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Basic Properties of Tempered ν-Sequence Spaces

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Highlights

- This paper focuses on building new tempered sequence spaces.
- The directed preserving generator $(d.p.g.)$ is used to compose some new tempered sequence spaces.
- The basic properties of these new tempered sequence spaces are obtained.

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Abstract

In this paper, we will introduce tempered ν-sequence spaces generated by directed preserving generator ν. After building the spaces, we investigate and show tempered ν-sequence spaces are Banach spaces. In addition, we also find that there is an isomorphism between tempered ν-sequence spaces and the classical one. The direct implication is that some tempered ν-sequence spaces have a Schauder basis.

1. INTRODUCTION

The notion of tempered sequence spaces appears firstly in [1], where the authors propose the solution of an infinite differential equation. The simplest Cauchy problem of an infinite system to describe this situation is

$$\xi_n'(t) = \xi_n(t)$$

with the initial condition

$$\xi_n'(0) = 0$$

where $t \in [0, T]$. We easily see that the solution to the above problem is

$$x(t) = \big(\xi_n(t)\big) = (ne^t)_{n\in\mathbb{N}}.$$

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Therefore, x does not belong to any classical sequence space. However, we can find a positive sequence such that x belong to one of the classical sequence spaces. This sequence is called tempering sequence. To see this, set the tempering sequence as $\zeta = (\zeta_n) = \left(\frac{1}{n^3}\right)$, then

$$\sup_n\{\zeta_n|\xi_n|: n = 1,2,\dots\} = \sup_n\left\{\frac{1}{n^3}|ne^t|: n = 1,2,\dots\right\}$$
$$= \sup_n\left\{\frac{e^t}{n^2}: n = 1,2,\dots\right\}.$$

Letting $n \to \infty$, we get $\lim_n\left\{\sup\left\{\frac{e^t}{n^2}\right\}\right\} < \infty$. Thus $\zeta x = (\zeta_n\xi_n)_{n\in\mathbb{N}}$ is bounded. In this way, Banaś and Krajewska introduce the following set

$$\ell_\infty^\zeta = \left\{x = \left(\dot{\xi}_n\right)_{n\in\mathbb{N}}: \left(\dot{\zeta}_n\dot{\xi}_n\right)\ is\ bounded\right\}$$

which later is extended to the other sequence spaces (see [2]).

These new types of sequence spaces can be used to solve the Cauchy problem for ordinary differential equations and fractional ones [3-7]. The first difference between tempered sequence spaces and traditional ones is that they are more abstract since they are defined in terms of the growth rate of sequences of functions rather than their pointwise behavior [8]. The second difference is the involving concept of measure of noncompactness, i.e., the mathematical tools to quantify the degree of noncompactness of a set [2, 7]. Measures of noncompactness are not typically used in the study of classical sequence spaces.

On the other hand, Grossman and Katz proposed an alternative field that is generated by a special function α. A function $\alpha: \mathbb{R}(\mathbb{C}) \to \mathbb{R}(\mathbb{C})$ is called a *generator function* if it is a one-one function [9]. Using this generator, one can translate the classical field to a new field with properties that differ from the classical one. The authors introduce new non-Newtonian calculi calculus based on this new field (cf. [10]). Later, many studies used this generator to introduce concepts in normed spaces (see [11-15]).

Recently, Rohman and Eryılmaz [16] built a generator called a *directed preserving generator* ($d.p.g.$) which has stronger properties than the previous one. A function $\nu: \mathbb{R} \to \mathbb{R}$ is called $d.p.g.$ if it satisfies: (1) injective and continue; (2) for any $a, b \in \mathbb{R}$ with $a \leq b$, we have $\nu(a) \dot{\leq} \nu(b)$ in ${}_\nu\mathbb{R}$; and (3) for any $a, b \in \mathbb{R}$, there exists $\nu(c) \in {}_\nu\mathbb{R}$ such that $\nu(a) \dot{\leq} \nu(c)$ and $\nu(b) \dot{\leq} \nu(c)$. The last condition of $d.p.g.$ ensures that the new field generated by a $d.p.g.$ is going to infinity. Later, the author introduced a notion ν-normed spaces based on this field and gave some basic properties of these spaces. In this paper, we will discuss new tempered sequence spaces based on this $.p.g.\ \nu$.

