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Research Article

A new distribution with four parameters: Properties and applications

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ABSTRACT

In this paper, a new lifetime distribution called compounded geometric-mixture exponential distribution is proposed by compounding the mixture exponential and geometric distributions. Some properties of the new distribution such as survival function, hazard function, moments, Lorenz and Bonferroni curves, etc. are obtained. The estimations of four parameters of the new model are studied by several methods. A Monte Carlo simulation study is performed to understand the behavior of estimators. Two real data applications are also provided.

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INTRODUCTION

Distribution theory is one of the most essential areas of statistics. There is a need to develop new statistical distributions since there is a lot of random phenomena waiting for modeling. Many new statistical distributions are introduced to model these events. Some of these can be given as [1]-[5]. Also, in the past two decades, there are many lifetime distributions introduced by using different compounding or mixing procedures. One of the most used compounding methodologies is the maximum or minimum of the sample with random size. Adamidis and Loukas [6] proposed a two-parameter exponential-geometric (EG) distribution by compounding an exponential distribution with a geometric distribution. Kuş [7] compounded exponential distribution with a truncated Poisson distribution called exponential-Poisson distribution. Tahmasbi and Rezaei [8] introduced an exponential-logarithmic distribution by compounding an exponential distribution with a logarithmic distribution. Other similar studies about compounding by using maximum or minimum methodology have been considered by [9]-[13].

In this paper, mixture exponential and geometric distribution are compounded. According to our knowledge, there is no work on compounding mixture distributions. In Section "Compounded Geometric-Mixture Exponential Distribution", a new distribution is introduced and distributional characteristics such as hazard rate function, r-th moment, skewness, kurtosis, Bonferroni and Lorenz curves, etc. are studied. In Section "Point Estimation", the statistical inference on the distribution parameter is studied by several methods of estimates such as maximum likelihood, least squares, weighted least squares, Anderson Darling and Cramér-von Mises. A simulation study is conducted to see the performance of all estimates in Section "Simulation Study" based on 5000 repetitions. In Section "Real Data Application", two practical examples are also

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provided based on real data to show the modeling ability of introduced distribution.

Compounded Geometric-Mixture Exponential Distribution

Let $Y_1, Y_2, \ldots, Y_N$ be N independent and identically distributed (iid) random variables having a mixture exponential distribution with the cumulative distribution function (cdf):

$$F_Y(y) = p(1-\exp(-y/\lambda_1)) + (1-p)(1-\exp(-y/\lambda_2)), \ \ y > 0, \qquad (1)$$

where $\lambda_1, \lambda_2 > 0$ are scale parameter and p ä $(0,1)$ is mixture parameter and N be a geometric random variable, independent of Y, with the probability mass function:

$$P(N = n) = (1-\theta)\theta^{(n-1)}, \ \ n = 1,2,\ldots, \qquad (2)$$

where θ ä $(0,1)$. Now, let us define a random variable as $X = \max(Y_1, Y_2, \ldots, Y_N)$. It is noticed that X denotes the lifetime of the parallel system with components distributed as a mixture of exponentials. Then cdf of X is obtained, respectively, by

$$\begin{aligned} F_X(x) &= \sum_{n=1}^{\infty} P(X \le x \mid N = n) P(N = n) \\ &= \sum_{n=1}^{\infty} \{p(1-\exp(-x/\lambda_1)) + (1-p)(1-\exp(-x/\lambda_2))\}^n (1-\theta)\theta^{(n-1)} \\ &= \frac{(1-\theta)\{1 - p\exp(-x/\lambda_1) - (1-p)\exp(-x/\lambda_2)\}}{1-\theta\{1 - p\exp(-x/\lambda_1) - (1-p)\exp(-x/\lambda_2)\}}. \end{aligned} \qquad (3)$$

Then probability density function (pdf), hazard rate and survival function of X can be obtained, respectively, by

$$f_X(x) = \frac{(1-\theta)\{p\lambda_2 \exp(-x/\lambda_1) + (1-p)\lambda_1 \exp(-x/\lambda_2)\}}{\lambda_1\lambda_2\{1-\theta+\theta[p\exp(-x/\lambda_1) + (1-p)\exp(-x/\lambda_2)]\}^2}, \qquad (4)$$

$$h(x) = \frac{(1-\theta)\{p\lambda_2 \exp(-x/\lambda_1) + (1-p)\lambda_1 \exp(-x/\lambda_2)\}}{\lambda_1\lambda_2\{p\exp(-x/\lambda_1) + (1-p)\exp(-x/\lambda_2)\}} \times \frac{1}{\{1-\theta+\theta[p\exp(-x/\lambda_1) + (1-p)\exp(-x/\lambda_2)]\}}, \qquad (5)$$

and

$$S(x) = \frac{p\exp(-x/\lambda_1) + (1-p)\exp(-x/\lambda_2)}{1-\theta\{1 - p\exp(-x/\lambda_1) - (1-p)\exp(-x/\lambda_2)\}}.$$

The distribution with pdf (4) is called compounded geometric-mixture exponential (CGME) distribution. Some submodel of CGME distribution are given as follow:

- If $\theta \to 0$ then CGME distribution tends to mixture exponential distribution.
- If $p \to 0$ or $p \to 1$ goes to one then CGME distribution tends to exponential geometric distribution introduced by Adamidis and Loukas [6].
- If $(\theta \to 0$ and $p \to 0)$ or $(\theta \to 0$ and $p \to 1)$ then CGME distribution tends to the exponential distribution.

