 \cdot (\frac{n}{2} - 1) + 2 = \frac{5n}{2} - 3 = max\{\chi^+(C^{x_i}_{n,c}) : \forall C^{x_i}_{n,c}\}$. □

3.2. On bipartite graphs

It is well known that a graph G is bipartite if and only G contains no odd cycle. Up to isomorphism a graph (connected) which contains no odd cycle has a unique vertex-set partition say, X, Y such that if $v, u \in X$ then $vu \notin E(G)$ and similarly for vertices in Y. If $|X|$ and $|Y|$ are even we say that the partition is *even balanced*, else if $|X|$ and $|Y|$ are odd it is *odd balanced*. We also say that the ordered pairs (a, b) and (c, d) are *bi-distinct* if and only if $a \neq c$ and $b \neq d$.

Theorem 3.8. *If G is isomorphic to a balanced bipartite graph $B_{n,m}$ with $n \geq m$ then G has a complete extended graph G_c^x.*

Proof. Case 1: Consider the even balanced bipartite graph $B_{n,m}$ which is isomorphic to G. Without loss of generality assume that $|X| = n \geq m = |Y|$. Label the vertices in X to be, $v_1, v_2, v_3 \ldots, v_n$ and those in Y to be, $u_1, u_2, u_3, \ldots, u_m$. Identify the maximum number of bi-distinct non-adjacent vertex pairs (v_i, u_j) in G. Assume that there are ℓ such bi-distinct pairs.

Subcase 1(i): If ℓ is even, add the edges $v_i u_j$ for all ℓ bi-distinct pairs. Clearly, $n - \ell$ vertices in X can be paired into bi-distinct vertex pairs. Add an edge for each bi-distinct vertex pair. Similarly, $m - \ell$ vertices in Y can be paired into bi-distinct vertex pairs. Add those corresponding edges as well. The new graph obtained through this construction is a complete degree-extension G_c^x of G.

Subcase 1(ii): If ℓ is odd select any $\ell - 1$ bi-distinct vertex pairs. Construct a new graph similar to subcase 1(i) which is clearly G_c^x.

Case 2: Consider the odd balanced bipartite graph $B_{n,m}$ which is isomorphic to G. As before, assume that $|X| = n \geq m = |Y|$ and label the vertices similarly.

Subcase 2(i): If ℓ is even select any $\ell - 1$ bi-distinct vertex pairs. Construct a new graph similar to subcase 1(i) which is clearly, G_c^x.

Subcase 2(ii): If ℓ is odd, construct a new graph similar to subcase 1(ii) which is clearly, G_c^x. □

We note that in all cases above the subgraph, $G - \{v_i, u_j : \forall \ell \text{ or } \ell - 1, (v_i, u_j) \text{ is a bi-distinct non-adjacent pair of vertices}\}$, is a complete bipartite subgraph.

Theorem 3.9. *If G is isomorphic to a balanced bipartite graph $B_{n,m}$ with $|X| = n \geq m = |Y|$ and G has a maximum ℓ bi-distinct pairs of vertices (v_i, u_j), $v_i \in X$, $u_j \in Y$, then,*

Case 1: If both n, m are even:
(i) If ℓ is even, $\chi'(G_c^x) = \frac{3(n-\ell)}{2} + \frac{7(m-\ell)}{2} + 3\ell$.
(ii) If ℓ is odd, $\chi'(G_c^x) = \lfloor \frac{n-\ell}{2} \rfloor + 2 \cdot \lceil \frac{n-\ell}{2} \rceil + \frac{7(m-\ell+1)}{2} + 3\ell$.

Case 2: If both n, m are odd:
(i) If ℓ is odd, $\chi'(G_c^x) = \frac{3(n-\ell)}{2} + \frac{7(m-\ell)}{2} + 3\ell$.
(ii) If ℓ is even, $\chi'(G_c^x) = \lfloor \frac{n-\ell}{2} \rfloor + 2 \cdot \lceil \frac{n-\ell}{2} \rceil + \frac{7(m-\ell+1)}{2} + 3\ell$.

Proof. Case 1: Assume that both $|X|$ and $|Y|$ are even. Label the vertices in X to be, $v_1, v_2, v_3 \ldots, v_n$ and those in Y to be, $u_1, u_2, u_3, \ldots, u_m$.