2. TEMPERED ν-SEQUENCE SPACES

In this section, we will introduce sequence spaces ${}_\nu c_0^\zeta$, ${}_\nu c^\zeta$, and ${}_\nu \ell_{\dot{p}}^\zeta$ over the field ${}_\nu\mathbb{R}$. Before going further, let X be the set of all ν-real valued sequences, i.e.

$$X = \left\{x = \left(\dot{\xi}_n\right)_{n\in\mathbb{N}}: \dot{\xi}_n \in {}_\nu\mathbb{R}\ for\ all\ n \in \mathbb{N}\right\}.$$

Let $\zeta = \left(\dot{\zeta}_n\right)_{n\in\mathbb{N}}$ be a ν-real-valued sequence such that $\zeta_n \dot{>} \dot{0}$ and $\dot{\zeta}_{n+1} \dot{\leq} \dot{\zeta}_n$ for all $n \in \mathbb{N}$ and later ζ will be called a tempering sequence. If we define

$${}_\nu c_0^\zeta = \left\{x = \left(\dot{\xi}_n\right)_{n\in\mathbb{N}} \in X: \dot{\zeta}_n\dot{\xi}_n \overset{\nu}{\to} \dot{0}\right\},$$

then the arithmetic of ${}_\nu\mathbb{R}$ and the continuity of d.p.g. ν (see [15]) together imply that for any $y = (\dot{\eta}_n)_{n\in\mathbb{N}} \in {}_\nu c_0^\zeta$, we have

$$\dot{\lambda}x + y = (\dot{\lambda}\dot{\zeta}_n\dot{\xi}_n) \dot{+} (\dot{\zeta}_n\dot{\eta}_n) = \dot{\lambda}(\dot{\zeta}_n\dot{\xi}_n) \dot{+} (\dot{\zeta}_n\dot{\eta}_n) \xrightarrow{\nu} \dot{0}$$

and hence $\dot{\lambda}x + y \in {}_{\nu}c_0^{\zeta}$. Therefore, ${}_{\nu}c_0^{\zeta}$ is a linear space.

Theorem 2.1. ${}_{\nu}c_0^{\zeta}$ is a normed space under the norm defined as

$$\dot{\|}\, x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \doteq \dot{\|}\, (\dot{\xi}_n) \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \doteq \sup_n \{\dot{\zeta}_n \dot{|}\dot{\xi}_n\dot{|} : n = 1,2,\dots\}.$$

Proof. Let $x = (\dot{0}, \dot{0}, \dots)$, then $\dot{\|}\, x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \doteq \sup_n \{\dot{\zeta}_n \dot{|}\dot{0}\dot{|} : for\ all\ n\} \doteq \dot{0}$. Conversely, let $\dot{\|}\, x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \doteq \dot{0}$. Since ζ is a positive sequence, then $x = (\dot{0}, \dot{0}, \dots)$. Therefore, $\dot{\|}\, x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \doteq \dot{0}$ if and only if x is a zero sequence. On the other hand, using the preliminary result in [15],

$$\begin{aligned} \dot{\|}\, \dot{\lambda}x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} &\doteq \sup_n \{\dot{\zeta}_n \dot{|}\dot{\lambda}\dot{\xi}_n\dot{|}\} \\ &\doteq \dot{|}\dot{\lambda}\dot{|} \sup_n \{\dot{\zeta}_n \dot{|}\dot{\xi}_n\dot{|}\} \doteq \dot{|}\dot{\lambda}\dot{|}\, \dot{\|}\, x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \end{aligned}$$