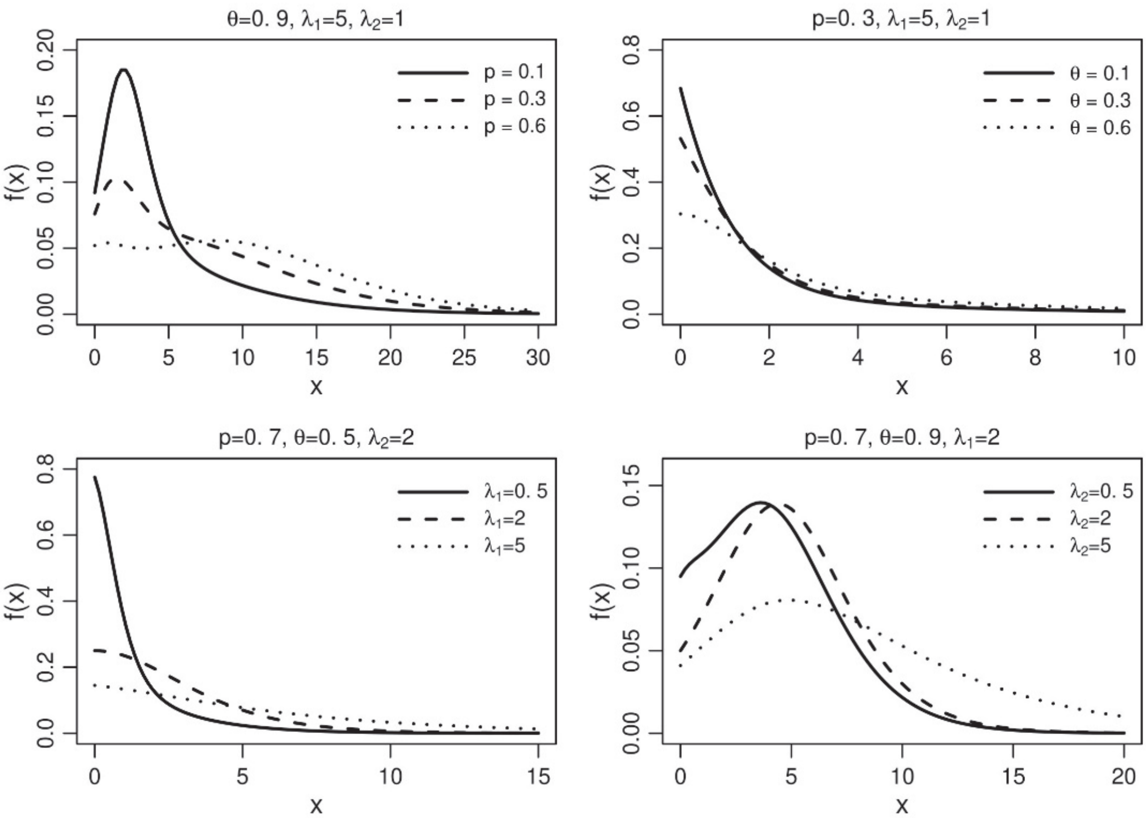

Figure 1. Probability density function of the CGME distribution for selected parameters values

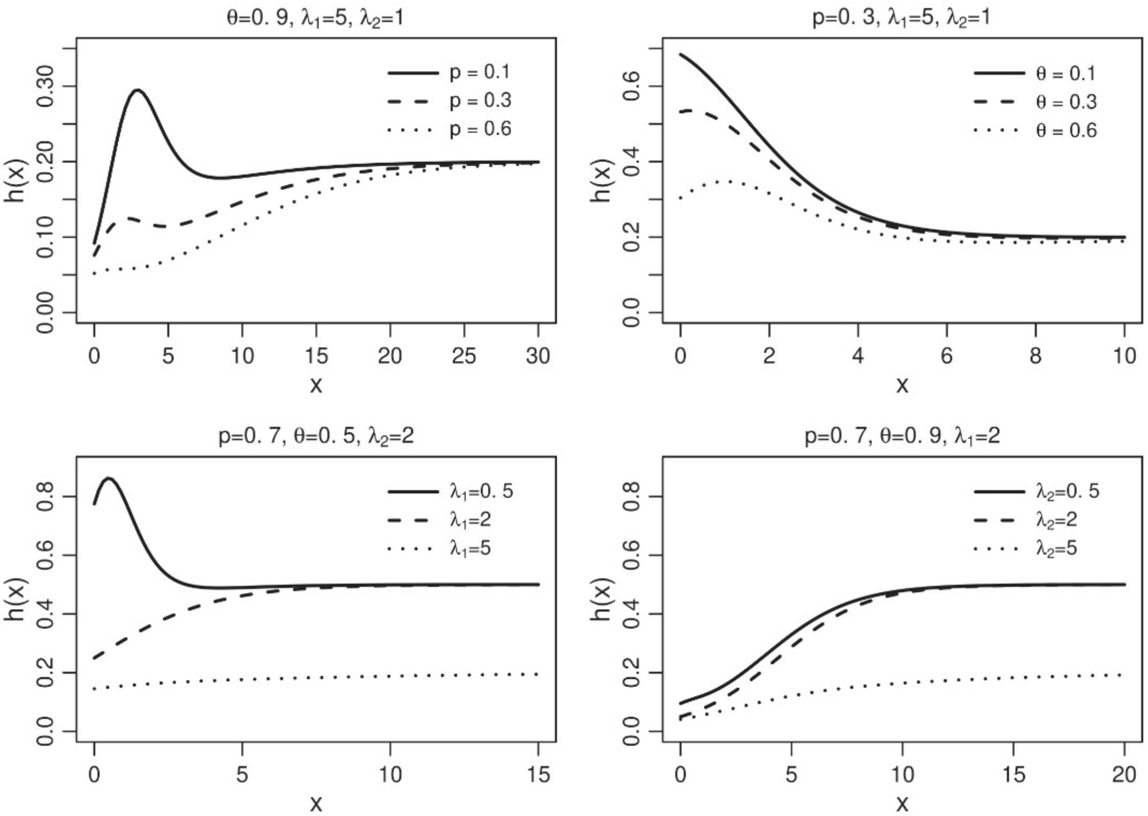

Figure 2. Hazard function of the CGME distribution for selected parameters values

- If $\lambda_1 = \lambda_2$ then CGME distribution tends to exponential geometric distribution introduced by Adamidis and Loukas [6].

From Figure 1, the pdf of CGME can be unimodal or bimodal as well as decreasing.

From Figure 2, the hazard function of introduced distribution exhibits decreasing, bath-tube, up-side bath tube shapes. This is desired property for a lifetime distribution and it gives flexibility to modeling real data. It is noticed that we have

$$\lim_{x\to\infty} h(x) = \begin{cases} \frac{1}{\lambda_1} & , \quad \lambda_1 \geq \lambda_2 \\ \frac{1}{\lambda_2} & , \quad \lambda_1 < \lambda_2. \end{cases} \tag{6}$$

It can be concluded that if $X \sim CGME(p,\theta,\lambda_1,\lambda_2)$ then

$$\lim_{t\to\infty} f_{X|X>t} = \begin{cases} \frac{1}{\lambda_1}\exp\left(-\frac{x}{\lambda_1}\right) & , \quad \lambda_1 \geq \lambda_2 \\ \frac{1}{\lambda_2}\exp\left(-\frac{x}{\lambda_2}\right) & , \quad \lambda_1 < \lambda_2. \end{cases}$$

