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Research Article

Inhomogeneous quantum group of Q-fermions

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ABSTRACT

In this work, we introduce a nonstandard algebra of q-fermions where q is a nonzero complex deformation parameter for the algebra of the commuting fermions. In order to show that q-fermions provides a proper generalization of the algebra of usual commuting fermions, we prove that there is an inhomogeneous quantum structure associated with q-fermions for a complex number q with |q| = 1.

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INTRODUCTION

Many people consider that quantum spaces provide a paradigm for the general frame of quantum deformed physics [1]. As a special case of quantum space, a quantum plane is studied in [2] by constructing the group of linear transformations on it, which is called quantum group of quantum matrices. In fact, the notion of quantum group (q-deformed Lie group), also known as a noncommutative or a noncocommutative generalization of a Hopf algebra, was first used by V.G. Drinfeld [3] and independently discovered by M. Jimbo [4] at the same time. Inspired by their pioneering works, many works were done in the different directions of mathematics and physics.

It is well known that the q-boson algebra can be used to construct the highest weight representations the quantum group SUq(2) (see [5,6]). In addition to the boson algebra, many people studied fermions by distinct type algebras. For example, an algebra of q-fermion creation and annihilation operators is given in [7] by following commutation relations :

$$a_q a_q^* + \sqrt{q} a_q^* a_q = q^{-N_a/2}$$

where N_a satisfies

$$[N_a, a_q^*] = a_q^*, \quad [N_a, a_q] = -a_q.$$

It is also shown in [7] that if , then many q-fermions can occupy a given state in contrast to the case of ordinary fermions. Furthermore, in [8], the algebra of commuting fermions is described as an associative algebra generated by creation and annihilation operators which are subject to the following relations:

$$c_i c_j + \sigma_{ij} c_j c_i = 0 \tag{1.1}$$

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$$c_i c_j^* + \sigma_{ij} c_j^* c_i = \delta_{ij} \qquad i, j = 1, 2, \ldots, d, \tag{1.2}$$

where each $c_i = \underbrace{1 \otimes \ldots \otimes c}_{i} \otimes \overbrace{1 \otimes \ldots \otimes 1}^{d-i}$ is called commuting fermions and σ_{ij} is defined as follows

$$\sigma_{ij} = \begin{cases} 1, & i = j \\ -1, & i < j. \end{cases} \tag{1.3}$$

Note that letting $i = j$ in (1.1) we obtain $c_i^2 = 0,\ c_i^{*2} = 0$, meaning that the Pauli exclusion principle is satisfied.

By the approach used in [2], Altıntaş et al. presented the inhomogenous quantum symmetry group of the commuting fermions in [8]. In this study, we define a new algebra of q-commuting fermions containing the algebra of commuting fermions given by the relations in (1.1) and (1.2). Moreover, we show that the algebra of such q-fermions admits an inhomogenous quantum symmetry group whenever $|q| = 1$. Now we need to recall from [2, 8, 9], some basic definitions and notions which will be necessary to present our results:

Let A be a vector space over $K = \square$ or $\square$ and $m : \mathsf{A} \otimes \mathsf{A} \to K$ (the multiplication), $\eta : K \to \mathsf{A}$ (the unit mapping) be two linear mappings. Then the triple (A, m, η) is called an *algebra* if the following conditions hold:

$$\begin{aligned} m \circ (m \otimes \mathrm{id}) &= m \circ (\mathrm{id} \otimes m) \\ m \circ (\eta \otimes \mathrm{id}) &= m \circ (\mathrm{id} \otimes \eta) \end{aligned} \tag{1.4}$$

where id is the identity mapping. A *coalgebra* A over K is a K vector space A, together with two linear mappings $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ and $\varepsilon : \mathsf{A} \to K$ satisfying the following rules:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{1.5}$$

$$m \circ ((\varepsilon \otimes \mathrm{id}) \circ \Delta) = \mathrm{id} = m \circ ((\mathrm{id} \otimes \varepsilon) \circ \Delta). \tag{1.6}$$

A *bialgebra* is a unital associative algebra endowed with a coalgebra structure admitting the compatibility conditions, that is, Δ and ε are both algebra homomorphisms with $\Delta(1_{\mathsf{A}}) = 1_{\mathsf{A}} \otimes 1_{\mathsf{A}}$ and $\varepsilon(1_{\mathsf{A}}) = 1_K$.