and the triangle inequality in ${}_{\nu}\mathbb{R}$ ([5], Lemma 3.1) gives

$$\begin{aligned} \dot{\|}\, x + y \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} &\doteq \sup_n \{\dot{\zeta}_n \dot{|}\dot{\xi}_n \dot{+} \dot{\eta}_n\dot{|}\} \\ &\dot{\leq} \sup_n \{\dot{\zeta}_n \dot{|}\dot{\xi}_n\dot{|} \dot{+} \dot{\zeta}_n \dot{|}\dot{\eta}_n\dot{|}\} \doteq \dot{\|}\, x \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \dot{+} \dot{\|}\, y \,\dot{\|}_{{}_{\nu}c_0^{\zeta}}. \end{aligned}$$

The last two results show that ${}_{\nu}c_0^{\zeta}$ is a normed space. ∎

Theorem 2.2. ${}_{\nu}c_0^{\zeta}$ is a Banach space.

Proof. Let $x^k = \left(\dot{\xi}_1^{(k)}, \dot{\xi}_2^{(k)}, \dot{\xi}_3^{(k)}, \dots\right)$ be a Cauchy sequence in ${}_{\nu}c_0^{\zeta}$. For a fixed $n_0 \in \mathbb{N}$ denote the n_0th term of each x^k as $\dot{\xi}_{n_0}^{(k)}$. Since x^k is a Cauchy sequence, for any $\dot{\varepsilon} \dot{>} \dot{0}$ there exists $N \in \mathbb{N}$ such that for every $j, k \geq N$, we have

$$\dot{\|}\, x^k - x^j \,\dot{\|}_{{}_{\nu}c_0^{\zeta}} \doteq \sup \left\{\dot{\zeta}_{n_0} \dot{|}\dot{\xi}_{n_0}^{(k)} \dot{-} \dot{\xi}_{n_0}^{(j)}\dot{|}\right\} \dot{<} \dot{\varepsilon}$$

and hence $\left(\dot{\xi}_{n_0}^{(k)}\right)$ is a Cauchy sequence in ${}_{\nu}\mathbb{R}$. The completeness of ${}_{\nu}\mathbb{R}$ implies that $\left(\dot{\xi}_{n_0}^{(k)}\right)$ converges to a number, say, $\dot{\xi}_{n_0} \in {}_{\nu}\mathbb{R}$. Since n_0 was arbitrary, if we apply this process for all $n \in \mathbb{N}$, then $x^k \in {}_{\nu}c_0^{\zeta}$ is a convergent sequence. Set $x = (\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dots)$ and assume the x^k converges to x. Since $x^k, x - x^k \in {}_{\nu}c_0^{\zeta}$ and ${}_{\nu}c_0^{\zeta}$ is a linear space, we see that $x = x - x^k + x^k$ and hence $x \in {}_{\nu}c_0^{\zeta}$. The proof is complete. ∎

If we define

$${}_{\nu}c^{\zeta} = \left\{x = (\dot{\xi}_n)_{n\in\mathbb{N}} \in X : (\dot{\zeta}_n\dot{\xi}_n)\ converges\ to\ a\ finite\ limit\right\}$$

And

$${}_{\nu}\ell_{\infty}^{\zeta} = \left\{x = (\dot{\xi}_n)_{n\in\mathbb{N}} \in X : (\dot{\zeta}_n\dot{\xi}_n)\ is\ a\ bounded\ sequence\right\},$$

then, in a similar way as in ${}_{\nu}c_0^{\zeta}$ space, one can see that ${}_{\nu}c^{\zeta}$ and ${}_{\nu}\ell_{\infty}^{\zeta}$ are normed spaces under the supremum norm, that is

$$\dot{\|}\ x\ \dot{\|}_{\ _\nu\ell_\infty^\zeta} \dot{=} \dot{\|}\ (\dot{\xi}_n)\ \dot{\|}_{\ _\nu\ell_\infty^\zeta} \dot{=} \sup_n\{\dot{\zeta}_n \dot{|}\dot{\xi}_n\dot{|} : n = 1,2,\dots\}.$$

As in $_\nu c_0^\zeta$, it is easy to show that $_\nu c^\zeta$ and $_\nu \ell_\infty^\zeta$ are complete normed spaces. However, we will show the completeness of $_\nu \ell_\infty^\gamma$.