Subcase 1(i): Let ℓ be even. Without loss of generality label the vertices of X, Y which belong to the ℓ bi-distinct non-adjacent pairs as $v_1, v_2, v_3, \ldots, v_\ell$ and $u_1, u_2, u_3, \ldots, u_\ell$, respectively such that (v_i, u_j), $1 \leq i, j \leq \ell$ are non-adjacent. Add edges $v_i u_i$, $1 \leq i \leq \ell$ and assign the colors $v_i \to c_1, u_i \to c_2$, $1 \leq i \leq \ell$. Label the rest of the vertices of X and Y to be $v_{\ell+1}, v_{\ell+2} \ldots, v_n$ and $u_{\ell+1}, u_{\ell+2}, \ldots, u_m$, respectively. Now, add the edges $v_{\ell+1}v_{\ell+2}, u_{\ell+1}u_{\ell+2}$ and assign the colors $v_{\ell+1} \to c_1, v_{\ell+2} \to c_3, u_{\ell+1} \to c_2, u_{\ell+2} \to c_4$. Note that, $v_{\ell+1}, v_{\ell+2}, u_{\ell+1}, u_{\ell+1}$ forms a complete graph of order 4 (K_4) in G_c^x since we have chosen ℓ to be maximum. Recursively proceed with the edge-adding and coloring protocols until only the $(n - m)$ vertices $v_{m+1}, v_{m+2}, \ldots, v_n$ are left. Add the edges $v_{m+1}v_{m+2}, v_{m+3}v_{m+4}, \ldots v_{n-1}v_n$, and assign the colors, alternating $v_{m+1} \to c_1, v_{m+2} \to c_2, \ldots v_{n-1} \to c_1, v_n \to c_2$. Clearly, the assignment of colors is a minimum proper coloring of G_c^x.

It follows that, $\theta(c_1) = \frac{n-\ell}{2} + \ell$, $\theta(c_2) = \frac{n-\ell}{2} + \ell$, $\theta(c_3) = \frac{m-\ell}{2}$, $\theta(c_4) = \frac{m-\ell}{2}$.

Therefore, $\chi'(G_c^x) = min\{\sum\limits_{i=1}^{k} i \cdot \theta(c_i) : \forall \text{ minimum proper colorings of } G^x\} = \theta(c_1) + 2 \cdot \theta(c_2) + 3 \cdot \theta(c_3) + 4 \cdot \theta(c_4) = \frac{3(n-\ell)}{2} + \frac{7(m-\ell)}{2} + 3\ell$.

Subcase 1(ii): Follows similar to Case (i) by considering only $\ell - 1$ of the maximum bi-distinct non-adjacents pairs of vertices.

Case 2: Assume that both $|X|$ and $|Y|$ are odd.

Subcase 2(i): Let ℓ be odd and label the vertices as in subcase 1(i). Without loss of generality label the vertices of X, Y which belong to the ℓ bi-distinct non-adjacent pairs as $v_1, v_2, v_3, \dots, v_\ell$ and $u_1, u_2, u_3, \dots, u_\ell$, respectively such that (v_i, u_j), $1 \leq i, j \leq \ell$ are non-adjacent. Add edges v_iu_i, $1 \leq i \leq \ell$ and assign the colors $v_i \to c_1, u_i \to c_2$, $1 \leq i \leq \ell$. Label the rest of the vertices of X and Y to be $v_{\ell+1}, v_{\ell+2} \dots, v_n$ and $u_{\ell+1}, u_{\ell+2}, \dots, u_m$, respectively. Now, add the edges $v_{\ell+1}v_{\ell+2}, u_{\ell+1}u_{\ell+2}$ and assign the colors $v_{\ell+1} \to c_1, v_{\ell+2} \to c_3, u_{\ell+1} \to c_2, u_{\ell+2} \to c_4$. Note that, $v_{\ell+1}, v_{\ell+2}, u_{\ell+1}, u_{\ell+1}$ forms a complete graph of order 4 (K_4) in G_c^x since we have chosen ℓ to be maximum. Recursively proceed with the edge-adding and coloring protocols until only the $(n-m)$ vertices $v_{m+1}, v_{m+2}, \dots, v_n$ are left. Add the edges $v_{m+1}v_{m+2}, v_{m+3}v_{m+4}, \dots v_{n-1}v_n$, and assign the colors, alternating $v_{m+1} \to c_1, v_{m+2} \to c_2, \dots v_{n-1} \to c_1, v_n \to c_2$. Clearly, the assignment of colors is a minimum proper coloring of G_c^x.

It follows that, $\theta(c_1) = \frac{n-\ell}{2} + \ell$, $\theta(c_2) = \frac{n-\ell}{2} + \ell$, $\theta(c_3) = \frac{m-\ell}{2}$, $\theta(c_4) = \frac{m-\ell}{2}$.

