Sigma J Eng Nat Sci, Vol. 41, No. 2, pp. 256–265, April, 2023

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2023.00037

Research Article

Cubic B-spline finite element method for generalized reaction-diffusion equation with delay

Gemeda Tolessa LUBO[1,*], Gemechis File DURESSA[2]

[1]Department of Mathematics, Dambi Dollo University, Dambi Dollo, 381149, Ethiopia
[2]Department of Mathematics, Jimma University, Jimma, 1000, Ethiopia

ARTICLE INFO

Article history
Received: 05 April 2021
Revised: 14 June 2021
Accepted: 12 August 2021

Keywords:
Generalized Reaction-Diffusion Equation; Cubic B-Spline; Finite Element; Crank-Nicolson Method; Convergence Analysis

ABSTRACT

In this paper, a cubic B -spline finite element method is constructed based on redefined cubic B-spline basis functions for solving the generalized reaction-diffusion equations with delay. The time discretization process is based on Crank-Nicolson method. Examples are worked out to validate the theoretical convergence analysis. The numerical results given in graphs and tables demonstrate that the present method approximates the exact solution very well. The accurateness of the numerical scheme is confirmed by computing L_2 and L_∞ error norms.

Cite this article as: Lubo GT, Duressa GF. Cubic B-spline finite element method for generalized reaction-diffusion equation with delay. Sigma J Eng Nat Sci 2023;41(2):256–265.

INTRODUCTION

In this paper, we consider the generalized reaction diffusion equation with delay of the following type

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + b\frac{\partial^2 u(x,t-\tau)}{\partial x^2} + cu(x,t) + du(x,t-\tau), t > 0, 0 < x < \pi, \\ u(x,t) = \varphi(x,t), -\tau \le t \le 0, \quad 0 \le x \le \pi, \\ u(0.t) = u(\pi.t) = 0, t \ge -\tau, \end{cases}$$

where $a > 0$ and $b > 0$ represent the diffusion coefficients $c, d \in \mathbb{R}$ and $\tau > 0$ is a delay constant. The generalized reaction-diffusion equations with delay have attracted a significant interest in the last several decades due to their frequent occurrence in real life situations [1-7]. The governing equation has been studied widely in the case where $c = d = 0$ [8-13] and $b = c = 0$ [14-16]. Equation of the type (1) has been considered equation [17,18]. A delay term could not only make the numerical solutions difficult to be obtained but also change the dynamical properties of a system [16]. This is the reason why such equations became the attention of researchers in numerical analysis and simulations. The accessibility of exact solutions, as well as efficient methods to obtain numerical approximations of the required precision, are very important. However, few delay generalized reaction-diffusion equations with a certain simple setting permit an analytic solution. Hence, computing approximate solutions essential are essential. Up to now, many of the numerical methods available to approximate the diffusion term are based on some classical numerical methods, for example, finite difference method [19], compact finite difference method [20,21], discontinuous Galerkin methods[22], spectral method [23], Haar Wavelet [24], variational method [25], waveform relation method [26], and so on.

In [27], an analytical technique for the nonlinear dynamic problem with delayed is solved. It is well known

***Corresponding author.**
*E-mail address: gemedatolesa@gmail.com

This paper was recommended for publication in revised form by Regional Editor Hijaz Ahmad

Published by Yıldız Technical University Press, İstanbul, Turkey

that the finite element method (FEM) can be easily designed for high order accuracy in space by taking higher-order polynomial as the basis (shape) function. The advantage of the FEM over the finite difference method is its ability to deal with the modeling of complex geometries and irregular shapes. The use of various degrees of B-spline functions to obtain the numerical solutions of some PDEs has been shown to provide easy and simple algorithms, for instance, quadratic B-spline least-square method [28], quadratic B -spline method [29,30], cubic B-spline finite element method [31-37], quartic B spline method [38], quintic B-spline finite element method [39,40], and so on. To the best knowledge of the author, the redefined cubic B -spline finite element method is not considered for finding the approximate solution of delay PDEs. In this paper, we have applied a redefined cubic B-spline FEM to find numerical solutions of the reaction-diffusion with the delay of the type

(1). Our selection of redefined cubic B-spline basis functions improves accuracy to fourth-order. The proposed method give better convergence result than the linear θ –method without increasing the computational cost. Our best concern in this study is to formulate a numerical scheme with a higher order of accuracy by using the redefined cubic B-spline shape functions.

Notations: Denote $\|.\|$ and $\|.\|r$ as the norm $L_2 = L_2(\Omega)$ and the sobolov space $H^r = H^r(\Omega) = W_2(\Omega)$ respectively, so that the real valued v,

$$\|v\| = \|v\|_{L_2} := \left(\int_\Omega |v(x)|^2\, dx\right)^{\frac{1}{2}}$$

and for a positive integer r

$$\|v\|_r = \|v\|_{H^r} := \left(\sum_{i\le r}\left\|\frac{\partial^i v(x)}{\partial x^i}\right\|^2\right)^{\frac{1}{2}}.$$

Let $v(x), w(x) (x \in \Omega)$ be real - valued funnctions.

$$(v, w) := \int_\Omega v(x)\, w(x) dx, (\nabla v, \nabla w) := \int_\Omega \frac{\partial v(x)}{\partial x}\frac{\partial w(x)}{\partial x} dx\,,$$

and C denotes a positive not necessarily the same at different occurrances, which may depend on a, b, c, d and t of (1) but

independent of h and Δt (the stepsizes in t – direction). We denote $u(t, x)$ by u or $u(t)$.

Assumptions: In this paper, assume that $u(t)$: = $u(t, .)$, $u_t(t)$: = $u_t(t, :)$, $u_{tt}(t)$: = $u_{tt}(t, :)$, $u_{ttt}(t)$: = $u_{ttt}(t, :) \in H_0^1(\Omega)$, and at $t = n\tau (n = -1, 0.1 \ldots .)$,the derivative of t is denotes as the left derivative.