Theorem 2.3. $_\nu \ell_\infty^\zeta$ is a complete normed space.

Proof. Let $x^k = \left(\dot{\xi}_1^{(k)}, \dot{\xi}_2^{(k)}, \dot{\xi}_3^{(k)}, \dots\right)$ be a Cauchy sequence in $_\nu \ell_\infty^\zeta$, i.e., for each $n \in \mathbb{N}$ we have

$$\lim_{j,k\to\infty} \dot{\|}\ x^k - x^j\ \dot{\|}_{\ _\nu\ell_\infty^\zeta} \dot{=} \lim_{j,k\to\infty}\left\{\sup_n\left\{\dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n^{(j)}\dot{|}\right\}\right\} \dot{=} \dot{0}.$$

Thus, for any $\dot{\varepsilon} \dot{>} \dot{0}$ there exists $N \in \mathbb{N}$ such that $\dot{\|}\ x^k - x^j\ \dot{\|}_{\ _\nu\ell_\infty^\zeta} \dot{<} \dot{\varepsilon}$ for all $j,k \geq N$. Hence, we can find $N_0 \in \mathbb{N}$ such that for all $j,k \geq N_0$ we get

$$\sup_n\left\{\dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n^{(j)}\dot{|}\right\} \dot{<} \dot{\varepsilon}/\dot{3}$$

and hence $\dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n^{(j)}\dot{|} \dot{<} \dot{\varepsilon}/\dot{3}$ for all $n \in \mathbb{N}$. Therefore, for a fixed $n_0 \in \mathbb{N}$, the sequence of ν-real numbers $\left(\dot{\xi}_{n_0}^{(k)}\right) = \left(\dot{\xi}_{n_0}^{(1)}, \dot{\xi}_{n_0}^{(2)}, \dot{\xi}_{n_0}^{(3)}, \dots\right)$ is a Cauchy sequence in $_\nu\mathbb{R}$. Since $_\nu\mathbb{R}$ is complete, this sequence converges to $\dot{\xi}_{n_0} \in {}_\nu\mathbb{R}$, i.e.

$$\dot{\zeta}_{n_0} \dot{|}\dot{\xi}_{n_0}^{(k)} \dot{-} \dot{\xi}_{n_0}\dot{|} \xrightarrow{\nu} \dot{0}$$

whenever $j \to \infty$. Since n_0 was arbitrary, for each n we have

$$\dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n\ \dot{|} \dot{<} \dot{\varepsilon}/\dot{3}$$

by letting $j \to \infty$. This result is true if we take the supremum over $n \in \mathbb{N}$, i.e.

$$\sup_n\left\{\dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n\ \dot{|}\right\} \dot{<} \dot{\varepsilon}/\dot{3}$$

for all $k \geq N_0$. Consequently

$$\lim_{k\to\infty} \dot{\|}\ x^k - x\ \dot{\|}_{\ _\nu\ell_\infty^\zeta} \dot{=} \dot{0}$$

where $x = (\dot{\xi}_n)_{n\in\mathbb{N}}$. This shows that x^k converges to x. Since each $x^k \in {}_\nu\ell_\infty^\zeta$, for each k, there exists $M_k \in {}_\nu\mathbb{R}$ such that $\dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)}\dot{|} \dot{\leq} M_k$ for all n. Set $M \dot{=} \max_k\{M_k : k = 1,2,\dots\}$, then

$$\begin{aligned}\dot{\zeta}_n \dot{|}\dot{\xi}_n\ \dot{|} &\dot{=} \dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)} \dot{+} \dot{\xi}_n\ \dot{-} \dot{\xi}_n^{(k)}\dot{|} \\ &\dot{\leq} \dot{\zeta}_n \dot{|}\dot{\xi}_n^{(k)}\dot{|} \dot{+} \dot{\zeta}_n \dot{|}\dot{\xi}_n\ \dot{-} \dot{\xi}_n^{(k)}\dot{|} \dot{<} M \dot{+} \dot{\varepsilon}/\dot{3}\end{aligned}$$

and hence $x = (\dot{\xi}_n)_{n\in\mathbb{N}}$ is a bounded sequence, i.e., $x \in {}_\nu\ell_\infty^\zeta$. This result completes the proof. ■