A *Hopf algebra* is a bialgebra A endowed with an algebra antihomomorphism from $\mathsf{A} \to \mathsf{A}$ satisfying the following condition

$$m \circ ((S \otimes \mathrm{id}) \circ \Delta) = \eta \circ \varepsilon = m \circ ((\mathrm{id} \otimes S) \circ \Delta). \tag{1.7}$$

A *quantum plane* is identified with an associative polynomial algebra generated by x and y satisfying q-commutation rule

$$xy = qyx \tag{1.8}$$

where q is a nonzero complex parameter. Moreover, the homogeneous quantum symmetry group of this quantum plane is considered in [2] as a group of linear transformations acting on a quantum plane as follows

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{1.9}$$

where x' and y' are q-commutative in the sense of (1.8), and the matrix entries a, b, c, d are commutative with the generators x, y. Thus, an element of the quantum group acting on the quantum plane is defined as a matrix M of the following form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.10}$$

where a, b, c, d hold the following noncommutative relations:

$$\begin{aligned} &ab = qba, ac = qca, ad = da + (q - q^{-1})bc, \\ &bc = cb, bd = qdb, cd = qdc. \end{aligned} \tag{1.11}$$

Manin also defined the determinant of such a quantum matrix as follows

$$det_q M = ad - qbc = da - q^{-1}bc. \tag{1.12}$$

So, if $det(M) \neq 0$, then its inverse is described as follows

$$M^{-1} = \frac{1}{det(M)} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \tag{1.13}$$

The Quantum invariance group of q-commuting fermions

Let A be an associative algebra generated by c_i's and their hermitian conjugates c_i^*'s which are subject to the following relations

$$c_i c_j + \sigma_{ij} c_j c_i = 0 \tag{2.1}$$

$$c_i c_j^* + \sigma_{ij} c_j^* c_i = \delta_{ij} \qquad i,j = 1,2,\ldots,d \tag{2.2}$$

where σ_{ij} is defined as follows

$$\sigma_{ij} = \begin{cases} 1, & i=j \\ -q, & i<j \end{cases} \text{ and } \sigma_{ij} = \sigma_{ji}^{-1}, \tag{2.3}$$

for a complex number q with $|q| = 1$. Note that this algebra becomes the algebra of commuting fermions given in [8] when $q = 1$. We also remark that one can obtain c_i^2 0, c_i^{*2} 0 whenever $i = j$ in (2.1). Thus it is seen that the Pauli exclusion principle is also satisfied for this algebra, as well. Now, following the approach used in [8], one can consider the inhomogeneous linear quantum transformation acting on the generators c_i and c_i^* as follows

$$c_{i'} = \sum_{j=1}^{d} \alpha_{ij} \otimes c_j + \sum_{j=1}^{d} \beta_{ij} \otimes c_j^* + \gamma_i \otimes 1 \tag{2.4}$$

$$c_i^{*'} = \sum_{j=1}^{d} \alpha_{ij}^* \otimes c_j^* + \sum_{j=1}^{d} \beta_{ij}^* \otimes c_j + \gamma_i^* \otimes 1. \tag{2.5}$$

This transformation matrix is expressed in the matrix form as follows

$$T = \begin{pmatrix} \alpha_{ij} & \beta_{ij} & \gamma_i \\ \beta_{ij}^* & \alpha_{ij}^* & \gamma_i^* \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & \Gamma \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

$$A = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ \beta_{ij}^* & \alpha_{ij}^* \end{pmatrix}, \tag{2.7}$$

and

$$\Gamma = \begin{pmatrix} \gamma_i \\ \gamma_i^* \end{pmatrix}. \tag{2.8}$$

Here the matrices $(\alpha_{ij}), (\alpha_{ij}^*), (\beta_{ij}), (\beta_{ij}^*)$ are $d \times d$, A is $2d \times 2d$, $(\gamma_i), (\gamma_i^*)$ are $d \times 1$ and Γ is $2d \times 1$. If we require that the new elements c_i and c_i^* hold the commutation relations (2.1) and (2.2), one can obtain the following relations among the entries of T matrix:

$$\begin{aligned}
&\alpha_{ij}\alpha_{kl} = \sigma_{ik}\sigma_{lj}\alpha_{kl}\alpha_{ij}, && \beta_{ij}\beta_{kl} = \sigma_{ik}\sigma_{lj}\beta_{kl}\beta_{ij}, \\
&\alpha_{ij}\alpha_{kl}^* = \sigma_{ik}\sigma_{lj}\alpha_{kl}^*\alpha_{ij}, && \beta_{ij}\beta_{kl}^* = \sigma_{ik}\sigma_{lj}\beta_{kl}^*\beta_{ij}, \\
&\alpha_{ij}\beta_{kl} = \sigma_{ik}\sigma_{lj}\beta_{kl}\alpha_{ij}, && \beta_{ij}\gamma_k = -\sigma_{ik}\gamma_k\beta_{ij}, \\
&\alpha_{ij}\beta_{kl}^* = \sigma_{ik}\sigma_{lj}\beta_{kl}^*\alpha_{ij}, && \beta_{ij}\gamma_k^* = -\sigma_{ik}\gamma_k^*\beta_{ij}, \\
&\alpha_{ij}\gamma_k = -\sigma_{ik}\gamma_k\alpha_{ij}, && \gamma_i\gamma_k + \sigma_{ik}\gamma_k\gamma_i + \sum_{j=1}^{d}\alpha_{ij}\beta_{kj} + \sigma_{ik}\sum_{j=1}^{d}\alpha_{kj}\beta_{ij} = 0 \\
&\alpha_{ij}\gamma_k^* = -\sigma_{ik}\gamma_k^*\alpha_{ij}, && \gamma_i\gamma_k^* + \sigma_{ik}\gamma_k^*\gamma_i + \sum_{j=1}^{d}\alpha_{ij}\alpha_{kj}^* + \sigma_{ik}\sum_{j=1}^{d}\beta_{kj}^*\beta_{ij} = \delta_{ik}.
\end{aligned} \tag{2.9}$$

Let B be an algebra generated by the entries of T with the commutation relations in (2.9). Using the definitions $\Delta(T) = T \dot{\otimes} T$, $\varepsilon(T) = I$, we obtain the action of coproduct and counit on the generators of B as follows

$$\begin{aligned}
\Delta(\alpha_{ij}) &= \sum_n \alpha_{in} \otimes \alpha_{nj} + \sum_n \beta_{in} \otimes \beta_{nj}^*, \\
\Delta(\alpha_{ij}^*) &= \sum_n \alpha_{in}^* \otimes \alpha_{nj}^* + \sum_n \beta_{in}^* \otimes \beta_{nj}, \\
\Delta(\beta_{ij}) &= \sum_n \alpha_{in} \otimes \beta_{nj} + \sum_n \beta_{in} \otimes \alpha_{nj}^*, \\
\Delta(\beta_{ij}^*) &= \sum_n \alpha_{in}^* \otimes \beta_{nj}^* + \sum_n \beta_{in}^* \otimes \alpha_{nj}, \\
\Delta(\gamma_i) &= \sum_n \alpha_{in} \otimes \gamma_n + \sum_n \beta_{in} \otimes \gamma_n^* + \gamma_i \otimes 1, \\
\Delta(\gamma_i^*) &= \sum_n \alpha_{in}^* \otimes \gamma_n^* + \sum_n \beta_{in}^* \otimes \gamma_n + \gamma_i^* \otimes 1, \\
\Delta(1) &= 1 \otimes 1,
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\varepsilon(\alpha_{ij}) = 1, \varepsilon(\beta_{ij}) = 0, \varepsilon(\gamma_i) = 0, \\
&\varepsilon(\beta_{ij}^*) = 0, \varepsilon(\alpha_{ij}^*) = 1, \varepsilon(\gamma_i^*) = 0.
\end{aligned} \tag{2.11}$$

Thus, it is clear that the condition (1.5) holds for the matrix T. Furthermore, one can straightforwardly show that Δ leaves invariant the relations in (2.9). Indeed, for example, we show this for some relations given in (2.9):