Cubic B-spline Finite Element Method

Let $\Delta t = \tau/(m+1)$ be a given step size with m ≥ 1, the grid points $t_n = n\Delta t$ (n = 0, 1, …) and be a given be the approximation in S_h^T of u(t) at t = t_n = nΔt. For positive integer N, let $\{x_k\}_{k=0}^{N+1}$ be a uniform partition on [0, π] in the x direction, such that $x_k = kh$, where h = π/(N + 1) is the step size. Define the space

$$S_3 = \left\{v: v \in C^2([0,\pi]), v|_{[x_{k-1,}x_k]} \in P_3, 1 \le k \le N+1\right\}$$

where P_3 is the space of all polynomial of degree ≤ 3. Extending the partition $\{x_k\}_{k=0}^{N+1}$ using $xk = kh$, k = −3, −2. −1, N + 2, N + 2, N + 4.. As a basis for s,we choose the B -splines, $\{Q_j\}_{j=-1}^{N+1}$, where

$$Q_j(x) = \begin{cases} h^{-3} f_1(x - x_{j-2}), & x \in [x_{j-2}, x_{j-1}]\,, \\ f_2\left(\frac{x - x_{j-1}}{h}\right), & x \in [x_{j-1}, x_j], \\ f_2\left(\frac{x_{j+1} - x}{h}\right), & x \in [x_j, x_{j+1}], \\ h^{-3} f_1(x_{j-2} - x), & x \in [x_{j+1}, x_{j+2}], \\ 0, & x \notin [x_{j-2}, x_{j+2}], \end{cases} \quad (2)$$

and let $f1(x) = x^3$, $f1(x) = 1 + 3x + 3x^2 - 3x^3$. We want to construct a basis for the space

$$S_h^T = \{v: v \in s_3, v(0) = v(\pi) = 0\}. \quad (3)$$

The cubic B -spline $\{\overline{Q_j}\}_{j=-1}^{N+1}$ can be redefined by

$$\begin{cases} \overline{Q_0} = Q_0 - 4Q_{-1}\,, & \overline{Q_1} = Q_1 - Q_{-1}\,, \\ \overline{Q_j} = Q_j\,, & j = 2,3,4, \ldots, N-1. \\ \overline{Q_N} = Q_N - 4Q_{N+1}\,, \overline{Q_N} = Q_{N+1} - 4Q_{N+2.} \end{cases} \quad (4)$$

As the cubic B-spline Q_j, the cubic $\overline{Q_j}$ have support of at least 4 subintervals. We want to find an approximation $U^n \in S_h^T$ such that

$$a(U^n, v) = l(v), v \in S_h^T. \quad (5)$$

The application of Galerkin Crank - Nicolson method to (1) leads to a numerical process of the following type

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, \chi\right) + a\left(\frac{\nabla U^n + \nabla U^{n-1}}{2}, \nabla\chi\right) + b\left(\frac{\nabla U^{n-m} + \nabla U^{n-m-1}}{2}, \nabla\chi\right) - c\left(\frac{U^n + U^{n-1}}{2}, \chi\right) - d\left(\frac{U^{n-m} + U^{n-m-1}}{2}, \chi\right) = 0, \text{for n} > 0 \quad (6)$$

Where $U^n(.) = \varphi(., t_n)$ for $-m \le n \le 0$.
Let

$$U^n(x) := \sum_{j=0}^{N+1} \overline{Q_j}\ (x)\alpha_j^n \quad (7)$$

Using (7) and choosing $\chi = \overline{Q_j}$; $i = 0,1, \ldots, N+1$. Then (6) can be reduced to

$$\frac{1}{\Delta t}\sum_{j=0}^{N+1}(\alpha_j^n - \alpha_j^{n-1})(\overline{Q_i}(x), \overline{Q_j}(x)) = -\frac{a}{2}\sum_{j=0}^{N+1}(\alpha_j^n + \alpha_j^{n-1})(\nabla\overline{Q_i}(x), \nabla\overline{Q_j}(x)) - \frac{b}{2}\sum_{j=0}^{N+1}(\alpha_j^{n-m} + \alpha_j^{n-m-1})(\nabla\overline{Q_i}(x), \nabla\overline{Q_j}(x))$$
$$+\frac{c}{2}\sum_{j=0}^{N+1}(\alpha_j^n + \alpha_j^{n-1})(\overline{Q_i}(x), \overline{Q_j}(x)) + \frac{d}{2}\sum_{j=0}^{N+1}(\alpha_j^{n-m} + \alpha_j^{n-m-1})(\overline{Q_i}(x), \overline{Q_j}(x)) \quad (8)$$

Equation (8) can be written

$$\frac{1}{\Delta t}\sum_{j=0}^{N+1}(\alpha_j^n-\alpha_j^{n-1})\int_0^{\pi}\overline{Q_i}(x)\overline{Q_j}(x)\,dx=-\frac{a}{2}\sum_{j=0}^{N+1}(\alpha_j^n+\alpha_j^{n-1})\int_0^{\pi}\overline{Q_i}'(x)\overline{Q_j}'(x)\,dx-\frac{b}{2}\sum_{j=0}^{N+1}(\alpha_j^{n-m}+\alpha_j^{n-m-1})\int_0^{\pi}\overline{Q_i}'(x)\overline{Q_j}'(x)\,dx$$
$$+\frac{c}{2}\sum_{j=0}^{N+1}(\alpha_j^n+\alpha_j^{n-1})\int_0^{\pi}\overline{Q_i}(x)\overline{Q_j}(x)\,dx+\frac{d}{2}\sum_{j=0}^{N+1}(\alpha_j^{n-m}+\alpha_j^{n-m-1})\int_0^{\pi}\overline{Q_i}(x)\overline{Q_j}(x)\,dx \qquad (9)$$

Defining the following matrices

$$E=\{e_{i,j}\}_{i,j=0}^{N+1}=\int_0^{\pi}\overline{Q_i}'(x)\overline{Q_j}'(x)\,dx \qquad (10)$$

$$D=\{d_{i,j}\}_{i,j=0}^{N+1}=\int_0^{\pi}\overline{Q_i}(x)\overline{Q_j}(x)\,dx \qquad (11)$$

Equation (9) is an (N + 2) × (N + 2) linear system

$$\begin{cases}\left(D+\frac{1}{2}a\Delta tE-\frac{1}{2}c\Delta tD\right)\alpha^n=\left(D-\frac{1}{2}a\Delta tE+\frac{1}{2}c\Delta tD\right)\alpha^{n-1}-\frac{1}{2}b\Delta tE(\alpha^{n-m}+\alpha^{n-m-1})+\frac{1}{2}d\Delta tD(\alpha^{n-m}+\alpha^{n-m-1})\\ \alpha^n=\gamma^n, for-m\le n\le 0.\end{cases} \qquad (12)$$