It is well known that for $1 \leq p < \infty$, ℓ_p space is complete. When we transfer the real field to ν-real field using $.p.g.\nu$, it is easy to see that $\dot{1} \dot{\leq} \dot{p} \dot{<} \infty$, $_\nu\ell_{\dot{p}}$ spaces are Banach spaces (cf. [5], Theorem 5.4 for a special $d.p.g.\alpha$). As in the previous discussion, let x be any ν-real-valued sequence and ζ be a tempering sequence. Define a set ${}_{\nu}\ell_{\dot{p}}^{\zeta}$ as

$$ {}_{\nu}\ell_{\dot{p}}^{\zeta} = \left\{ x = (\dot{\xi}_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\xi}_n\dot{|}^{\dot{p}} \mathrel{\dot{<}} \infty \right\}, $$

where $\dot{1} \mathrel{\dot{\leq}} \dot{p} \mathrel{\dot{<}} \infty$. It is easy to see that the set ${}_{\nu}\ell_{\dot{p}}^{\zeta}$ is a linear space. Define a norm on ${}_{\nu}\ell_{\dot{p}}^{\zeta}$ space as

$$ \dot{\|}\, x \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}} \doteq \left(\sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\xi}_n\dot{|}^{\dot{p}} \right)^{\dot{1}/\dot{p}}, $$

then ${}_{\nu}\ell_{\dot{p}}^{\zeta}$ is a normed space. Clearly $\dot{\|}\, x \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}} \doteq \dot{0}$ if and only if $x = (\dot{0}, \dot{0}, \dot{0}, \dots)$ and $\dot{\|}\, \dot{\lambda} x \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}} \doteq \dot{|}\dot{\lambda}\dot{|}\ \dot{\|}\, x \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}}$. Yet to be proved is the triangle inequality. By Lemma 3.3 in [5],

$$ \begin{aligned} \dot{\|}\, x + y \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}} &\doteq \left(\sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\xi}_n \dot{+} \dot{\eta}_n\dot{|}^{\dot{p}} \right)^{\dot{1}/\dot{p}} \\ &\mathrel{\dot{\leq}} \left(\sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\xi}_n\dot{|}^{\dot{p}} \right)^{\dot{1}/\dot{p}} \dot{+} \left(\sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\eta}_n\dot{|}^{\dot{p}} \right)^{\dot{1}/\dot{p}} \\ &\doteq \dot{\|}\, x \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}} \dot{+} \dot{\|}\, y \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}}. \end{aligned} $$

Theorem 2.4. ${}_{\nu}\ell_{\dot{p}}^{\zeta}$, $\dot{1} \mathrel{\dot{\leq}} \dot{p} \mathrel{\dot{<}} \infty$, is a complete normed space.

Proof. Let $(x^k) \subset {}_{\nu}\ell_{\dot{p}}^{\zeta}$ be a Cauchy sequence. Given any $\dot{\varepsilon} \mathrel{\dot{>}} \dot{0}$, there exists $N_0 \in \mathbb{N}$ such that for all $j, k \geq N_0$, $\dot{\|}\, x^k - x^j \,\dot{\|}_{{}_{\nu}\ell_{\dot{p}}^{\zeta}} \mathrel{\dot{<}} \dot{\varepsilon}$. Thus

$$ \sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n^{(j)}\dot{|}^{\dot{p}} \mathrel{\dot{<}} \dot{\varepsilon}^{\dot{p}}. $$