Therefore, $\chi'(G_c^x) = min\{\sum\limits_{i=1}^{k} i \cdot \theta(c_i) : \forall$ minimum proper colorings of $G^x\} = \theta(c_1) + 2 \cdot \theta(c_2) + 3 \cdot \theta(c_3) + 4 \cdot \theta(c_4) = \frac{3(n-\ell)}{2} + \frac{7(m-\ell)}{2} + 3\ell$.

Subcase 2(ii): Let ℓ be even and label the vertices as before. The proof follows from similar reasoning found in Subcase 1(i). □

Note that the maximum matching in a bipartite graph hence, the maximum bi-distinct non-adjacent pairs of vertices and the value ℓ can be determined by amongst others, the $n^{\frac{5}{2}}$-algorithm described in [6]. We now discuss a special case of this theorem where $|X| = |Y|$, which means both the partitions of the bipartite graph have equal cardinality. This class of graph includes all regular bipartite graphs. For this case we calculate both $\chi'(G^x)$ and $\chi^+(G_c^x)$.

Definition 3.10. Let G be a bipartite graph, with vertex partitions X and Y. Let S be a subset of X. For, $v \in S$ we define $N^c(v)$ to be the set of vertices in Y that are not adjacent to v. We define $N^c(S)$ to be $\bigcup_{v \in S} N^c(v)$.

Corollary 3.11. *Let G be a bipartite graph of even order n, with vertex partitions X and Y, such that $|X| = |Y|$. If for every $S \subseteq X$, $|N^c(S)| \geq |S|$ then,*
(i) $\chi'(G_c^x) = \frac{3n}{2}$.
(ii) $\chi^+(G_c^x) = \frac{5n}{2} - 3$.

Proof. Case (i): Note that in G, the sets X and Y are independent sets. Hence, in G^c vertices in $\langle X \rangle$ and $\langle Y \rangle$ will be cliques of size $|X|$ and $|Y|$, respectively. Let us denote these cliques as $K_{|X|}$ and $K_{|Y|}$ respectively. Also, note that the graph $G^c - E(K_{|X|}) - E(K_{|Y|})$ is bipartite. By Hall's Theorem (see [2]) the graph $G^c - E(K_{|X|}) - E(K_{|Y|})$ has a perfect matching M, since in G, for every $S \subseteq X$, $|N^c(S)| \geq |S|$. Thus, G can always be extended to a bipartite graph. Hence, $\chi'(G_c^x) = \frac{3n}{2}$.

Case (ii): Now, let us assign a different extension of G. Choose two arbitrary edges from M, x_1y_1 and x_2y_2, such that $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Now, for extending G, use the edges $M - x_1y_1 - x_2y_2 + x_1x_2 + y_1y_2$. This can always be done since the graph is bipartite. Now, we can color x_1 and y_1 using c_1. Vertices in $X - x_1$ and $Y - y_1$ can be colored using c_2 and c_3, respectively. This obviously shows that, $\chi^+(G_c^x) = \frac{5n}{2} - 3$. □

4. On some specific classes of graphs

Lemma 4.1. *Let G be a graph of order n, such that $\delta(G) > \frac{n}{2}$, then $diam(G) \leq 2$.*

Proof. Let A be the adjacency matrix of G. Every entry $a_{i,j}$ of A^2 is the scalar product of the i-th row and the j-th column of A. Since every row and column of A has at least $\frac{n}{2}$ entries equal to 1, every $a_{i,j} > 0$, $a_{i,j}$ an entry of A^2. Hence, for any two distinct vertices $v, u \in V(G)$ we have $d_G(v, u) \leq 2$. Therefore, $diam(G) \leq 2$. □

Theorem 4.2. *Let G be a graph of order n, such that $\delta(G) > \frac{n}{2}$. Then G is bipartite if and only it is triangle free and $K_{r,r}$ is the only such graph.*

Proof. *Necessary condition:* It follows from Lemma 4 that $diam(G) \leq 2$. Therefore, the largest holes G can have are of the kind, C_4. If G is triangle free it has no odd cycle hence, G is bipartite.

Sufficient condition: If G is bipartite it contains no odd cycle hence, G is triangle free.

Finally $K_{r,r}$ is the only r-regular bipartite graph on exactly (at most) $2r$ vertices. □

Lemma 4.3. *Let graphs G and H have minimum proper coloring sets $\mathcal{C}_G = \{c_1, c_2, c_3, \dots, c_{\chi(G)}\}$ and $\mathcal{C}_H = \{c_1, c_2, c_3, \dots, c_{\chi(H)}\}$, respectively. Assume $\chi(G) \leq \chi(H)$. Let $v_i \to c_k$, $v_i \in V(G)$ and $u_j \to c_t$, $u_j \in V(H)$ and both $c_k, c_t \in \mathcal{C}_G$ hence, $c_t \in \mathcal{C}_H$, as well.*

(i) If G and H are joined by merging vertices $v_i in V(G)$ and $u_j \in V(H)$ as a common vertex which is now assigned the color c_k; an equivalent minimum proper coloring is possible by re-assigning color c_k to all vertices in H which were assigned c_t; and assigning color c_t to all vertices in H which were assigned color c_k.