We can write the entries of the matrices E and D in (10) and (11) as stiffness matrix

$$E=\frac{3}{10h}\begin{bmatrix}80&43&-20&-1&0&\dots&\dots&\dots&0\\43&104&-14&-24&-1&\ddots&&&\vdots\\-20&-14&80&-15&-24&-1&\ddots&&\vdots\\-1&-24&-15&80&-15&-24&-1&\ddots&\vdots\\0&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&0\\\vdots&\ddots&-1&-24&-15&80&-15&-24&-1\\\vdots&&\ddots&-1&-14&-15&80&-14&-20\\\vdots&&&\ddots&-1&-24&-14&104&43\\0&\dots&\dots&\dots&0&-1&-20&43&80\end{bmatrix}$$

and the mass matrix

$$D=\frac{h}{140}\begin{bmatrix}496&773&116&1&0&\dots&\dots&\dots&0\\773&2296&1190&120&1&\ddots&&&\vdots\\116&1190&2416&1191&120&1&\ddots&&\vdots\\1&120&1191&2416&1191&120&1&\ddots&\vdots\\0&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&0\\\vdots&\ddots&1&120&1191&2416&1191&120&1\\\vdots&&\ddots&1&120&1191&2416&1190&116\\\vdots&&&\ddots&1&120&1190&2296&773\\0&\dots&\dots&\dots&0&1&116&773&496\end{bmatrix}$$

with Υ^n is the initial approximation of $\varphi(t_n)$ and $\alpha^n:=[\alpha_0,\alpha_1,\dots\alpha_{N+1}]^T$.

Convergence Analysis

We shall prove the optimal convergence order for the numerical scheme.

Let u_h and u be the solutions of (1). Then error can be quantified by bounding the norm of the error $u_h(t)-u(t)$ *in* terms of the mesh spacing h of the finite element mesh.

Remark 1. Using polynomial with degrees $p\ge 1$ as basis we expect an error bound of

$$\|u_h(t)-u(t)\|\le Ch^{p+1},$$

where C is a problem-dependent constant independent of h and the constant $p+1$ indicates the order of convergence of the FEM, as the mesh spacing h decreases.

Theorem 2 Let u and U^n be the exact and approximation solution of Eq.(1) respectively. Assume that $\|u(t)-R_hu(t)\|\le Ch^4\|u(t)\|_r$, $\|u_t(t)-R_hu_t(t)\|_i\le Ch^4\|u_t(t)\|_r$,$-\tau\le i\le 0$, and $\|\varphi_h(t)-\varphi(t)\|_i\le Ch^4$. *then*

$$\|U^n-u(t_n)\|\le C(h^4+(\Delta t)^2), for\ any\ n=1,2,\dots$$

Proof.Denote

$$U^n-u(t_n)=\left(U^n-D_hu(t_n)\right)+\left(D_hu(t_n)-u(t_n)\right)=\theta^n+\rho^n$$

and $\rho^n(t)=\rho(t_n)$ is bounded as in [26].

$$\left(\frac{\theta^n-\theta^{n-1}}{\Delta t},\chi\right)+a\left(\frac{\nabla\theta^n+\nabla\theta^{n-1}}{2},\nabla\chi\right)+b\left(\frac{\nabla\theta^{n-m}+\nabla\theta^{n-m-1}}{2},\nabla\chi\right)-c\left(\frac{\theta^n+\theta^{n-1}}{2},\chi\right)$$
$$-d\left(\frac{\theta^{n-m}+\theta^{n-m-1}}{2},\chi\right)=-(W^n,\ \chi),\forall\,\chi\in S_h^T \qquad (15)$$

where

$$W^n=\frac{D_hu(t_n)-D_hu(t_{n-1})}{\Delta t}-\frac{u_t(t_n)+u_t(t_{n-1})}{2}$$

$$=(D_h-I)\tilde{\partial}u(t_n)+\left(\tilde{\partial}u(t_n)-\frac{u_t(t_n)+u_t(t_{n-1})}{2}\right)=:W_1^n+W_2^n$$

Choosing $\chi=\frac{\theta^n+\theta^{n-1}}{2}$, gives

$$\left(\frac{\theta^n-\theta^{n-1}}{\Delta t},\frac{\theta^n+\theta^{n-1}}{2}\right)+a\left\|\frac{\theta^n+\theta^{n-1}}{2}\right\|_1^2+b\left(\frac{\nabla\theta^{n-m}+\nabla\theta^{n-m-1}}{2},\frac{\nabla\theta^n+\nabla\theta^{n-1}}{2}\right)-c\left(\frac{\theta^n+\theta^{n-1}}{2},\frac{\theta^n+\theta^{n-1}}{2}\right)$$
$$-d\left(\frac{\theta^{n-m}+\theta^{n-m-1}}{2},\frac{\theta^n+\theta^{n-1}}{2}\right)=-\left(W^n,\frac{\theta^n+\theta^{n-1}}{2}\right)$$

By using Schwartz inequalty

$$\left(\frac{\theta^n-\theta^{n-1}}{\Delta t},\frac{\theta^n+\theta^{n-1}}{2}\right)+\left\|\frac{\theta^n+\theta^{n-1}}{2}\right\|_1^2$$
$$\le C\left(\left\|\frac{\theta^{n-m}+\theta^{n-m-1}}{2}\right\|_1^2+\left\|\frac{\theta^n+\theta^{n-1}}{2}\right\|_1^2+\left\|\frac{\theta^{n-m}+\theta^{n-m-1}}{2}\right\|_1^2+\|W^n\|\left\|\frac{\theta^n+\theta^{n-1}}{2}\right\|_1^2\right)$$

So

$$\|\theta^n\|^2+\Delta t\left\|\frac{\theta^n+\theta^{n-1}}{2}\right\|_1^2$$
$$\le C\left(\|\theta^{n-1}\|^2+\Delta t\left\|\frac{\theta^{n-m}+\theta^{n-m-1}}{2}\right\|_1^2+\Delta t\left\|\frac{\theta^n+\theta^{n-1}}{2}\right\|_1^2+\Delta t\left\|\frac{\theta^{n-m}+\theta^{n-m-1}}{2}\right\|_1^2\right.$$
$$\left.+(\Delta t)^2\|W^n\|^2\right).$$

Without loss of generality ,assume that $n\in((k-1)m,km]$, $k\in N$.Then,

$$\|\theta^n\|^2\le C\left(\|\theta^{n-1}\|^2+\Delta t\left\|\frac{\theta^{n-m}+\theta^{n-m-1}}{2}\right\|_1^2+\Delta t\left\|\frac{\theta^{n-m}+\theta^{n-m-1}}{2}\right\|_1^2+(\Delta t)^2\|W^n\|^2\right).$$

$$\le C(\|\theta^{n-1}\|^2+\|\theta^{n-m-1}\|^2+\Delta t\|\theta^{n-2m}+\theta^{n-2m-1}\|_1^2+(\Delta t)^2(\|W^n\|^2+\|W^{n-m}\|^2))$$

$$\le\dots\le C\left(\sum_{i=0}^{k-1}\left\|\theta^{n-im-1}\right\|^2+\Delta t\left\|\theta^{n-km}+\theta^{n-km-1}\right\|_1^2+(\Delta t)^2\sum_{i=0}^{k-1}\left\|W^{n-im}\right\|^2\right)$$