Let the X have CGME with pdf (4). Then, for $r = 1,2,...,$ the r-th moment of X is given by

$$\begin{aligned} \mu_r &= \int_0^\infty x^r f(x)\,dx \\ &= \int_0^\infty x^r \frac{(1-\theta)\left\{p\lambda_2\exp\left(-\frac{x}{\lambda_1}\right)+(1-p)\lambda_1\exp\left(-\frac{x}{\lambda_2}\right)\right\}}{\lambda_1\lambda_2\left\{1-\theta+\theta\left(p\exp\left(-\frac{x}{\lambda_1}\right)+(1-p)\exp\left(-\frac{x}{\lambda_2}\right)\right)\right\}^2}dx \\ &= \int_{-1}^{1}\frac{2}{(1-y)^2}\left(\frac{1+y}{1-y}\right)^r \\ &\times\frac{(1-\theta)\left\{p\lambda_2\exp\left(-\frac{1}{\lambda_1}\left(\frac{1+y}{1-y}\right)\right)+(1-p)\lambda_1\exp\left(-\frac{1}{\lambda_2}\left(\frac{1+y}{1-y}\right)\right)\right\}}{\lambda_1\lambda_2\left\{1-\theta+\theta\left(p\exp\left(-\frac{1}{\lambda_1}\left(\frac{1+y}{1-y}\right)\right)+(1-p)\exp\left(-\frac{1}{\lambda_2}\left(\frac{1+y}{1-y}\right)\right)\right)\right\}^2}dy \\ &\cong \sum_{\ell=0}^{N}\varpi_l\frac{2}{(1-y_\ell)^2}\left(\frac{1+y_\ell}{1-y_\ell}\right)^r \\ &\times\frac{(1-\theta)\left\{p\lambda_2\exp\left(-\frac{1}{\lambda_1}\left(\frac{1+y_\ell}{1-y_\ell}\right)\right)+(1-p)\lambda_1\exp\left(-\frac{1}{\lambda_2}\left(\frac{1+y_\ell}{1-y_\ell}\right)\right)\right\}}{\lambda_1\lambda_2\left\{1-\theta+\theta\left(p\exp\left(-\frac{1}{\lambda_1}\left(\frac{1+y_\ell}{1-y_\ell}\right)\right)+(1-p)\exp\left(-\frac{1}{\lambda_2}\left(\frac{1+y_\ell}{1-y_\ell}\right)\right)\right)\right\}^2} \end{aligned} \tag{7}$$

where $f(\cdot)$ is pdf given in Eq. (4), y_l and ϖ_ℓ are the zeros and the corresponding Christoffel numbers of the Legendre-Gauss quadrature formula on the interval $(-1,1)$, respectively, see [14]. It is also noted ϖ_ℓ is calculated by

$$\varpi_\ell = \frac{2}{(1-y_\ell)^2\left[L'_{N+1}(y_\ell)\right]^2}, \tag{8}$$

where

$$L'_{N+1}(y_\ell) = \frac{dL_{N+1}(y)}{dy} \tag{9}$$

at $y = y_\ell$ and $L_{N+1}(\cdot)$ is the Legendre polynomial of degree N Eq. 7 is calculated by **lgwt** command in Matlab. We observe that $N = 30$ is sufficient to obtain the satisfied approximation to true moments. The mean, variance, skewness and kurtosis of CGME distribution are presented in Tables 1-2. The values of N has been taken to be $N = 30$ in the numerical calculations. By increasing p, θ, λ_1 and λ_2 the mean and variance increase. By increasing p and θ, skewness and kurtosis decrease.

The Bonferroni and Lorenz curves introduced by Bonferroni [15] have applications in economics to study income and poverty, reliability, demography, insurance and medicine. The Bonferroni curve $(BC(\xi))$ and Lorenz curve $(LC(\xi))$ for a positive random variable X are described as follows:

$$BC(\xi) = \frac{1}{\xi\mu_1}\int_0^q tf(t)\,dt \tag{10}$$

and

$$LC(\xi) = \frac{1}{\xi\mu_1}\int_0^q tf(t)\,dt \tag{11}$$

respectively, where $\mu_1 = E(T)$ and $q = F^{-1}(\xi)$.

In the following, we give the Bonferroni and Lorenz curves of CGME with pdf (4). Let the random variable X have CGME with pdf (4). Then, $BC(\xi)$ and $LC(\xi)$ are given, respectively, by

$$\begin{aligned} BC(\xi) &= \frac{q^2}{4\xi\mu_1}\sum_{\ell=0}^{N}\varpi_\ell(y_\ell+1)f\left(\frac{q}{2}(y_\ell+1)\right) \\ &= \frac{q^2}{4\xi\mu_1}\sum_{\ell=0}^{N}\varpi_\ell(y_\ell+1) \\ &\times\frac{(1-\theta)\left\{p\lambda_2\exp\left(-\frac{1}{\lambda_1}\left(\frac{q}{2}(y_\ell+1)\right)\right)+(1-p)\lambda_1\exp\left(-\frac{1}{\lambda_2}\left(\frac{q}{2}(y_\ell+1)\right)\right)\right\}}{\lambda_1\lambda_2\left\{1-\theta+\theta\left(p\exp\left(-\frac{1}{\lambda_1}\left(\frac{q}{2}(y_\ell+1)\right)\right)+(1-p)\exp\left(-\frac{1}{\lambda_2}\left(\frac{q}{2}(y_\ell+1)\right)\right)\right)\right\}^2} \end{aligned} \tag{12}$$

and

$$\begin{aligned} LC(\xi) &= \frac{q^2}{4\mu_1}\sum_{\ell=0}^{N}\varpi_\ell(y_\ell+1)f\left(\frac{q}{2}(y_\ell+1)\right) \\ &= \frac{q^2}{4\mu_1}\sum_{\ell=0}^{N}\varpi_\ell(y_\ell+1) \\ &\times\frac{(1-\theta)\left\{p\lambda_2\exp\left(-\frac{1}{\lambda_1}\left(\frac{q}{2}(y_\ell+1)\right)\right)+(1-p)\lambda_1\exp\left(-\frac{1}{\lambda_2}\left(\frac{q}{2}(y_\ell+1)\right)\right)\right\}}{\lambda_1\lambda_2\left\{1-\theta+\theta\left(p\exp\left(-\frac{1}{\lambda_1}\left(\frac{q}{2}(y_\ell+1)\right)\right)+(1-p)\exp\left(-\frac{1}{\lambda_2}\left(\frac{q}{2}(y_\ell+1)\right)\right)\right)\right\}^2} \end{aligned} \tag{13}$$

Table 1. The mean and variance of CGME distribution for different values of p, θ, λ_1 and λ_2

