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Research Article

On innovations of the multivariable fractional Hardy-type inequalities on time scales

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ABSTRACT

Fractional integral-type inequalities, dynamic equations, integral operators and variable exponents have an important place in time scales theory and harmonic analysis. Our main goal in this study is to obtain the multivariable fractional Hardy-type integral inequality using a new version of Jensen's inequality for super-quadratic and sub-quadratic functions on time scales with variable exponents.

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INTRODUCTION

Firstly, we can inform readers about the historical development of Hardy inequality as follows. The discrete Hardy inequality was proved the following by G.H. Hardy [1]. Let (b_m) be a sequence of non-negative real numbers and for $c > 1$, $c \in \mathbb{R}$, then

$$\sum_{m=1}^{\infty}\left(\frac{b_1+b_2+b_3+\cdots+b_m}{m}\right)^c \le \left(\frac{c}{c-1}\right)^c \sum_{m=1}^{\infty} b_m^c. \tag{1}$$

The classical Hardy inequality was proved the following by G.H. Hardy [2]. If g^c is integrable, then we have

$$\int_0^{\infty}\left(\frac{1}{s}\int_0^{s} g(\tau)d\tau\right)^c ds \le \left(\frac{c}{c-1}\right)^c \int_0^{\infty} g^c(s)ds, \tag{2}$$

for $c > 1$ and $g \ge 0$.

Later, inequality (2) has been generalized the following by G.H. Hardy [3]. If g integrable on $(0, \infty)$, then we have

$$\int_0^{\infty}\left(\frac{1}{t^n}\int_t^{\infty} g(\tau)d\tau\right)^c dt \le \left(\frac{c}{1-n}\right)^c \int_0^{\infty} \frac{1}{t^{n-c}} g^c(t)dt, \quad n<1, \tag{3}$$

$$\int_0^{\infty}\left(\frac{1}{t^n}\int_0^{t} g(\tau)dr\right)^c dt \le \left(\frac{c}{n-1}\right)^c \int_0^{\infty} \frac{1}{t^{n-c}} g^c(t)dt, \quad n>1, \tag{4}$$

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for $c > 1$ and $g(t) > 0$.

Hardy and Littlewood [4] demonstrated the following discrete versions of inequalities (3) and (4). Let $c > 1$, if (b_n) is a sequence of non-negative terms, then we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\sum_{m=n}^{\infty} b_m \right)^c \leq M \sum_{n=1}^{\infty} \frac{1}{n^{k-c}} b_n^c, \qquad k < 1, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\sum_{m=1}^{n} b_m \right)^c \leq M \sum_{n=1}^{\infty} \frac{1}{n^{k-c}} b_n^c, \qquad k > 1, \tag{6}$$

where M is a non-negative constant.

Oguntuase and Persson [5] presented a number of Hardy-type inequalities on time scales using super-quadraticity technique which is based on the application of Jensen dynamic inequality. Fabelurin et al. [6] proved a new Jensen inequality for multivariate super-quadratic functions. For some recent developments of Hardy-type integral inequalities on time scales and related results we refer interested reader to the book [7].

Fractional Hardy-type integral inequalities also play an important role in time scales. Let H_β and $\widetilde{H}_\beta$ be the fractional Hardy operator and its adjoint on $(0, \infty)$,

$$H_\beta f(t) = \frac{1}{t^{1-\beta}} \int_0^t f(s) ds, \quad \widetilde{H}_\beta f(t) = \int_t^{\infty} \frac{1}{t^{1-\beta}} f(s) ds, \tag{7}$$

where $0 \leq \beta < 1$ (for details see [8]). When $\beta = 0$, we denote H_0 as H and $\widetilde{H}_0$ as $\widetilde{H}$. Hardy [9-11] established the following Hardy integral inequalities

$$\int_0^{\infty} |Hf(x)|^p dx \leq (p')^p \int_0^{\infty} |f(x)|^p dx, \quad p > 1, \tag{8}$$

and

$$\int_0^{\infty} \left|\widetilde{H} f(x)\right|^p dx \leq p^p \int_0^{\infty} |f(x)|^p dx, \qquad p > 1, \tag{9}$$

where ${}^1/_p + {}^1/_{p'} = 1$. Heinig et al. [12] proved the following n-dimensional fractional order Hardy-type integral inequality. Let $1 < q < \infty$, $n \geq 1$. If $\gamma q > 1$, then we have