Since this is true for each n, we see that for a fixed n_0

$$ \dot{\zeta}_{n_0} \dot{|}\dot{\xi}_{n_0}^{(k)} \dot{-} \dot{\xi}_{n_0}^{(j)}\dot{|} \mathrel{\dot{<}} \dot{\varepsilon} $$

whenever $j, k \geq N_0$. Hence, $\left(\dot{\xi}_{n_0}^{(k)}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in ${}_{\nu}\mathbb{R}$. The completeness of ${}_{\nu}\mathbb{R}$ implies the convergence of $\left(\dot{\xi}_{n_0}^{(k)}\right)$. Assume that $\dot{\xi}_{n_0}^{(k)} \xrightarrow{\nu} \dot{\xi}_{n_0}$. Since n_0 is arbitrary, by letting $j \to \infty$, we see that

$$ \sum_{n=1}^{\infty} \dot{\zeta}_n{}^{\dot{p}} \,\dot{|}\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n\dot{|}^{\dot{p}} \mathrel{\dot{<}} \dot{\varepsilon}^{\dot{p}} $$

whenever $k \geq N_0$. This result implies that for each k, the sequence $\left(\dot{\xi}_n^{(k)} \dot{-} \dot{\xi}_n\right)_{n\in\mathbb{N}}$ is an element of ${}_{\nu}\ell_{\dot{p}}^{\zeta}$. Put $x = (\dot{\xi}_n)_{n\in\mathbb{N}}$. Since for each k, $x^k = \left(\dot{\xi}_n^{(k)}\right)_{n\in\mathbb{N}} \in {}_{\nu}\ell_{\dot{p}}^{\zeta}$, by Minkowski's inequality for ν-real scalars,

$$\begin{aligned}\left(\dot{\|}\ x\ \dot{\|}_{\ {}_{\nu}\ell^{\zeta}_{\dot{p}}}\right)^{\dot{p}} &\dot{=} \sum_{n=1}^{\infty} \dot{\zeta}_n^{\ \dot{p}}\ \dot{|}\dot{\xi}_n\dot{|}^{\dot{p}} \\ &\dot{=} \sum_{n=1}^{\infty} \dot{\zeta}_n^{\ \dot{p}}\ \dot{|}\dot{\xi}_n^{(k)}\dot{+}\dot{\xi}_n\dot{-}\dot{\xi}_n^{(k)}\dot{|}^{\dot{p}} \\ &\dot{\leq} \sum_{n=1}^{\infty} \dot{\zeta}_n^{\ \dot{p}}\ \dot{|}\dot{\xi}_n^{(k)}\dot{|}^{\dot{p}}\dot{+}\sum_{n=1}^{\infty} \dot{\zeta}_n^{\ \dot{p}}\ \dot{|}\dot{\xi}_n\dot{-}\dot{\xi}_n^{(k)}\dot{|}^{\dot{p}} \\ &\dot{<} M\dot{+}\dot{\varepsilon}^{\dot{p}}\end{aligned}$$

for a finite $M \in {}_{\nu}\mathbb{R}$. Therefore, $x \in {}_{\nu}\ell^{\zeta}_{\dot{p}}$. On the other hand,

$$\dot{\|}\ x^k - x\ \dot{\|}_{\ {}_{\nu}\ell^{\zeta}_{\dot{p}}} \dot{=} \left(\sum_{n=1}^{\infty} \dot{\zeta}_n^{\ \dot{p}}\ \dot{|}\dot{\xi}_n^{(k)}\dot{-}\dot{\xi}_n\dot{|}^{\dot{p}}\right)^{\dot{1}/\dot{p}} \dot{<} \dot{\varepsilon}$$

whenever $k \geq N_0$, i.e., $x^k \to x$ in ${}_{\nu}\ell^{\zeta}_{\dot{p}}$ and so the proof is complete. ■