Therefore,

$$\|\theta^n\|^2 \le C\left(\sum_{l=0}^{k-1}\|\theta^{n-im-1}\|^2 + \Delta t\,\|\theta^{n-km}+\theta^{n-km-1}\|_1^2 + (\Delta t)^2\sum_{i=0}^{k-1}\|W^{n-im}\|^2\right)$$

By the assumption of the theorem and using the discrete Gronwall inequality (see [42])

$$\|\theta^n\|^2 \le C\left(\|\theta^0\|^2 + \Delta t\|\theta^{n-km}+\theta^{n-km-1}\|_1^2 + (\Delta t)^2\textstyle\sum_{i=0}^{k-1}\|W^{n-im}\|^2\right) \quad (16)$$

We write

$$W_1^n = (D_h - I)\tilde{\partial}u(t_n) = (\Delta t)^{-1}\int_{t_{n-1}}^{t_n}(D_h - I)u_t(t)dt$$

So

$$(\Delta t)^2\textstyle\sum_{i=1}^{k-1}\|W^{n-im}\|^2 \le \sum_{i=1}^{k-1}\left(\int_{t_{n-im-1}}^{t_{n-im}} Ch^4\|u_t(t)\|_4 dt\right)^2 \le Ch^4 \quad (17)$$

Further,

$$\|\Delta t W_2^i\| = \left\|u(t_i) - u(t_{i-1}) - \Delta t\frac{u_t(t_i)+u_t(t_{i-1})}{2}\right\| \le C(\Delta t)^2\int_{t_{i-1}}^{t_i}\|u_{ttt}(t)\|dt$$

So that

$$(\Delta t)^2\sum_{i=1}^{k-1}\|W^{n-im}\|^2 \le C(\Delta t)^4\sum_{i=1}^{k-1}\left(\int_{t_{im-1}}^{t_{n-im}}\|u_{ttt}(t)\|dt\right)^2 \le C(\Delta t)^4, \quad (18)$$

$$\|U^n - u(t_n)\| \le C(h^4 + (\Delta t)^2),$$

which together with Eq. (17) and Eq. (18) and the assumption of the theorem

Numerical Simulation

We carry out numerical experiments to illustrate the theoretical results. To evaluate errors, L_2 *and* $L\infty$ norms are applied as follows:

$$L_2 = \|u(t_n) - U^n\|_2 = \sqrt{h\sum_{n=0}^{N+1}|u(t_n) - U^n|^2}. \quad L_\infty = \|u(t_n) - U^n\|_\infty = Max|u(t_n) - U^n|$$

Order of convergence is denoted by

$$\text{Order (r)} = \frac{\log(\text{Error (N)/Error (2N)})}{\log(2 - \frac{1}{N+1})}$$

Example 1 We consider the following equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a\frac{\partial^3 u(x,t)}{\partial x^2} + b\frac{\partial^3 u(x,t-\tau)}{\partial x^2} + cu(x,t) + du(x,t-\tau) \\ t > 0, 0 < x < \pi \\ u(x,t) = \sin(x), -\tau \le t \le 0, 0 \le x \le \pi \\ u(0,t) = \varphi(\pi,t) = 0, t \ge -\tau \end{cases} \quad (19)$$

We set the parameters $a = 2$, $b = 0.5$, $c = 1$, $d = 0.5$, $\tau = 1$ and $T = 2$ and solve problem Eq.(19) on $[0, \pi] \times [0,2]$ with different temporal and spatial step sizes ($\Delta t = \tau/(m+1)$, $h = \pi/(N+1)$). The exact solution is $u(x.t) = exp(-t)sin(x)$.

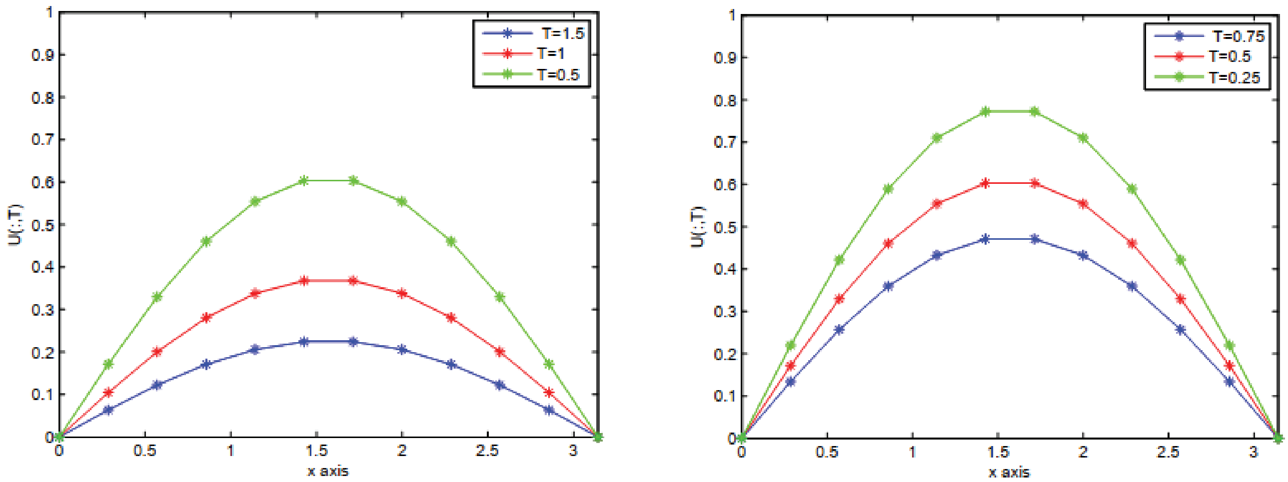
Numerical errors and the corresponding orders are listed in Table 1 - 5. There is a noticeable decrease in both error norms when mesh sizes decrease. These results confirm that the numerical method is convergent. The results in Tables 3, 4 and 5 show that our result is more accurate than in comparison with those obtained by [18]. In Figure 1 and 5, the graph of approximation solutions for example 1 and 2 at different times respectively are given. In figures 2 and 6, the approximation and exact solution are drawn on the same coordinate axis. Approximation and exact solution are depicted in Figures 3, 4,7, and 8. It can be observed that the graphs are similar.