1- $\Delta(\alpha_{ij}\alpha_{kl}) = \Delta(\alpha_{ij})\Delta(\alpha_{kl})$

$$\begin{aligned}
&= (\sum_n \alpha_{in} \otimes \alpha_{nj} + \sum_n \beta_{in} \otimes \beta_{nj}^*)(\sum_m \alpha_{km} \otimes \alpha_{ml} + \sum_m \beta_{km} \otimes \beta_{ml}^*) \\
&= \sum_{m,n} (\alpha_{in}\alpha_{km} \otimes \alpha_{nj}\alpha_{ml} + \alpha_{in}\beta_{km} \otimes \alpha_{nj}\beta_{ml}^* + \beta_{in}\alpha_{km} \otimes \beta_{nj}^*\alpha_{ml} + \beta_{in}\beta_{km} \otimes \beta_{nj}^*\beta_{ml}^*) \\
&= \sum_{m,n} [\sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\alpha_{km}\alpha_{in} \otimes \alpha_{mn}\alpha_{lj}) + \sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\beta_{km}\alpha_{in} \otimes \beta_{ml}^*\alpha_{nj}) \\
&\quad + \sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\alpha_{km}\beta_{in} \otimes \alpha_{ml}\beta_{nj}^*) + \sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\beta_{km}\beta_{in} \otimes \beta_{ml}^*\beta_{nj}^*)] \\
&= \sigma_{ik}\sigma_{lj}(\sum_m \alpha_{km} \otimes \alpha_{ml} + \sum_m \beta_{km} \otimes \beta_{ml}^*)(\sum_n \alpha_{in} \otimes \alpha_{nj} + \sum_n \beta_{in} \otimes \beta_{nj}^*) \\
&= \sigma_{ik}\sigma_{lj}\Delta(\alpha_{kl})\Delta(\alpha_{ij}) \\
&= \sigma_{ik}\sigma_{lj}\Delta(\alpha_{kl}\alpha_{ij})
\end{aligned}$$

$$\Delta(\alpha_{ij}\alpha_{kl}) = \sigma_{ik}\sigma_{lj}\Delta(\alpha_{kl}\alpha_{ij})$$

2- $\Delta(\beta_{ij}\gamma_k) = \Delta(\beta_{ij})\Delta(\gamma_k)$

$$\begin{aligned}
&= (\sum_n \alpha_{in} \otimes \beta_{nj} + \sum_n \beta_{in} \otimes \alpha_{nj}^*)(\sum_m \alpha_{km} \otimes \gamma_m + \sum_m \beta_{km} \otimes \gamma_m^* + \gamma_k \otimes 1) \\
&= \sum_{m,n} (\alpha_{in}\alpha_{km} \otimes \beta_{nj}\gamma_m + \alpha_{in}\beta_{km} \otimes \beta_{nj}\gamma_m^* + \alpha_{in}\gamma_k \otimes \beta_{nj} + \beta_{in}\alpha_{km} \otimes \alpha_{nj}^*\gamma_m \\
&\quad + \beta_{in}\beta_{km} \otimes \alpha_{nj}^*\gamma_m^* + \beta_{in}\gamma_k \otimes \alpha_{nj}^*)
\end{aligned}$$

$$= \sum_{m,n}[-\sigma_{ik}\sigma_{mn}\sigma_{nm}(\alpha_{km}\alpha_{in}\otimes\gamma_m\beta_{nj}) - \sigma_{ik}\sigma_{mn}\sigma_{nm}(\beta_{km}\alpha_{in}\otimes\gamma_m^*\beta_{nj}) - \sigma_{ik}\gamma_k\alpha_{in}\otimes\beta_{nj}$$
$$-\sigma_{ik}\sigma_{mn}\sigma_{nm}(\alpha_{km}\beta_{in}\otimes\gamma_m\alpha_{nj}^*) - \sigma_{ik}\sigma_{mn}\sigma_{nm}(\beta_{km}\beta_{in}\otimes\gamma_m^*\alpha_{nj}^*) - \sigma_{ik}\gamma_k\beta_{in}\otimes\alpha_{nj}^*]$$
$$= -\sigma_{ik}(\sum_m \alpha_{km}\otimes\gamma_m + \beta_{km}\otimes\gamma_m^* + \gamma_k\otimes 1)(\sum_n \alpha_{in}\otimes\beta_{nj} + \beta_{in}\otimes\alpha_{nj}^*)$$
$$= -\sigma_{ik}\Delta(\gamma_k)\Delta(\beta_{ij})$$
$$= -\sigma_{ik}\Delta(\gamma_k\beta_{ij})$$