After discussing the tempered ν- sequence spaces, it is natural to ask about the relation of these spaces with the classical sequence spaces. Firstly, define a mapping $T\colon {}_{\nu}\ell^{\zeta}_{\infty} \to {}_{\nu}\ell_{\infty}$ by $T(x) = T\left(\left(\dot{\xi}_n\right)\right) = \left(\dot{\zeta}_n\dot{\xi}_n\right)$. Take arbitrary $x, y \in {}_{\nu}\ell^{\zeta}_{\infty}$, then

$$\begin{aligned}\dot{\|}\ T(x) - T(y)\ \dot{\|}_{\ {}_{\nu}\ell_{\infty}} &\dot{=} \dot{\|}\ T\left(\left(\dot{\xi}_n\right)\right) - T\left((\dot{\eta}_n)\right)\ \dot{\|}_{\ {}_{\nu}\ell_{\infty}} \\ &\dot{=} \dot{\|}\ \left(\dot{\zeta}_n\dot{\xi}_n\right) - \left(\dot{\zeta}_n\dot{\eta}_n\right)\ \dot{\|}_{\ {}_{\nu}\ell_{\infty}} \\ &\dot{=} \sup_n\left\{\dot{|}\left(\dot{\zeta}_n\dot{\xi}_n\right)\dot{-}\left(\dot{\zeta}_n\dot{\eta}_n\right)\dot{|}\colon n = 1,2,\dots\right\} \\ &\dot{=} \sup_n\left\{\dot{\zeta}_n\dot{|}\dot{\xi}_n\dot{-}\dot{\eta}_n\dot{|}\colon n = 1,2,\dots\right\} \\ &\dot{=} \dot{\|}\ \left(\dot{\xi}_n\right) - (\dot{\eta}_n)\ \dot{\|}_{\ {}_{\nu}\ell^{\zeta}_{\infty}} \\ &\dot{=} \dot{\|}\ x - y\ \dot{\|}_{\ {}_{\nu}\ell^{\zeta}_{\infty}}.\end{aligned}$$

In addition,

$$\begin{aligned}T\left(\dot{\lambda}x + y\right) &= \left(\dot{\lambda}\dot{\zeta}_n\dot{\xi}_n\dot{+}\dot{\zeta}_n\dot{\eta}_n\right) \\ &= \dot{\lambda}\left(\dot{\zeta}_n\dot{\xi}_n\right) + \left(\dot{\zeta}_n\dot{\eta}_n\right) \\ &= \dot{\lambda}T(x) + T(y).\end{aligned}$$

It is easy to see that $T(x) \neq T(y)$ whenever $x \neq y$. These inspections show that T is an isometric isomorphism. This result also true for ${}_{\nu}c^{\zeta}_0$ and ${}_{\nu}c^{\zeta}$, since both of them are closed subspaces of ${}_{\nu}\ell^{\zeta}_{\infty}$. Similarly, we can find an isometric isomorphism between ${}_{\nu}\ell^{\zeta}_{\dot{p}}$ and ${}_{\nu}\ell_{\dot{p}}$. Therefore, we can summarize the following theorem.

Theorem 2.5. ${}_{\nu}c^{\zeta}_0$, ${}_{\nu}c^{\zeta}$, and ${}_{\nu}\ell^{\zeta}_{\dot{p}}$, $\dot{1} \dot{\leq} \dot{p} \dot{\leq} \infty$, respectively are isometrically isomorphic with ${}_{\nu}c_0$, ${}_{\nu}c$, and ${}_{\nu}\ell_{\dot{p}}$.

It is well known that for $1 \leq p < \infty$, ℓ_p space has a Schauder basis. Recall that we can find a field isomorphism between ${}_{\nu}\mathbb{R}$ and $\mathbb{R}$ [15], the following is an immediate consequence of the last theorem.

Corollary 2.6. ${}_{\nu}\ell^{\zeta}_{\dot{p}}$ space, $\dot{1} \dot{\leq} \dot{p} \dot{\leq} \infty$, has a Schauder basis.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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