		(λ_1, λ_2)							
		(1,0.5)		**(2,1.5)**		**(3,2.5)**		**(5,4)**	
θ	p	**Mean**	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**
0.1	0.2	0.6337	0.4672	1.6864	2.7842	2.7398	7.2079	4.4261	18.9298
0.3	0.4	0.8411	0.7429	2.0249	3.5864	3.2125	8.7991	5.2371	23.5834
0.5	0.6	1.1292	1.0813	2.5041	4.6420	3.8872	10.9359	6.3906	29.7827
0.7	0.8	1.5763	1.5226	3.2808	6.1646	4.9962	14.1049	8.2758	38.8760
0.9	0.9	2.4730	2.1805	5.0101	8.6875	7.5614	19.6417	12.5692	54.4005

Table 2. The skewness and kurtosis of CGME distribution for different values of p, θ, λ_1 and λ_2

		(λ_1, λ_2)							
		(1,0.5)		**(2,1.5)**		**(3,2.5)**		**(5,4)**	
θ	p	**Skewness**	**Kurtosis**	**Skewness**	**Kurtosis**	**Skewness**	**Kurtosis**	**Skewness**	**Kurtosis**
0.1	0.2	2.6248	14.7971	2.0271	9.4176	1.9609	8.8577	1.9816	9.0284
0.3	0.4	2.2770	11.2397	1.8669	8.4404	1.7972	7.9412	1.8202	8.1042
0.5	0.6	1.8370	8.1771	1.6294	7.1193	1.5802	6.8295	1.5970	6.9282
0.7	0.8	1.3793	5.9152	1.3111	5.6832	1.2884	5.5814	1.2962	5.6155
0.9	0.9	0.8524	4.2737	0.8354	4.2542	0.8263	4.2304	0.8293	4.2360

where ϖ_ℓ is given by (8).

We also give a result on the stochastic ordering features of the introduced. These features have applications in various fields, including reliability, insurance, economics and actuaries. Further detailed information about stochastic ordering readers can be read in [16]. The following theorem shows that the CGME random variables can be ordered for the likelihood ratio.

Theorem 1. Let $X \sim CGME(p,\theta_1,\lambda_1,\lambda_2)$ with pdf $f_X(x)$ and $Y \sim CGME(p,\theta_2,\lambda_1,\lambda_2)$ with pdf $f_Y(x)$. If $\theta_1 < \theta_2$ then X is smaller than Y in the likelihood ratio order, i.e. the ratio function of the corresponding pdfs is decreasing in x.

Proof. For any $x > 0$, the ratio of the two densities is given by

$$
\begin{aligned}
g(x) &= \frac{f_X(x)}{f_Y(x)} \\
&= \frac{(1-\theta_1)\left\{1-\theta_2+\theta_2\left(p\exp\left(-\frac{x}{\lambda_1}\right)+(1-p)\exp\left(-\frac{x}{\lambda_2}\right)\right)\right\}^2}{(1-\theta_2)\left\{1-\theta_1+\theta_1\left(p\exp\left(-\frac{x}{\lambda_1}\right)+(1-p)\exp\left(-\frac{x}{\lambda_2}\right)\right)\right\}^2}
\end{aligned}
$$

Let us consider the first order derivative of $\log(g(x))$ in x

$$
\begin{aligned}
\frac{d\log(g(x))}{dx} &= \frac{2\overbrace{(\theta_1-\theta_2)}^{-}\overbrace{\left\{\lambda_1(1-p)\exp\left(-\frac{x}{\lambda_2}\right)+\lambda_2 p\exp\left(-\frac{x}{\lambda_1}\right)\right\}}^{+}}{\underbrace{\lambda_1\lambda_2}_{+}\left\{\underbrace{\theta_1(1-p)\exp\left(-\frac{x}{\lambda_2}\right)}_{+}+\underbrace{1-\theta_1}_{+}+\underbrace{\theta_1 p\exp\left(-\frac{x}{\lambda_1}\right)}_{+}\right\}} \\
&\times\frac{1}{\left\{\underbrace{\theta_2(1-p)\exp\left(-\frac{x}{\lambda_2}\right)}_{+}+\underbrace{1-\theta_2}_{+}+\underbrace{\theta_2 p\exp\left(-\frac{x}{\lambda_1}\right)}_{+}\right\}}<0
\end{aligned}
$$

for $\theta_1 < \theta_2$. This proof is completed.

Corollary 1. It follows from Shaked and Shanthikumar [16] that X is also smaller than Y in the hazard ratio, mean residual life and stochastic orders under the conditions given in Theorem 1.

Point Estimations

In this section, we discuss the estimation of unknown parameters p,θ,λ_1 and λ_2 of the CGME distribution by maximum likelihood, least square, weighted least square, Cramer-von Mises and Anderson-Darling methodology.

Let $x_1,x_2,\ldots,x_n$ be a realization of a random sample from the $CGME(\Xi)$ distribution and $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ denotes the corresponding observed order statistics. Then the log-likelihood function is given by

$$
\begin{aligned}
\ell(\Xi) &= n\log(\theta-1)-n\log(\lambda_1\lambda_2) \\
&+n\sum_{i=1}^{n}\log(\{p\lambda_2\exp(-x_i/\lambda_1)+(1-p)\lambda_1\exp(-x_i/\lambda_2)\}) \\
&-2n\sum_{i=1}^{n}\log(\{1-\theta+\theta[p\exp(-x_i/\lambda_1)+(1-p)\exp(-x_i/\lambda_2)]\}),
\end{aligned} \tag{14}
$$

Hence, the maximum likelihood estimate (MLE) of Ξ is given by

$$\hat{\Xi}_1 = \arg\max_{\Xi}\{\ell(\Xi)\},$$

where $\Xi = (p,\theta,\lambda_1,\lambda_2)$ is parameter vector and $\hat{\Xi}_1 = (\hat{p},\hat{\theta},\hat{\lambda}_1,\hat{\lambda}_2)$ is MLE of Ξ. Let us define the following functions which are used to define the different types of estimators:

$$
\begin{aligned}
Q_{LS}(\Xi) &= \sum_{i=1}^{n}\left(F(x_{(i)})-\frac{i}{n+1}\right)^2, \\
Q_{WLS}(\Xi) &= \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(F(x_{(i)})-\frac{i}{n+1}\right)^2, \\
Q_{CvM}(\Xi) &= \frac{1}{12n}+\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{2n}\right)^2,
\end{aligned}
$$

and

$$Q_{AD}(\Xi) = -n-\frac{1}{n}\sum_{i=1}^{n}\{(2i-1)\log(F(x_{(i)}))\}+\frac{1}{n}\sum_{i=1}^{n}\log\{1-F(x_{(i)})\},$$

where $F(\cdot)$ is cdf of $CGME(\Xi)$ distribution given in (3). Then, the least squares estimator (LSE), weighted least squares estimator (WLSE), Anderson Darling estimator (ADE) and the Cramér-von Mises estimator (CvME) of Ξ are given, respectively, by

$$\hat{\Xi}_2 = \arg\min_{\Xi}\{Q_{LS}(\Xi)\}, \tag{15}$$

$$\hat{\Xi}_3 = \arg\min_{\Xi}\{Q_{WLS}(\Xi)\}, \tag{16}$$

$$\hat{\Xi}_4 = \arg\min_{\Xi}\{Q_{AD}(\Xi)\}, \tag{17}$$

$$\hat{\Xi}_5 = \arg\min_{\Xi}\{Q_{CvM}(\Xi)\}. \tag{18}$$

CG method is used for optimization purposes in obtaining the estimators mentioned above. The CG method can be easily employed by **optim** function in R.