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^q}{|x|^{\gamma q n}} dx \right)^{\frac{1}{q}} \leq \frac{2^{\frac{n(1+\gamma q)}{q}} n^{\frac{1}{q}} [q(1+\gamma) - 1]}{|D^{n-1}|^{\frac{1}{q}} (\gamma q - 1)} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{n(1+\gamma q)}} dy dx \right)^{\frac{1}{q}}, \tag{10}$$

where $x, y \in \mathbb{R}^n$ and D^{n-1} is the unit sphere in $\mathbb{R}^n$.

Dyda [13] proved the following fractional order Hardy-type integral inequality.

Let $S \subset \mathbb{R}^n, n \geq 1$ be an open set and let $\varepsilon_S(x) = \inf\{|x-y| : y \in S^k\}$. For $0 < \beta, q < \infty$ and for all $f \in C_k(S)$, then

$$\int_S \frac{|f(x)|^q}{\varepsilon_S(x)^\beta} dx \leq k \int_S \int_S \frac{|f(x) - f(y)|^q}{|x-y|^{n+\beta}} dx dy, \tag{11}$$

where $k = k(S, \beta, n, q)$ and $k < \infty$ is a constant that depends only on S, β, n, q

Loss and Sloane [14] have proved the following fractional Hardy inequality

$$\frac{1}{2} \int_{S \times S} \frac{\left(f(x) - f(y)\right)^2}{|x-y|^{n+\beta}} dx dy \geq \Lambda_{n,\beta} \int_S \frac{f(x)^2}{dist(x, S^k)^\beta} dx, \quad f \in C_k(S) \tag{12}$$

for convex domain $S \subset \mathbb{R}^n$ and $1 < \beta < 2$, where $\Lambda_{n,\beta} = \mu^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{1+\beta}{2}\right) B\left(\frac{1+\beta}{2}, \frac{2-\beta}{2}\right) - 2^\beta}{\Gamma\left(\frac{n+\beta}{2}\right) \beta 2^\beta}$ is the best constant, B is the Euler beta function, and $C_k(S)$ denotes the class of all continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ with compact support in S. Dyda [15] proved the following fractional Hardy-type integral inequality.

Let $1 < \beta < 2$, $a < b$ and $a, b \in (-\infty, \infty)$. For for all $f \in C_k(a, b)$ the following inequality is provided.

$$\frac{1}{2} \int_a^b \int_a^b \frac{\left(f(x) - f(y)\right)^2}{|x-y|^{\beta+1}} dx dy \geq \left(\Lambda_{1,\beta} + \frac{4 - 2^{3-\beta}}{\beta(b-a)} \right) \int_a^b f(x)^2 \left(\frac{b-a}{(x-a)(b-x)} \right)^{\beta-1} dx. \tag{13}$$

Bogdan and Dyda [16] proved the following Hardy-type inequality in the half-space $S = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. If $f \in C_k(S)$, then we have

$$\frac{1}{2} \int_S \int_S \frac{\left(f(x) - f(y)\right)^2}{|x-y|^{n+\beta}} dx dy \geq \Lambda_{n,\beta} \int_S \frac{f(x)^2}{x_n^\beta} dx, \tag{14}$$

where $\Lambda_{n,\beta} = \mu^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{1+\beta}{2}\right) B\left(\frac{1+\beta}{2}, \frac{2-\beta}{2}\right) - 2^\beta}{\Gamma\left(\frac{n+\beta}{2}\right) \beta 2^\beta}$.

Sloane [17] established the following a fractional Hardy-Sobolev-Maz'ya inequality.