Table 1. Error norms and convergence orders for $T = 2, \tau = 1(\Delta t \approx \Delta x^2)$

N	L_2	Order	L_∞	Order
7	7.8685E-04	-	8.5506E-04	-
14	6.4117E-05	3.9887	6.9156E-05	4.006
28	4.4172E-06	4.0580	4.7832E-06	4.0520
56	2.7781E-07	4.0937	3.0115E-07	4.0921
112	1.7472E-08	4.0424	1.8946E-8	4.0419

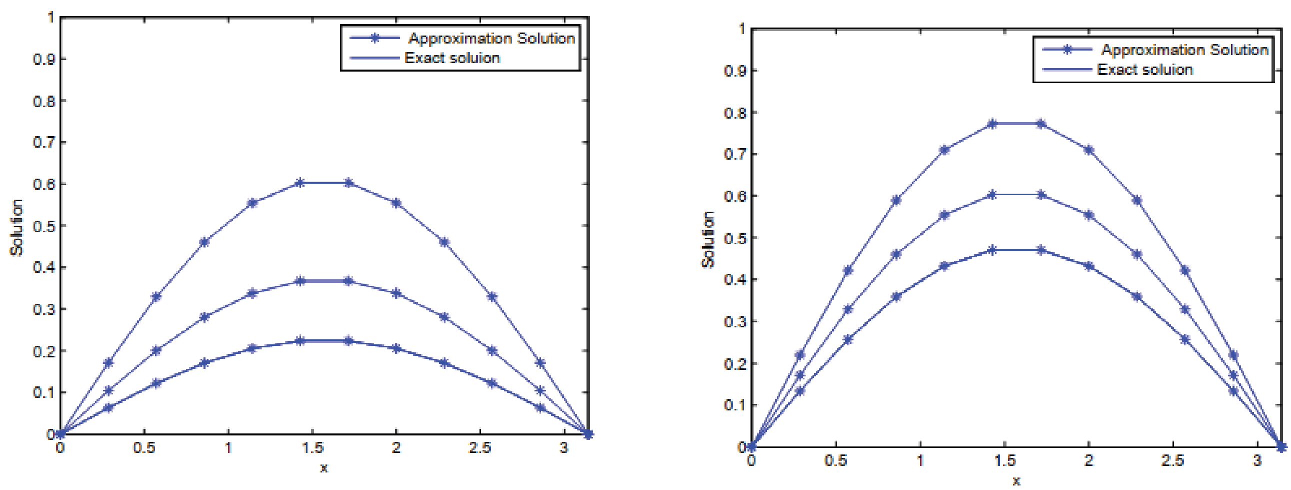
Table 2. Error norms and convergence orders for $T = 2, \tau = 1(\Delta t \approx \Delta x^2)$

N	L_2	Order	L_∞	Order
10	2.3378E-04	-	2.5206E-04	-
20	1.6027E-05	4.1447	1.7332E-05	4.1340
40	1.0649E-06	4.0539	1.1530E-06	4.0508
80	6.7117E-8	4.0585	7.2770E-8	4.0577



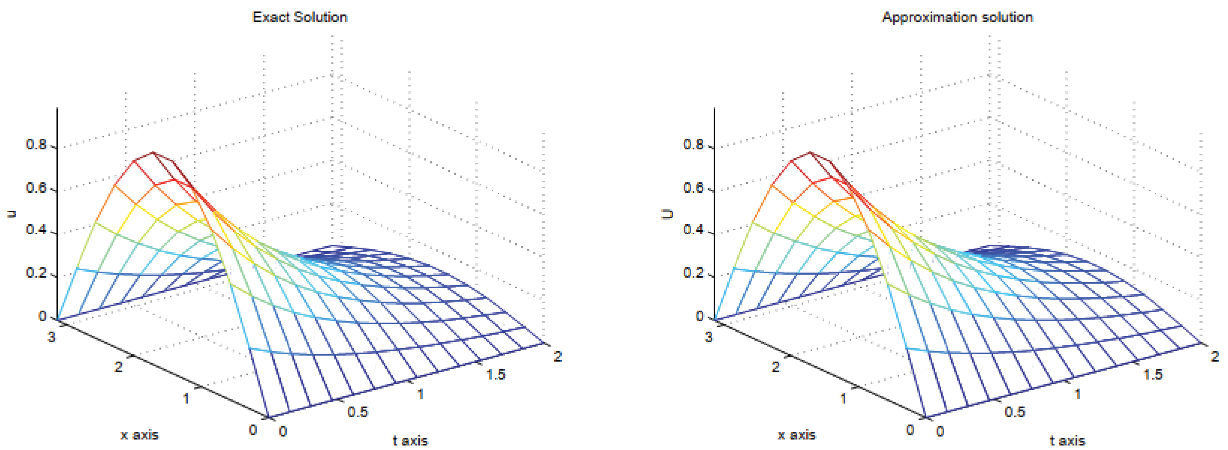
(a) **(b)**

Figure 1. Approximate solutions for fixed time and step sizes ($N = 10$, $m = 100$) for example 1.



(a) **(b)**

Figure 2. Comparison between approximate and exact solutions for example 1 where (a) $T = 1.5, T = 1, T = 0.5$ with step sizes ($N = 10, m = 100$). (b) $T = 0.75. T = 0.5, T = 0.25$ with step sizes ($N = 10, m = 100$).



(a) **(b)**

Figure 3. Approximation solution and analytical solution ($N = 10, m = 5, and\ \tau = 1$).

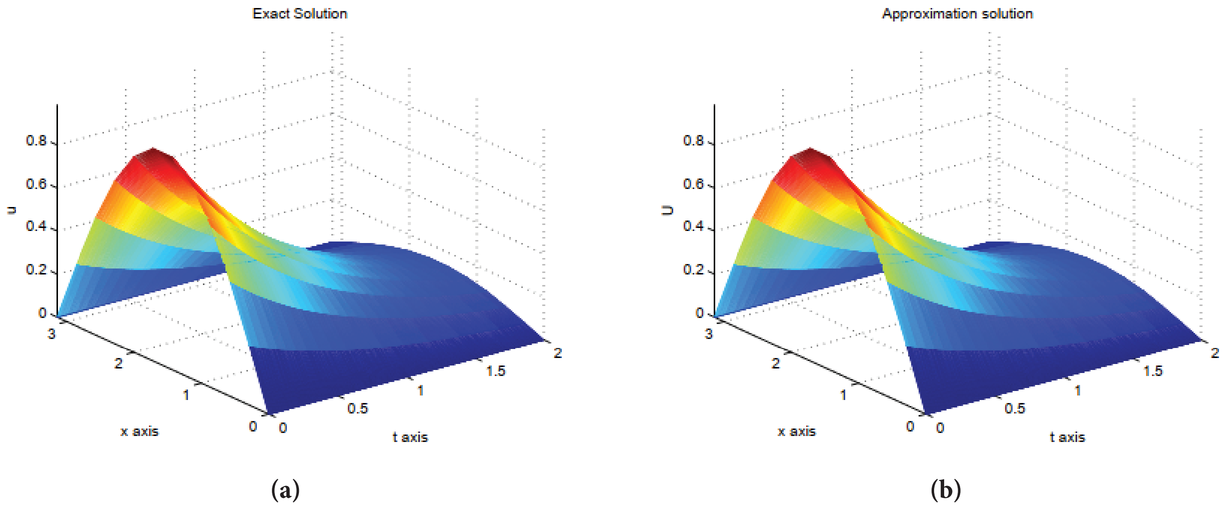


Figure 4. Comparisons between approximate and exact solutions of example 1 with step size ($N = 10, m = 200$).