$$\Delta(\beta_{ij}\gamma_k) = -\sigma_{ik}\Delta(\gamma_k\beta_{ij})$$

3- $\Delta(\alpha_{ij}\beta_{kl}^*) = \Delta(\alpha_{ij})\Delta(\beta_{kl}^*)$

$$= (\sum_n \alpha_{in}\otimes\alpha_{nj} + \sum_n \beta_{in}\otimes\beta_{nj}^*)(\sum_m \alpha_{km}^*\otimes\beta_{ml}^* + \sum_m \beta_{km}^*\otimes\alpha_{ml})$$
$$= \sum_{m,n}(\alpha_{in}\alpha_{km}^*\otimes\alpha_{nj}\beta_{ml}^* + \alpha_{in}\beta_{km}^*\otimes\alpha_{nj}\alpha_{ml} + \beta_{in}\alpha_{km}^*\otimes\beta_{nj}^*\beta_{ml}^*$$
$$+\beta_{in}\beta_{km}^*\otimes\beta_{nj}^*\alpha_{ml})$$

$$= \sum_{m,n}[\sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\alpha_{km}^*\alpha_{in}\otimes\beta_{ml}^*\alpha_{nj}) + \sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\beta_{km}^*\alpha_{in}\otimes\alpha_{ml}\alpha_{nj})$$
$$+\sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\alpha_{km}^*\beta_{in}\otimes\beta_{ml}^*\beta_{nj}^*) + \sigma_{ik}\sigma_{mn}\sigma_{nm}\sigma_{lj}(\beta_{km}^*\beta_{in}\otimes\alpha_{ml}\beta_{nj}^*)]$$
$$= \sigma_{ik}\sigma_{lj}(\sum_m \alpha_{km}^*\otimes\beta_{ml}^* + \sum_m \beta_{km}^*\otimes\alpha_{ml})(\sum_n \alpha_{in}\otimes\alpha_{nj} + \sum_n \beta_{in}\otimes\beta_{nj}^*)$$
$$= \sigma_{ik}\sigma_{lj}\Delta(\beta_{kl}^*)\Delta(\alpha_{ij})$$
$$= \sigma_{ik}\sigma_{lj}\Delta(\beta_{kl}^*\alpha_{ij})$$

$$\Delta(\alpha_{ij}\beta_{kl}^*) = \sigma_{ik}\sigma_{lj}\Delta(\beta_{kl}^*\alpha_{ij})$$

4- $\Delta(\gamma_i\gamma_k + \sigma_{ik}\gamma_k\gamma_i) = \Delta(\gamma_i)\Delta(\gamma_k) + \sigma_{ik}\Delta(\gamma_k)\Delta(\gamma_i)$

$$= (\sum_n \alpha_{in}\otimes\gamma_n + \sum_n \beta_{in}\otimes\gamma_n^* + \gamma_i\otimes 1)(\sum_m \alpha_{km}\otimes\gamma_m + \sum_m \beta_{km}\otimes\gamma_m^* + \gamma_k\otimes 1)$$
$$+\sigma_{ik}(\sum_m \alpha_{km}\otimes\gamma_m + \sum_m \beta_{km}\otimes\gamma_m^* + \gamma_k\otimes 1)(\sum_n \alpha_{in}\otimes\gamma_n + \sum_n \beta_{in}\otimes\gamma_n^* + \gamma_i\otimes 1)$$

$$= \sum_{n,m}[\alpha_{in}\alpha_{km}\otimes\gamma_n\gamma_m + \alpha_{in}\beta_{km}\otimes\gamma_n\gamma_m^* + \alpha_{in}\gamma_k\otimes\gamma_n] + \sum_{n,m}[\beta_{in}\alpha_{km}\otimes\gamma_n^*\gamma_m$$
$$+\beta_{in}\beta_{km}\otimes\gamma_n^*\gamma_m^* + \beta_{in}\gamma_k\otimes\gamma_n^*] + \sum_m[\gamma_i\alpha_{km}\otimes\gamma_m + \gamma_i\beta_{km}\otimes\gamma_m^* + \gamma_i\gamma_k\otimes 1]$$
$$+\sigma_{ik}(\sum_{n,m}[\alpha_{km}\alpha_{in}\otimes\gamma_m\gamma_n + \alpha_{km}\beta_{in}\otimes\gamma_m\gamma_n^* + \alpha_{km}\gamma_i\otimes\gamma_m] + \sum_{n,m}[\beta_{km}\alpha_{in}\otimes\gamma_m^*\gamma_n$$
$$+\beta_{km}\beta_{in}\otimes\gamma_m^*\gamma_n^* + \beta_{km}\gamma_i\otimes\gamma_m^*] + \sum_n[\gamma_k\alpha_{in}\otimes\gamma_n + \gamma_k\beta_{in}\otimes\gamma_n^* + \gamma_k\gamma_i\otimes 1])$$