Simulation Study

In this section, a simulation study is performed with 5000 repetitions are conducted for some parameters settings and *n*=100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000. In the simulation study, the bias and mean square error (MSE) of MLE, LSE, WLSE, ADE and CvME estimates are given in Tables 3-10. In simulation study, the CG algorithm is used while obtaining MLE, LSE, WLSE, ADE, and CvME estimators. The parameters of the distribution in which the sample is produced are used as the initial value. In the following, two different algorithms are recommended to generate the random sample from $CGME(p,\theta,\lambda_1,\lambda_2)$ distribution.

Algorithm 1.

i. Generate $N_j \sim Geo(\theta)$, $j=1,2,\ldots,k$

ii. Generate $U_{1i} \sim Uniform(0,1)$, $i = 1,2,\ldots,N_j$,

iii. If $U_{i1} < p$ then $X_i = -\lambda_1 \log(1-U_{2i})$ otherwise $X_i = -\lambda_2 \log(1-U_{2i})$ where p is the mixture parameter and $U_{2i} \sim Uniform(0,1)$.

iv. Let $Z_j = \max(X_1, X_2, X_{N_j})$, $j=1,2,...,k$, where Z_j follows from $CGME(p,\theta,\lambda_1,\lambda_2)$.

Algoritm 2.

i. Generate $U_i \sim Uniform(0,1)$, $i=1,2,\ldots,k$,

ii. Set Z_i as the root of the equation

$$\frac{(1-\theta)\left\{1-p\exp\left(-\frac{Z_i}{\lambda_1}\right)-(1-p)\exp\left(-\frac{Z_i}{\lambda_2}\right)\right\}}{1-\theta\left\{1-p\exp\left(-\frac{Z_i}{\lambda_1}\right)-(1-p)\exp\left(-\frac{Z_i}{\lambda_2}\right)\right\}} = U_i,\ i=1,2,\ldots,k.$$

Then $Z_1, Z_2, \ldots, Z_k$ are iid sample from CGME. It is noted that Algorithm 1 is based on the motivation of distribution and Algorithm 2 is based on the inverse transform method of generating random data. It is seen that Algorithm 1 gives much faster than Algorithm 2.

Table 3. Average bias and MSEs for parameter p when $\Xi = (0.95, 0.8, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	0.0314	0.0070	0.0253	0.0099	0.0302	0.0091	0.0308	0.0080	0.0113	0.0092
200	-0.0180	0.0026	-0.0132	0.0040	-0.0157	0.0049	-0.0158	0.0026	-0.0060	0.0038
300	-0.0107	0.0015	-0.0084	0.0025	-0.0087	0.0024	-0.0095	0.0015	-0.0039	0.0023
400	-0.0070	0.0011	-0.0035	0.0021	-0.0061	0.0014	-0.0057	0.0012	-0.0007	0.0031
500	-0.0038	0.0008	-0.0014	0.0014	-0.0020	0.0012	-0.0026	0.0009	0.0014	0.0014
600	-0.0030	0.0008	-0.0034	0.0011	-0.0028	0.0009	-0.0034	0.0008	-0.0010	0.0011
700	-0.0023	0.0007	-0.0029	0.0009	-0.0031	0.0008	-0.0033	0.0007	-0.0014	0.0009
800	-0.0011	0.0007	-0.0026	0.0008	-0.0019	0.0007	-0.0022	0.0006	-0.0009	0.0008
900	-0.0006	0.0006	-0.0032	0.0007	-0.0018	0.0006	-0.0025	0.0005	-0.0015	0.0008
1000	0.0006	0.0006	-0.0023	0.0007	-0.0024	0.0006	-0.0014	0.0005	-0.0011	0.0007

Table 4. Average bias and MSEs for parameter θ when $\Xi = (0.95, 0.8, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	0.0300	0.0062	-0.0023	0.0041	0.0060	0.0050	0.0103	0.0047	0.0043	0.0040
200	0.0161	0.0032	-0.0014	0.0019	0.0031	0.0026	0.0046	0.0022	0.0021	0.0019
300	0.0105	0.0022	-0.0005	0.0014	0.0013	0.0017	0.0032	0.0016	0.0018	0.0014
400	0.0081	0.0017	-0.0001	0.0010	0.0019	0.0012	0.0024	0.0011	0.0014	0.0011
500	0.0060	0.0014	-0.0013	0.0009	0.0005	0.0009	0.0009	0.0010	0.0000	0.0009
600	0.0043	0.0012	-0.0008	0.0007	0.0003	0.0008	0.0010	0.0008	0.0001	0.0007
700	0.0034	0.0010	-0.0011	0.0005	0.0002	0.0007	0.0007	0.0006	0.0002	0.0005
800	0.0023	0.0009	-0.0012	0.0005	-0.0001	0.0006	0.0001	0.0006	-0.0004	0.0005
900	0.0017	0.0008	-0.0006	0.0004	-0.0002	0.0005	0.0007	0.0005	0.0001	0.0004
1000	0.0010	0.0007	-0.0006	0.0004	0.0012	0.0005	0.0003	0.0005	0.0002	0.0004