Table 3. Error norms and convergence orders for $T = 2, \tau = 0.5(\Delta t \approx 0.5\Delta x^2)$

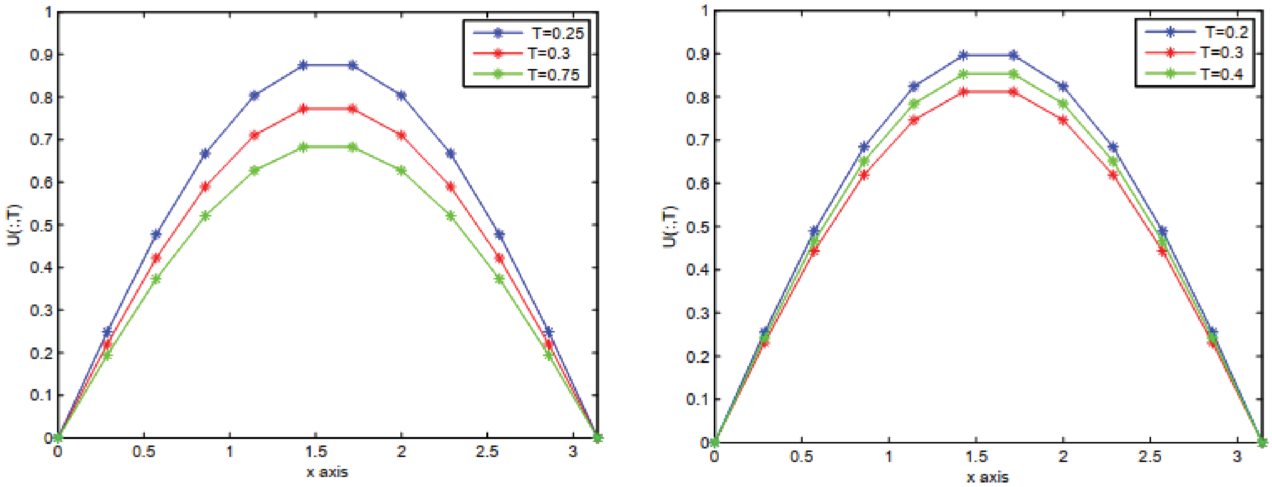
N	Compact θ−*method* (θ=1)[18]				*Present method*			
	L_2	*Order*	L_∞	*Order*	L_2	*Order*	L_∞	*Order*
10	2.96E-04	4.20	3.36E-04	4.13	6.11E-05	4.45	6.27E-05	4.31
20	1.82E-05	4.03	1.45E-05	4.03	4.05E-06	4.20	4.33E-06	4.14
40	1.13E-06	4.00	9.05E-07	4.00	2.67E-07	4.06	2.88E-07	4.05
80	7.09E-08	4.00	5.65E-08	4.00	1.68E-08	4.06	1.82E-08	4.06

Table 4. Error norms and convergence orders for $T = 2, \tau = 0.5$ $(\Delta t \approx \Delta x^2)$

N	Linear θ−*method* (θ=1)[18]				*Present method*			
	L_2	*Order*	L_∞	*Order*	L_2	*Order*	L_∞	*Order*
10	4.34E-03	2.31	3.47E-03	2.24	2.13E-04	3.85	2.11E-04	3.65
20	9.83E-04	2.14	7.84E-04	2.14	1.64E-05	3.96	1.73E-05	3.86
40	2.38E-04	2.04	1.90E-04	2.04	1.06E-06	4.00	1.14E-06	4.07
80	5.94E-05	2.01	4.72E-05	2.01	6.70E-08	4.05	7.25E-08	4.04

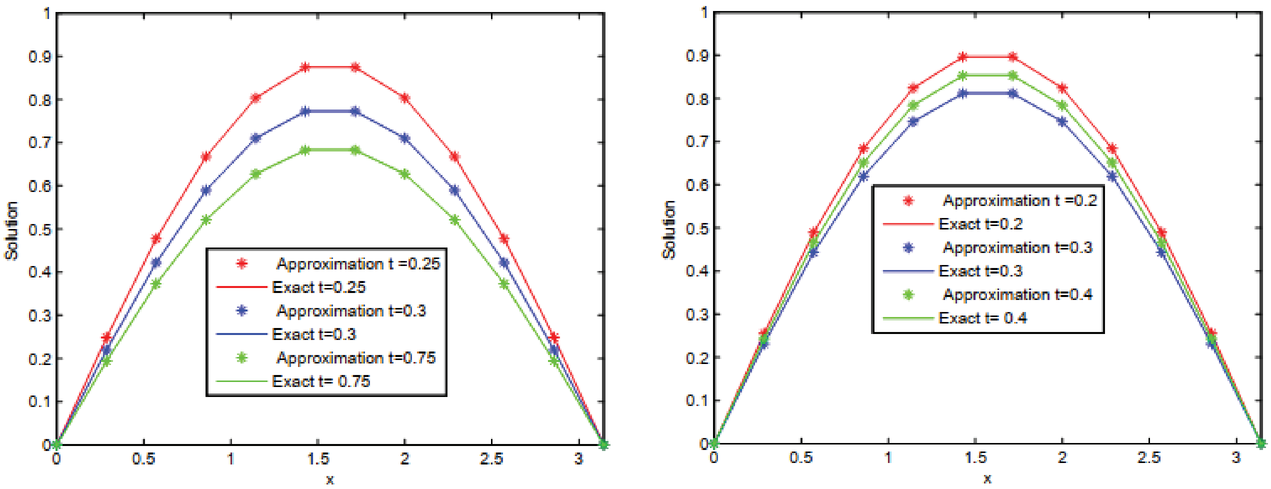
Table 5. Error norms and convergence orders for $T = 2, \tau = 0.5$ $(\Delta t \approx 0.5\Delta x^2)$