Let $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$ be the upper half-space, and let D be a domain in $\mathbb{R}^n$ with non-empty boundary. Then, there exists a fractional Hardy inequality on $\mathbb{R}_+^n$ which states that there exists $\Lambda_{n,q,\beta} > 0$ so that for all $g \in C_k(\mathbb{R}_+^n)$

$$\int_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \frac{\left(g(x) - g(y)\right)^q}{|x-y|^{n+\beta}} dx dy \geq \Lambda_{n,q,\beta} \int_{\mathbb{R}_+^n} \frac{g(x)^q}{x_n^\beta} dx, \tag{15}$$

where $1 \leq q < \infty$, $0 < \beta < q$ and $\beta \neq 1$.

Dyda and Frank [18] demonstrated the following a fractional version of the Hardy-Sobolev-Maz'ya inequality.

Let $d \geq 2$, $2 \leq q < \infty$ and $t \in (0,1)$ with $1 < qt < d$. There is a constant $\Theta_{d,q,t} > 0$ such that

$$\int_D \int_D \frac{|f(x)-f(y)|^q}{|x-y|^{d+qt}} dxdy - \Lambda_{d,q,t} \int_D \frac{|f(x)|^q}{m_{qt} x^{qt}} dx \geq \Theta_{d,q,t} \left(\int_D |f(x)|^p \, dx \right)^{\frac{q}{p}}, \quad (16)$$

for all open $D \subsetneq \mathbb{R}^d$ and for all $f \in W_0^{t,q}(D)$, where $p = dq/(d-qt)$.

Edmunds et al. [19] established the following fractional Hardy-type inequalities. Let $0 < t < 1$ be such that $\frac{1}{p} - \frac{1}{q} < \frac{t}{d}$, where $1 < p, q < \infty$. Assume that $D \in \mathbb{R}^d$ a bounded domain for $d > 2$, whose complement is (t,p) locally uniformly fat with constants γ, s_0. Then, for all $f \in C_0^\infty(D)$,

$$\int_D \frac{|f(x)|^q}{dist(x,\partial D)^{q(t+d(1/q-1/p))}} dx \leq k \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+pt}} dydx \right)^{\frac{q}{p}}, \quad (17)$$

where constant k depends on t, d, p, q, γ, $s0$ and $diam(D)$. Ihnatsyeva et al. [20] proved the following fractional order Hardy inequalities. Let $0 < t < 1$, $1 < q < \infty$ satisfy $0 < tq < d$, and let $\Omega \subset \mathbb{R}^d$ be an open set. Suppose that there exist $d - tq < \gamma \leq d$ and $C_0 > 0$, $k \geq 1$ such that

$$H_\infty^\gamma\left(\partial_{x,k}^{vis}\Omega\right) \geq C_0 dist(x,\partial\Omega)^\gamma, \qquad for\ all\ x \in \Omega. \quad (18)$$

Then Ω admits an (t, q)-Hardy inequality, where fractional (t, q)-Hardy inequality is

$$\int_\Omega \frac{|f(x)|^q}{dist(x,\partial\Omega)^{qt}} dx \leq k \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{d+qt}} dydx. \quad (19)$$

In this study we prove the multivariable fractional Hardy-type integral inequality using new version Jensen's inequality for multivariable super-quadratic and sub-quadratic functions on time scales.

Now let's give the concepts of time scales to prove our results.

MATHEMATICAL BACKGROUND

The founder of time scales calculus is German mathematician Stefan Hilger [21]. For a quarter century, integral-type inequalities and dynamic equations in time scales have gained a very important place in the scientific world.

In this section, we will give some concepts that will be necessary for us to prove our results (for details [22-27]). A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers $\mathbb{R}$. The $(0,\infty)\mathbb{T}$ is denoted by $(0,\infty) \cap \mathbb{T}$. The mappings $\sigma, \rho: \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}: s > t\}$, for $t \in \mathbb{T}$. Respectively, $\sigma(t)$ is forward jump operator and $\rho(t)$ is backward jump operator. If $\sigma(t) > t$, then t is right-scattered and if $\sigma(t) = t$, then t is called right-dense. If $\rho(t) < t$, then t is left-scattered and if $\rho(t) = t$, then t is called left-dense. Let two mappings μ, ϑ: $\mathbb{T} \to \mathrm{R}^+$ such that $\mu(t) = \sigma(t) - t$, $\vartheta(t) = t - \rho(t)$ are called graininess mappings.