Table 5. Average bias and MSEs for parameter λ_1 when $\Xi = (0.95, 0.8, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	-0.2539	0.1824	-0.0625	0.0450	-0.1182	0.0765	-0.1392	0.0768	-0.0679	0.0453
200	-0.1381	0.0900	-0.0296	0.0249	-0.0618	0.0440	-0.0677	0.0399	-0.0336	0.0271
300	-0.0908	0.0603	-0.0240	0.0225	-0.0352	0.0296	-0.0460	0.0293	-0.0265	0.0228
400	-0.0646	0.0457	-0.0118	0.0203	-0.0274	0.0233	-0.0284	0.0226	-0.0133	0.0197
500	-0.0469	0.0382	-0.0004	0.0148	-0.0115	0.0204	-0.0147	0.0194	-0.0012	0.0146
600	-0.0359	0.0322	-0.0068	0.0102	-0.0122	0.0164	-0.0167	0.0152	-0.0064	0.0098
700	-0.0324	0.0285	-0.0063	0.0067	-0.0144	0.0126	-0.0169	0.0121	-0.0097	0.0071
800	-0.0208	0.0256	-0.0022	0.0063	-0.0073	0.0119	-0.0090	0.0107	-0.0029	0.0060
900	-0.0155	0.0233	-0.0058	0.0059	-0.0062	0.0107	-0.0118	0.0102	-0.0060	0.0062
1000	-0.0108	0.0210	-0.0056	0.0066	-0.0159	0.0117	-0.0089	0.0093	-0.0070	0.0064

Table 6. Average bias and MSEs for parameter $_2$ when $\Xi = (0.95, 0.8, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	0.0318	0.0917	0.0048	0.0014	0.0143	0.0032	0.0175	0.0044	0.0027	0.0011
200	0.0249	0.0469	0.0007	0.0004	0.0059	0.0014	0.0066	0.0009	0.0000	0.0006
300	0.0175	0.0292	0.0008	0.0002	0.0027	0.0006	0.0045	0.0005	0.0004	0.0002
400	0.0160	0.0228	0.0004	0.0002	0.0020	0.0003	0.0026	0.0006	-0.0001	0.0003
500	0.0130	0.0142	-0.0012	0.0001	0.0008	0.0003	0.0006	0.0003	-0.0014	0.0002
600	0.0072	0.0120	-0.0006	0.0001	0.0004	0.0002	0.0010	0.0002	-0.0008	0.0001
700	0.0079	0.0104	-0.0001	0.0000	0.0009	0.0001	0.0014	0.0001	0.0000	0.0001
800	0.0067	0.0113	-0.0004	0.0000	0.0004	0.0001	0.0005	0.0001	-0.0004	0.0000
900	0.0035	0.0119	-0.0001	0.0000	-0.0002	0.0001	0.0004	0.0001	-0.0002	0.0000
1000	0.0040	0.0104	-0.0001	0.0000	0.0009	0.0002	0.0004	0.0001	-0.0001	0.0000

Table 7. Average bias and MSEs for parameter p when $\Xi = (0.6, 0.9, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	0.0111	0.0523	-0.0258	0.0336	-0.0163	0.0361	-0.0098	0.0376	-0.0017	0.0321
200	-0.0093	0.0402	-0.0186	0.0213	-0.0118	0.0230	-0.0089	0.0265	-0.0054	0.0201
300	-0.0173	0.0318	-0.0167	0.0138	-0.0132	0.0173	-0.0106	0.0201	-0.0093	0.0139
400	-0.0134	0.0247	-0.0087	0.0097	-0.0109	0.0147	-0.0060	0.0112	-0.0027	0.0096
500	-0.0131	0.0214	-0.0069	0.0088	-0.0070	0.0095	-0.0050	0.0097	-0.0015	0.0089
600	-0.0158	0.0206	-0.0110	0.0088	-0.0091	0.0092	-0.0098	0.0112	-0.0065	0.0085
700	-0.0132	0.0167	-0.0108	0.0079	-0.0082	0.0152	-0.0087	0.0081	-0.0053	0.0071
800	-0.0109	0.0147	-0.0053	0.0056	-0.0062	0.0058	-0.0041	0.0063	-0.0018	0.0053
900	-0.0097	0.0125	-0.0039	0.0043	-0.0068	0.0064	-0.0028	0.0051	-0.0014	0.0045
1000	-0.0097	0.0108	-0.0060	0.0042	-0.0076	0.0048	-0.0056	0.0051	-0.0036	0.0041

Table 8. Average bias and MSEs for parameter θ when $\Xi = (0.6, 0.9, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	0.0023	0.0021	-0.0112	0.0024	-0.0092	0.0025	-0.0079	0.0026	-0.0056	0.0020
200	-0.0013	0.0014	-0.0069	0.0012	-0.0052	0.0012	-0.0051	0.0013	-0.0040	0.0011
300	-0.0019	0.0010	-0.0048	0.0007	-0.0040	0.0009	-0.0037	0.0007	-0.0031	0.0007
400	-0.0017	0.0008	-0.0030	0.0005	-0.0033	0.0005	-0.0023	0.0005	-0.0019	0.0005
500	-0.0013	0.0007	-0.0023	0.0004	-0.0022	0.0004	-0.0017	0.0004	-0.0012	0.0004
600	-0.0015	0.0006	-0.0027	0.0004	-0.0021	0.0004	-0.0023	0.0004	-0.0019	0.0004
700	-0.0014	0.0005	-0.0027	0.0004	-0.0018	0.0003	-0.0020	0.0004	-0.0016	0.0003
800	-0.0009	0.0004	-0.0015	0.0003	-0.0014	0.0002	-0.0012	0.0003	-0.0008	0.0002
900	-0.0009	0.0004	-0.0012	0.0002	-0.0015	0.0002	-0.0009	0.0002	-0.0007	0.0002
1000	-0.0011	0.0003	-0.0016	0.0002	-0.0016	0.0002	-0.0014	0.0002	-0.0010	0.0002

Table 9. Average bias and MSEs for parameter λ_1 when $\Xi = (0.6, 0.9, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	-0.0736	0.0978	-0.0260	0.0100	-0.0233	0.0160	-0.0228	0.0138	-0.0176	0.0093
200	-0.0329	0.0478	-0.0115	0.0052	-0.0100	0.0065	-0.0106	0.0071	-0.0089	0.0048
300	-0.0251	0.0294	-0.0084	0.0032	-0.0081	0.0051	-0.0053	0.0054	-0.0052	0.0035
400	-0.0174	0.0242	-0.0043	0.0022	-0.0036	0.0043	-0.0029	0.0030	-0.0020	0.0025
500	-0.0165	0.0225	-0.0036	0.0026	-0.0028	0.0026	-0.0024	0.0034	-0.0026	0.0025
600	-0.0137	0.0204	-0.0037	0.0020	-0.0026	0.0030	-0.0025	0.0032	-0.0030	0.0024
700	-0.0106	0.0162	-0.0033	0.0017	-0.0030	0.0026	-0.0018	0.0024	-0.0032	0.0016
800	-0.0115	0.0150	-0.0029	0.0012	-0.0024	0.0022	-0.0023	0.0021	-0.0015	0.0012
900	-0.0101	0.0106	-0.0021	0.0013	-0.0010	0.0023	-0.0022	0.0018	-0.0014	0.0014
1000	-0.0071	0.0108	-0.0018	0.0013	-0.0008	0.0019	-0.0007	0.0022	-0.0011	0.0014