N	Linear θ−*method* (θ=1/2)[18]				*Present method*			
	L_2	*Order*	L_∞	*Order*	L_2	*Order*	L_∞	*Order*
5	1.40E-03	-	1.06E-03	-	9.17E-04	-	8.55E-04	-
10	3.81E-04	1.87	3.04E-04	1.80	6.11E-05	4.45	6.27E-05	4.31
20	9.67E-05	1.98	7.72E-05	1.98	4.05E-06	4.20	4.33E-06	4.13
40	2.61E-05	1.89	2.09E-05	1.89	2.67E-07	4.06	2.88E-07	4.05
80	6.15E-06	2.09	4.91E-06	2.09	1.68E-08	4.06	1.82E-08	4.06



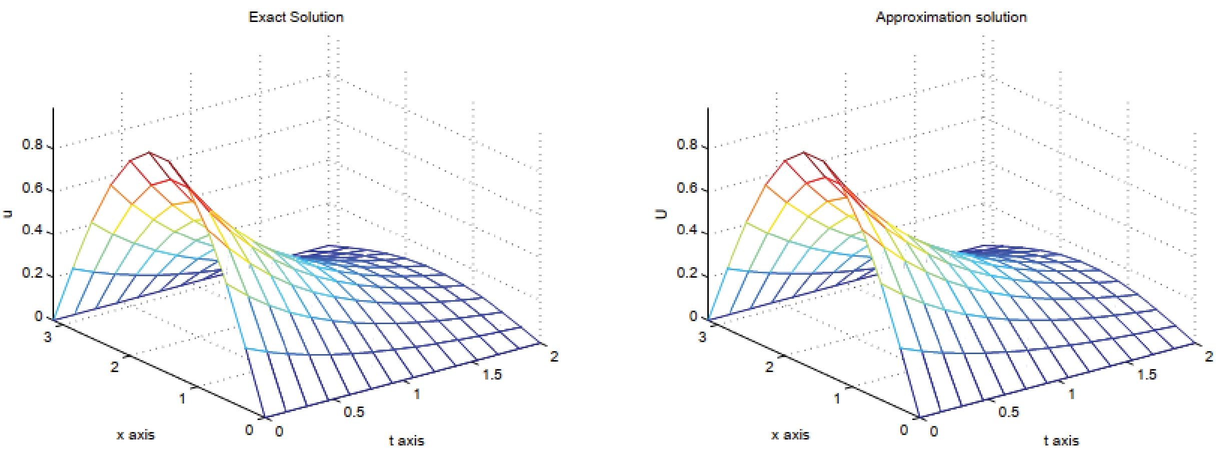
(a) (b)

Figure 5. Approximate solutions for fixed time and step sizes (N = 10, m = 100) for example 2.



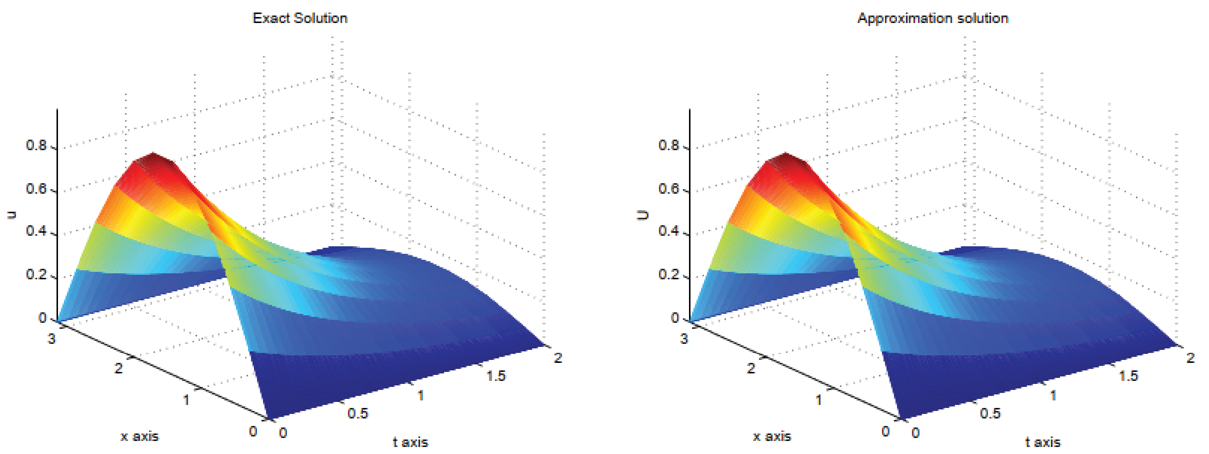
(a) (b)

Figure 6. Comparisons between approximate and exact solutions of example 2 (N = 10, m = 100).



(a) (b)

Figure 7. Approximation solution and analytical solution (N = 10, m = 5, and τ = 1).



(a) (b)

Figure 8. Comparison between approximate and exact solutions for example 2 (N = 10; m = 100).

CONCLUSION

A finite element method based on redefined cubic B-spline shape function has been developed to solve the generalized reaction-diffusion equation with delay. The convergence analysis is studied. From this result, we can conclude that the cubic B-spline finite element method is feasible. The approximation solutions are tested by comparing with analytical solutions. Our selection of redefined cubic B-spline basis functions improves accuracy to fourth-order. The advantage of the FEM over the finite difference method is its ability to deal with the modeling of complex geometries and irregular shapes. The proposed method gives a better convergence result than the linear θ -method without increasing the computational costs. The numerical results are obtained by applying MATLAB software. This method can be easily extended to tackle a broad class of PDEs.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Li D, Zhang J, Zhang Z. Unconditionally optimal error estimates of a linearized Galerkin method for nonlinear time fractional reaction sub-diffusion equations. J Sci Comput 2018;76:848-866. [CrossRef]

[2] Li D, Zhang J. Efficient implementation to numerically solve the nonlinear time fractional parabolic problems on unbounded spatial domain. J Comput Phys 2016;322:415-428. [CrossRef]

[3] Li D, Wang J. Unconditionally optimal error analysis of Crank-Nicolson Galerkin FEMs for a strongly nonlinear parabolic system. J Sci Comput 2017;72:892-915. [CrossRef]

[4] Kongson J, Amornsamankul S. A model of the signal transduction process under a delay. East Asian J Appl Math 2017;7:741-751. [CrossRef]

[5] Culshaw RV, Ruan S, Webb G. A mathematical model of cell-to-cell spread of HIV-1 that includes a time delay. J Math Biol 2003;46:425-444. [CrossRef]

[6] Kumar R, Sharma AK, Agnihotri K. Dynamics of an innovation diffusion model with time delay. East Asian J Appl Math 2017;7:455-481. [CrossRef]