(ii) If graphs G and H are allowed to join by merging an edge each into a common edge, result (i) holds by applying it consecutively to the pairs of vertices (v_i, u_j) and (v_ℓ, u_m), $v_i v_\ell \in E(G)$ and $u_j u_m \in E(H)$.

Proof. The results are trivial. □

Now we introduce a family of *pattern structured* graphs. When a cluster of two copies of a graph H are allowed to merge at least one edge (not necessarily structurally equivalent edges) to share at least one common edge, the new graph is called a *H-gridlike cluster*. Two or more *H-gridlike clusters* are allowed to merge similarly to get an expanded *H-gridlike cluster*. When a cluster of two or more copies of a graph H are all allowed to merge a vertex (not necessarily structurally equivalent vertices) to share a common vertex, the new graph is called a *H-cloverlike cluster*. When a cluster of two or more copies of a graph H are all allowed to merge an edge (not necessarily structurally equivalent edges) to all share a common edge, the new graph is called a *H-booklike cluster*. When two copies H_1, H_2 are joined by a path $ve_0w_1e_1w_2e_2w_3\dots e_{m-1}w_me_mu$, $v \in V(H_1), u \in V(H_2)$ we say they are adjacent. If a graph G^* is the composition of the aforesaid then; a vertex, a H-gridlike cluster, a H-cloverlike cluster, a H-booklike cluster are called H-elements of G^*. If G^* has a cycle between at least two H-elements G^* is called cyclic, else it is called acyclic or *H-treelike*.

Theorem 4.4. *For a H-treelike graph G^* with $\chi(H) \geq 2$ we have, $\chi(G^*) = \chi(H)$.*

Proof. Consider any H-treelike graph G^*.

Case (i): Consider a H-gridlike element and without loss of generality consider any pair H_1, H_2 sharing a common edge say, uv. If in a minimum proper coloring of H_1 the vertex coloring is $v \to c_i, u \to c_j$ then, after applying Lemma 4.3 the χ-number of the partial H-gridlike element remains the same.

Case (ii): Consider a H-cloverlike element and without loss of generality assume the common vertex is v. If in a minimum proper coloring of H_1 the vertex coloring is $v \to c_i$ then after applying Lemma 4.3(i) the χ-number in the partial H-cloverlike element remains the same. By iteratively applying Lemma 4.3(i) to all pairs, the χ-number of the whole H-cloverlike element remains the same.

Case (iii): Consider a H-booklike element. Similar reasoning as in Case (i) follows.

Case (iv): First consider adjacent H-elements H_1 and H_2 as the vertex join between H_1 and an end vertex of a path and apply Lemma 4.3(i). Thereafter, consider the vertex join between the other end vertex of the path and H_2 and apply Lemma 4.3(i) again. Clearly, the χ-number remains the same.

Invoking Cases (i) to (iv) throughout a H-treelike graph G^* settles the result in general. $\square$

5. Conclusion

We recall that an *almost regular* graph G is such that, $\Delta(G) - \delta(G) = 1$. A *partial extension* of an almost regular graph G is defined to be the graph G^{px} obtained by adding edges to G such that, G^{px} is $\Delta(G)$-regular. A path is such an almost regular graph so it is easy to see that, $P_n^{px} = C_n$ therefore, $\chi'(P_n^{px}) = \chi'(C_n)$ and $\chi^+(P_n^{px}) = \chi^+(C_n)$. There is scope to research $\chi'(G_n^{px})$ and $\chi^+(G_n^{px})$ for other classes of almost regular graphs.

For an r-regular graph G on at least r^4 vertices the b-chromatic number is $\varphi(G) = r + 1$ (see [8]). This result has been improved in [3] by bounding the result to graphs having at least $2r^3$ vertices. So it is easy to see that for C_n, $n \geq 54$, we have, $\varphi(C_n) = 3$ and $\varphi(C_{n,c}^x) = 4$. In general, it will be worthy to research the relationship between $\varphi'(C_n)$ and $\varphi'(C_{n,c}^x)$ as well as that, between $\varphi^+(C_n)$ and $\varphi^+(C_{n,c}^x)$ and perhaps for other classes of regular graphs, if such exist.

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