Table 10. Average bias and MSEs for parameter λ_2 when $\Xi = (0.6, 0.9, 2, 4)$

	MLE		LSE		WLSE		ADE		CvME	
n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
100	0.0324	0.1095	0.0578	0.0361	0.0508	0.0419	0.0519	0.0411	0.0348	0.0278
200	0.0165	0.0635	0.0271	0.0140	0.0224	0.0171	0.0248	0.0189	0.0175	0.0121
300	0.0066	0.0441	0.0177	0.0080	0.0154	0.0110	0.0132	0.0114	0.0099	0.0073
400	0.0035	0.0358	0.0104	0.0050	0.0081	0.0087	0.0082	0.0072	0.0058	0.0049
500	0.0027	0.0331	0.0087	0.0046	0.0063	0.0056	0.0074	0.0070	0.0049	0.0044
600	0.0017	0.0302	0.0085	0.0044	0.0055	0.0063	0.0063	0.0067	0.0059	0.0045
700	0.0045	0.0238	0.0079	0.0036	0.0065	0.0045	0.0058	0.0053	0.0063	0.0034
800	0.0003	0.0217	0.0064	0.0025	0.0044	0.0042	0.0056	0.0040	0.0039	0.0026
900	-0.0013	0.0170	0.0053	0.0023	0.0025	0.0044	0.0053	0.0035	0.0030	0.0023
1000	-0.0001	0.0152	0.0058	0.0025	0.0027	0.0037	0.0043	0.0041	0.0039	0.0024

From Tables 3-10, it is observed that the bias and MSE's of the all estimates go to zero when n tends to infinity. It can be also concluded that all the estimates of all parameters are consistent.

The following comments can be reached from Tables 3-6 for $\Xi = (0.95, 0.8, 2, 4)$:

- For the p parameter, the bias of the CvME was observed to be smaller than the others, whereas the estimator with the smallest MSEs was ADE.
- For the θ parameter, the bias and MSEs of the CvME were observed to be smaller than the others.
- For the λ_1 parameter, the bias and MSEs of LSE were observed to be smaller than the others.
- For the λ_2 parameter, the bias and MSEs of CvME were observed to be smaller than the others.

The following comments can be reached from Tables 7-10 for $\Xi = (0.6, 0.9, 2, 4)$:

- For the p parameter, the bias and MSEs of the CvME were observed to be smaller than the others.
- For the θ parameter, the bias and MSEs of the MLE were observed to be smaller than the others.
- For the λ_1 parameter, the bias and MSEs of CvME were observed to be smaller than the others.
- For the λ_2 parameter, the bias and MSEs of MLE were observed to be smaller than the others.

Considering the results summarized above, it was seen that CvME for new distribution generally gave better results compared to other methods in terms of both bias and MSEs, compared to MLE, LSE, WLSE, and ADE. However, it can be said that the performances of LSE estimators and CvME are close to each other in terms of bias and MSEs. MLE, which is a very popular estimator, can have very high bias and MSEs for some parameters compared to other estimators. For this reason, according to the results obtained from the simulation studies conducted here, it would be appropriate to choose CvME or LSE among the estimators of MLE, LSE, WLSE, ADE and CvME in the estimation of the parameters of the CGME distribution.

Real Data Application

The CGME distribution is now fitted to the two data sets. Five distributions are fitted to all data sets with the likelihood principle. The MLEs of distribution parameters are obtained by numerical methods that try to maximize the log-likelihood. In most cases, we observe that the different initial values give different estimates, and one cannot conclude which one is treated as a MLE. Therefore, an algorithm is used to get the almost correct MLEs of parameters given in Tables 11 and 12. An algorithm is given as follows:

Algorithm 3.

A1. 2000 (it can be increased by optionally) initial values are uniformly generated from a subset of parameter space.

A2. Using initial values generated in Step **A1**, the numerical methods Nelder-Mead, BFGS, and CG are used to maximize the log-likelihood.

A3. The likelihoods for all estimates in Step **A2** are ordered from large to small.

A4. The estimates with the largest likelihoods are treated as MLEs of parameters.

It is also pointed out that two data sets are also analyzed for the comparison CGME distribution with exponential(E), Weibull(W), mixture exponential (ME) and exponential geometric(EG) (introduced by Adamidis and Loukas [6]) distributions are considered.

The MLEs of parameters and related standard errors for CGME, E, W, ME and EG distributions are obtained for all data sets. The log-likelihood ℓ, -2ℓ, AIC, BIC, CAIC, HQIC, Kolmogorov-Smirnov statistic (KS), Anderson-Darling statistic(AD), Cramér von Mises statistic(CvM) and related p-values(KS p-value, AD p-value and CvM p-value), the MLE $\hat{p}_i$ $(i = 1,2,3)$ of parameter p_i with standard error $se(\hat{p}_i)$ and AN intervals (LB_{p_i}, UB_{p_i}) are calculated and they are presented in Tables 11 and 12. It is noted that some lower limits of AN CI are below the lower bound of parameter space. It can be corrected with a lower bound of parameter space. In Tables 11-12, initial parameters, and the numerical methods are given to get MLEs for all models in the analysis. From Tables 11 and 12, the CGME distribution has the smallest values of -2ℓ, AIC, BIC, CAIC, HQIC, KS, AD and CvM. Furthermore, the goodness of fit tests KS, AD and CvM confirm the ULW model validity (p values>0.05). From these results, it is concluded that the CGME distribution is better than the others in terms of all criteria.

First real data set: The real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (see, Lee and Wang [17]): 0.08, 2.09, 3.48,4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82,0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65,12.63, 22.69.