[7] Mei M, Ou C, Zhao XQ. Global stability of mono stable traveling waves for nonlocal time delayed reaction-diffusion equations. SIAM J Math Anal 2010;42:2762-2790. [CrossRef]

[8] Adam AM, Bashier EB, Hashim MH, Patidar KC. Fitted Galerkin spectral method to solve delay partial differential equations. Math Methods Appl Sci 2016;39:3102-3115. [CrossRef]

[9] Reyes E, Rodrguez F, Martn JA. Analytic-numerical solutions of diffusion mathematical models with delays. Comput Math Appl 2008;56:743-753. [CrossRef]
[10] Wu F, Li D, Wen J, Duan J. Stability and convergence of compact finite difference method for parabolic problems with delay. Appl Math Comput 2018;322:129-139. [CrossRef]
[11] Zhang Q, Chen M, Xu Y, Xu D. Compact -method for the generalized delay diffusion equation. Appl Math Comput 2018;316:357-369. [CrossRef]
[12] Martn JA, Rodrguez F, Company R. Analytic solution of mixed problems for the generalized diffusion equation with delay. Math Comput Model 2004;40:361-369. [CrossRef]
[13] Garcia P, Castro MA, Martn JA, Sirvent A. Numerical solutions of diffusion mathematical models with delay. Math Comput Model 2009;50:860-868. [CrossRef]
[14] Huang C, Vandewalle S .Unconditionally stable difference methods for delay partial differential equations. Numer Math 2012;122:579-601. [CrossRef]
[15] Li D, Zhang C, Wen J. A note on compact finite difference method for reaction-diffusion equations with dela. Appl Math Model 2015;39:1749-1754. [CrossRef]
[16] Li D, Zhang C. On the long time simulation of reaction-diffusion equations with delay. Sci World J 2014;186802. [CrossRef]
[17] Blanco-Cocom L, Vila-Vales E. Convergence and stability analysis of the -method for delayed diffusion mathematical models. Appl Math Comput 2014;23:16-25. [CrossRef]
[18] Wu F, Wang Q, Cheng X, Chen X. Linear θ -Method and Compact Method for Generalised Reaction-Diffusion Equation with Delay. Intern J Differ Eq 2018;2018:6402576.
[19] Tian H. Asymptotic stability analysis of the linear θ -method for linear parabolic differential equations with delay. J Differ Equ Appl 2009;15:473-487. [CrossRef]
[20] Sun ZZ, Zhang ZB. A linearized compact difference scheme for a class of nonlinear delay partial differential equations. Appl Math Model 2013;37:742-752. [CrossRef]
[21] Zhang Q, Zhang C. A compact difference scheme combined with extrapolation techniques for solving a class of neutral delay parabolic differential equations. Appl Math Lett 2013;26:306-312. [CrossRef]
[22] Li D, Zhang C, Qin H. LDG method for reaction-diffusion dynamical systems with time delay. Appl Math Comput 2011;217:9173-9181. [CrossRef]
[23] Bhrawy AH, Abdelkawy MA, Mallawi F. An accurate Chebyshev pseudospectral scheme for multi-dimensional parabolic problems with time delays. Bound Value Probl 2015;1:1-20. [CrossRef]
[24] Aziz I, Amin R. Numerical solution of a class of delay differential and delay partial differential equations via Haar wavelet. Appl Math Model 2016;40:10286-10299. [CrossRef]
[25] Chen X, Wang L. The variational iteration method for solving a neutral functional-differential equation with proportional delays. Comput Math Appl 2010;59:2696-2702. [CrossRef]
[26] Jackiewicz Z, Zubik-Kowal B. Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations. Appl Numer Math 2006;56:433-443. [CrossRef]
[27] El-Dib YO. Periodic solution and stability behavior for nonlinear oscillator having a cubic nonlinearity time-delayed. Int Ann Sci 2018;5:12-25. [CrossRef]
[28] Kutluay SE, Esen AL, Dag I. Numerical solutions of the Burgers equation by the least-squares quadratic B-spline finite element method. J Comput Appl Math 2004;167:21-33. [CrossRef]
[29] Tasbozan O, Esen A. Quadratic B-spline Galerkin method for numerical solutions of fractional telegraph equations. Bull Math Sci Appl 2017;18:23-39. [CrossRef]
[30] Kutluay S, Ucar Y. Numerical solutions of the coupled Burgers equation by the Galerkin quadratic Bspline finite element method. Math Methods Appl Sci 2013;36:2403-2415. [CrossRef]
[31] Karako SBG, Zeybek H. A cubic B-spline Galerkin approach for the numerical simulation of the GEW equation. Stat Optim Inf Comput 2016;4:30-41. [CrossRef]
[32] Soliman AA. A Galerkin solution for Burgers' equation using cubic B-spline finite elements. Abstr Appl Anal 2012;2012:527467. [CrossRef]
[33] Goh J, Majid AA, Ismail AIM. Numerical method using cubic B-spline for the heat and wave equation. Comput Math Appl 2011;62:4492-4498. [CrossRef]
[34] Aksan EN. An application of cubic B-Spline finite element method for the Burgers equation. Therm Sci 2018;22:195-202. [CrossRef]
[35] Mirzaee F, Alipour S. Bi-cubic B-spline functions to solve linear two-dimensional weakly singular stochastic integral equation. Iran J Sci Technol Trans A Sci 2021;45:965-972. [CrossRef]
[36] Mirzaee F, Alipour S. Cubic B-spline approximation for linear stochastic integro-differential equation of fractional order. J Comput Appl Math 2020;366:112440. [CrossRef]
[37] Mirzaee F, Alipour S. An efficient cubic B-spline and bi-cubic B-spline collocation method for numerical solutions of multidimensional nonlinear stochastic quadratic integral equations. Math Methods Appl Sci 2020;43:384-397. [CrossRef]
[38] Korkmaz A, Akso AM, Dag I. Quartic B-spline differential quadrature method. Int J Nonlinear Sci 2011;11:403-411.

[39] Mirzaee F, Alipour S. Quintic B-spline collocation method to solve n-dimensional stochastic Volterra integral equations. J Comput Appl Math 2021;384:113-153. [CrossRef]

[40] Kutluay, S, Yagmurlu NM. The modified Bi-quintic B-spline base functions, an application to diffusion equation. Int J 2017;5:26-32.

[41] Liang H. Convergence and asymptotic stability of Galerkin methods for linear parabolic equations with delays. Appl Math Comput 2015;264:160-178. [CrossRef]

[42] Brunner H. Collocation methods for Volterra integral and related functional differential equations. Annotated ed. Cambridge: Cambridge University Press; 2004. [CrossRef]