$$= \sum_{n,m}[\alpha_{in}\alpha_{km}\otimes\gamma_n\gamma_m + \alpha_{in}\beta_{km}\otimes\gamma_n\gamma_m^* + \alpha_{in}\gamma_k\otimes\gamma_n + \beta_{in}\alpha_{km}\otimes\gamma_n^*\gamma_m$$
$$+\beta_{in}\beta_{km}\otimes\gamma_n^*\gamma_m^* + \beta_{in}\gamma_k\otimes\gamma_n^*] + \sum_m[\gamma_i\alpha_{km}\otimes\gamma_m + \gamma_i\beta_{km}\otimes\gamma_m^* + \gamma_i\gamma_k\otimes 1]$$
$$\sigma_{ik}(\sum_{n,m}[\sigma_{ki}\sigma_{nm}\alpha_{in}\alpha_{km}\otimes\gamma_m\gamma_n + \sigma_{ki}\sigma_{nm}\beta_{in}\alpha_{km}\otimes\gamma_m\gamma_n^* - \sigma_{ki}\gamma_i\alpha_{km}\otimes\gamma_m$$
$$+\sigma_{ki}\sigma_{nm}\alpha_{in}\beta_{km}\otimes\gamma_m^*\gamma_n + \sigma_{ki}\sigma_{nm}\beta_{in}\beta_{km}\otimes\gamma_m^*\gamma_n^* - \sigma_{ki}\gamma_i\beta_{km}\otimes\gamma_m^*]$$
$$+\sigma_{ik}\sum_n[-\sigma_{ki}\alpha_{in}\gamma_k\otimes\gamma_n - \sigma_{ki}\beta_{in}\gamma_k\otimes\gamma_n^* + \gamma_k\gamma_i\otimes 1])$$

$$= \sum_{n,m}[\alpha_{in}\alpha_{km}\otimes(\gamma_n\gamma_m + \sigma_{nm}\gamma_m\gamma_n) + \alpha_{in}\beta_{km}\otimes(\gamma_n\gamma_m^* + \sigma_{nm}\gamma_m^*\gamma_n)$$
$$+\beta_{in}\alpha_{km}\otimes(\gamma_n^*\gamma_m + \sigma_{nm}\gamma_m\gamma_n^*) + \beta_{in}\beta_{km}\otimes(\gamma_n^*\gamma_m^* + \sigma_{nm}\gamma_m^*\gamma_n^*)]$$
$$+(\gamma_i\gamma_k + \sigma_{ik}\gamma_k\gamma_i)\otimes 1$$

$$= \sum_{n,m}[\alpha_{in}\alpha_{km}\otimes\sum_j(-\alpha_{nj}\beta_{mj} - \sigma_{nm}\alpha_{mj}\beta_{nj}) + \alpha_{in}\beta_{km}\otimes$$
$$(\delta_{nm} - \sum_j(\alpha_{nj}\alpha_{mj}^* + \sigma_{nm}\beta_{mj}^*\beta_{nj}) + \sigma_{nm}\beta_{in}\alpha_{km}\otimes$$
$$(\delta_{mn} - \sum_j(\alpha_{mj}\alpha_{nj}^* + \sigma_{mn}\beta_{nj}^*\beta_{mj}) + \beta_{in}\beta_{km}\otimes\sum_j(-\beta_{nj}^*\alpha_{mj}^* - \sigma_{nm}\beta_{mj}^*\alpha_{nj}^*)]$$
$$+(-\sum_n\alpha_{in}\beta_{kn} - \sigma_{ik}\sum_n\alpha_{kn}\beta_{in})\otimes 1$$