Second real data set: The real data set (see, Ghitany et al. [18]) consists of 100 observations on waiting time (in minutes) before the customer received service in a bank. The data are: 0.8, 0.8,1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6,4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9,5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1,7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5,9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2,

Table 11. Data analysis results for the first data

	CGME	W	ME	EG	E
ℓ	-409.7614	-414.0869	-413.3001	-414.3419	-414.3419
-2ℓ	819.5229	828.1738	826.6001	828.6838	828.6838
AIC	827.5229	832.1738	832.6001	832.6838	830.6838
BIC	838.9310	837.8778	841.1562	838.3879	833.5358
CAIC	827.8481	832.2698	832.7937	832.7798	830.7155
HQIC	832.1581	834.4913	836.0765	835.0014	831.8426
KS	0.0348	0.0700	0.0971	0.0846	0.0846
AD	0.1294	0.9577	1.3218	1.1736	1.1736
CVM	0.0196	0.1537	0.2054	0.1788	0.1788
KS p value	0.9977	0.5570	0.1791	0.3184	0.3184
AD p value	0.9996	0.3801	0.2253	0.2777	0.2777
CVM p value	0.9975	0.3789	0.2575	0.3128	0.3128
$\hat{p}_1$	0.8998	9.5607	0.9461	0.1068	9.3656
$\hat{p}_2$	0.8085	1.0478	8.3872	0.0000	
$\hat{p}_3$	2.8807		26.5484		
$\hat{p}_4$	12.8552				
LB p_1	0.7984	7.8890	0.7440	0.0686	7.7431
LB p_2	0.5690	0.9154	6.1431	-0.5898	
LB p_3	0.7731		-20.1168		
LB p_4	4.4101				
UB p_1	1.0013	11.2324	1.1483	0.1450	10.9881
UB p_2	1.0480	1.1803	10.6313	0.5898	
UB p_3	4.9884		73.2136		
UB p_4	21.3002				
SE p_1	0.0518	0.8529	0.1031	0.0195	0.8278
SE p_2	0.1222	0.0676	1.1450	0.3009	
SE p_3	1.0754		23.8092		
SE p_4	4.3088				
Initial value for $\hat{p}_1$	0.7334	88.3825	0.9580	0.6711	0.8000
Initial value for $\hat{p}_2$	0.8926	84.7931	8.0167	0.8521	
Initial value for $\hat{p}_3$	12.7750		18.5367		
Initial value for $\hat{p}_4$	11.0346				

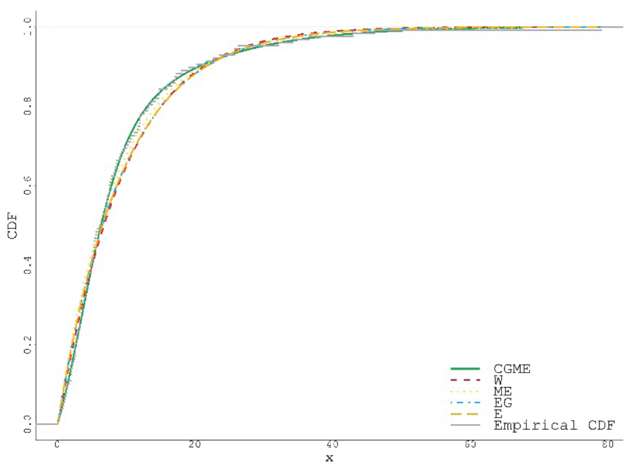

Figure 3. Fitted and empirical cdf plots for the first data

11.2, 11.5, 11.9,12.4, 12.5, 2.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4,15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3,21.4, 21.9, 23, 27, 31.6, 33.1, 38.5. This data also analyzed by [19]. CGME distribution also better modelled than all distribution given in [19] from this data sets.

CONCLUSIONS

In this study, a new lifetime distribution is introduced by the compounding methodology. The geometric and mixture exponential distributions were used to obtain the new distribution. Some characteristic properties of the new distribution such as moments, stochastic order, skewness, kurtosis, etc. are examined. Some characteristic properties of the new distribution such as moments, stochastic order, skewness, kurtosis, etc. were examined. Different

Table 12. Data analysis results for the second data

	CGME	W	ME	EG	E
ℓ	-316.8625	-318.2710	-328.0033	-328.0033	-328.0033
-2ℓ	633.7250	636.5420	656.0065	656.0065	656.0065
AIC	641.7250	640.5420	662.0065	660.0065	658.0065
BIC	652.1457	645.7523	669.8220	665.2169	660.6117
CAIC	642.1461	640.6657	662.2565	660.1302	658.0474
HQIC	645.9425	642.6507	665.1696	662.1153	659.0609
KS	0.0334	0.0607	0.1658	0.1658	0.1658
AD	0.1219	0.4402	4.0277	4.0277	4.0277
CVM	0.0161	0.0669	0.6716	0.6716	0.6716
KS p value	0.9999	0.8554	0.0082	0.0082	0.0082
AD p value	0.9998	0.8077	0.0085	0.0085	0.0085
CVM p value	0.9994	0.7716	0.0148	0.0148	0.0148
$\hat{p}_1$	0.9344	10.8298	0.5001	0.1023	9.7770
$\hat{p}_2$	0.9803	1.4414	9.7770	0.0000	
$\hat{p}_3$	1.2815		9.7770		
$\hat{p}_4$	5.7373				
LB p_1	0.8412	9.2723	-16468.3642	0.0654	7.8607
LB p_2	0.9467	1.2289	6.3469	-0.5138	
LB p_3	0.4872		6.3414		
LB p_4	3.8550				
UB p_1	1.0276	12.3873	16469.3645	0.1391	11.6933
UB p_2	1.0139	1.6538	13.2071	0.5138	
UB p_3	2.0757		13.2126		
UB p_4	7.6195				
SE p_1	0.0476	0.7947	8402.6362	0.0188	0.9777
SE p_2	0.0171	0.1084	1.7501	0.2621	
SE p_3	0.4052		1.7529		
SE p_4	0.9604				
Initial value for $\hat{p}_1$	0.2583	83.2927	0.2725	5.0286	0.8000
Initial value for $\hat{p}_2$	0.5572	71.8334	7.3973	0.8252	
Initial value for $\hat{p}_3$	8.4371		13.3400		
Initial value for $\hat{p}_4$	9.0949				

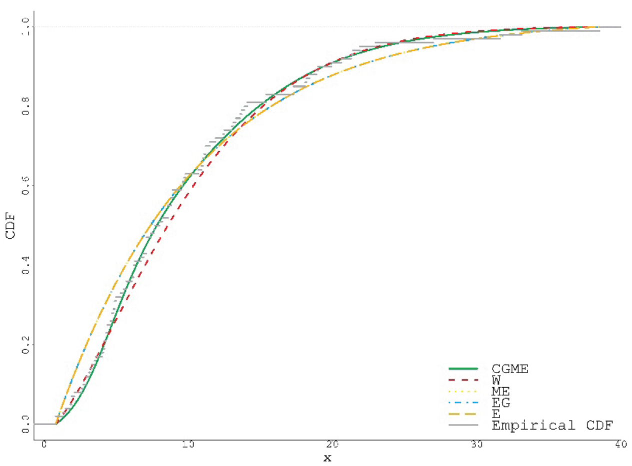

Figure 4. Fitted and empirical cdf plots for the second data

estimators have been studied to estimate the four unknown parameters of the new distribution. The performance of the estimators was examined with a Monte Carlo simulation. Two real data analyses are also examined to demonstrate the real-life usability of the distribution.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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