$$= \sum_j[\sum_{n,m}\alpha_{in}\alpha_{km}\otimes\alpha_{nj}\beta_{mj} + \alpha_{in}\alpha_{km}\otimes\beta_{nj}\alpha_{mj} + \alpha_{in}\beta_{km}\otimes\alpha_{nj}\alpha_{mj}^* + \alpha_{in}\beta_{km}\otimes\beta_{nj}\beta_{mj}^*]$$
$$+\sum_{n,m}\alpha_{in}\beta_{km}\otimes\delta_{nm} - \sum_j[\sum_{n,m}\beta_{in}\alpha_{km}\otimes\alpha_{nj}^*\alpha_{mj} + \beta_{in}\alpha_{km}\otimes\beta_{nj}^*\beta_{mj} + \beta_{in}\beta_{km}\otimes\beta_{nj}^*\alpha_{mj}^*$$
$$+\beta_{in}\beta_{km}\otimes\alpha_{nj}^*\beta_{mj}^*] + \sigma_{nm}\beta_{in}\alpha_{km}\otimes\delta_{mn} + (-\sum_n\alpha_{in}\beta_{kn} - \sigma_{ik}\sum_n\alpha_{kn}\beta_{in})\otimes 1$$

$$= \sum_j[\sum_{n,m}\alpha_{in}\alpha_{km}\otimes\alpha_{nj}\beta_{mj} + \alpha_{in}\alpha_{km}\otimes\beta_{nj}\alpha_{mj} + \alpha_{in}\beta_{km}\otimes\alpha_{nj}\alpha_{mj}^* + \alpha_{in}\beta_{km}\otimes\beta_{nj}\beta_{mj}^*$$
$$+\beta_{in}\alpha_{km}\otimes\alpha_{nj}^*\alpha_{mj} + \beta_{in}\alpha_{km}\otimes\beta_{nj}^*\beta_{mj} + \beta_{in}\beta_{km}\otimes\beta_{nj}^*\alpha_{mj}^* + \beta_{in}\beta_{km}\otimes\alpha_{nj}^*\beta_{mj}^*]$$

$$= -\sum_n[\sum_j\alpha_{in}\otimes\alpha_{nj} + \sum_j\beta_{in}\otimes\beta_{nj}^*]\sum_m[\sum_j\alpha_{km}\otimes\beta_{mj} + \sum_j\beta_{km}\otimes\alpha_{mj}^*]$$
$$-\sum_n[\sum_j\alpha_{in}\otimes\beta_{nj} + \sum_j\beta_{in}\otimes\alpha_{nj}^*]\sum_m[\sum_j\alpha_{km}\otimes\alpha_{mj} + \sum_j\beta_{km}\otimes\beta_{mj}^*]$$

$$= -\sum_j\Delta(\alpha_{ij})\Delta(\beta_{kj}) - \sum_j\Delta(\beta_{ij})\Delta(\alpha_{kj}) = \Delta(-\sum_j\alpha_{ij}\beta_{kj} - \sum_j\beta_{ij}\alpha_{kj})$$

$$= \Delta(-\sum_j\alpha_{ij}\beta_{kj} - \sigma_{ik}\sum_j\alpha_{kj}\beta_{ij})$$

$$\Delta(\gamma_i\gamma_k + \sigma_{ik}\gamma_k\gamma_i) = \Delta(-\sum_j\alpha_{ij}\beta_{kj} - \sigma_{ik}\sum_j\alpha_{kj}\beta_{ij})$$

CONCLUSIONS

Consequently, the condition (1.6) holds for the matrix T, implying that B is a bialgebra. Now, it remains to show that the condition (1.7) holds the matrix T. Really, by letting

$$\sigma_{ij} = q_{ij} = p_{ij}^{-1} = \begin{cases} 1, & i = j \\ -q, & i < j \end{cases} \qquad (2.12)$$

in the quantum group $Gl_{q_{ij},p_{ij}}(2d)$ of [10], we easily see that the matrix A given in (2.7) belongs to the quantum group $Gl_{q_{ij},p_{ij}}(2d)$. Thus, T is invertible, implying that the inverse of T is of the following form

$$T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{pmatrix}. \qquad (2.13)$$

That is, it is clear that $S(T) = T^{-1}$, implying that the coinverse condition (1.7) holds for such an S. Thus, as the algebra of entries of T acting on q-fermions, B corresponds to the inhomogeneous quantum group induced by the following comappings

$$\Delta(T) = T\dot{\otimes}T$$
$$\varepsilon(T) = I$$
$$S(T) = T^{-1}.$$

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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