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$. Briefly

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho \sup \mathbb{T}, \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty, \end{cases}$$

by the same way

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\mathbb{T})], & |\inf \mathbb{T}| < \infty, \\ \mathbb{T}, & \inf \mathbb{T} = -\infty. \end{cases}$$

Assume that $h: \mathbb{T} \to \mathrm{R}$ is a function and let $t \in \mathbb{T}^k (t \neq minT)$. If h is $\Delta-$ differentiable at point t, then h is continuous at point t and if h is left continuous at point t, t is right-scattered, then h is $\Delta-$ differentiable at point t

$$h^\Delta(t) = \frac{h^\sigma(t) - h(t)}{\mu(t)}.$$

Let t is right-dense. If h is $\Delta-$ differentiable at point t and $\lim_{s\to t} \frac{h(t)-h(s)}{t-s}$, then

$$h^\Delta(t) = \lim_{s\to t} \frac{h(t)-h(s)}{t-s}.$$

If h is $\Delta-$ differentiable at point t, then $h^\sigma(t) = h(t) + \mu(t)h^\Delta(t)$. If $\mathbb{T} = \mathbb{R}$, then $h^\Delta(t)(t) = h'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $h^\Delta(t)$ reduces to $\Delta h(t)$.

The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. Let $h: \mathbb{T} \to \mathrm{R}$ and $h^\sigma: \mathbb{T} \to \mathrm{R}$ by $h^\sigma(t) = h(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $h^\sigma = h \circ \sigma$ and let $h: \mathbb{T} \to \mathrm{R}$ and $h^\sigma: \mathbb{T} \to \mathrm{R}$ by $h^\sigma(t) = h(\rho(t))$ for all $t \in \mathbb{T}$, i.e., $h^\sigma = h \circ \rho$.

The Hilger derivative (also delta derivative) $h^\Delta(t)$ is defined as follows.

There exists a neighborhood V of t such that

$$\left|h(\sigma(t)) - \mathrm{h}(\mathrm{s}) - h^\Delta(t)(\sigma(t)-s)\right| \leq |\sigma(t) - s|, \quad (20)$$

for all $\varepsilon > 0$ and $s, t \in V$.

Suppose that $H: \mathbb{T} \to \mathrm{R}$ is defined by $\Delta-$ antiderivative of $h: \mathbb{T} \to \mathrm{R}$, then $H^\Delta = h(t)$ holds for all $t \in \mathbb{T}$. We define the Cauchy $\Delta-$integral of h by

$$\int_s^t h(\tau)\Delta\tau = H(t) - H(s),$$

for $s, t \in V$. If $a, b \in \mathbb{T}$ and $\mathrm{u}, \mathrm{v} \in C_{rd}(\mathbb{T})$, then

$$\int_a^b u(x)v^\Delta(x)\Delta x = [u(x)v(x)]_a^b - \int_a^b v^\sigma(x)u^\Delta(x)\Delta x. \quad (21)$$

Suppose that $f,\ g$: $\mathbb{R} \to \mathbb{R}$ is continuously Hilger (delta) differentiable, then $f \circ g$: $\mathbb{R} \to \mathbb{R}$ is Hilger (delta) differentiable

$$(f \circ g)^{\Delta}(x) = \left\{ \int_0^1 f'(g(x) + h\mu(x)g^{\Delta}(x))\, dh \right\} g^{\Delta}(x). \quad (22)$$

If f, g satisfy the conditions of [23, Theorem 1.90], then $f \circ g$: $\mathbb{T} \to \mathbb{R}$ is Hilger differentiable and there exists d in the real interval $[x, \sigma(x)]$ such that

$$(f \circ g)^{\Delta}(x) = f'\big(g(d)\big) g^{\Delta}(x). \quad (23)$$

If g, h: $\mathbb{T} \to \mathbb{R}$ continuous real-valued functions, $a. b \in T, p > 1$ and $1/_p + 1/_q = 1$, then

$$\int_a^b g(x)h(x)dx \le \left(\int_a^b \big(g(x)\big)^p dx \right)^{\frac{1}{p}} \left(\int_a^b \big(h(x)\big)^q dx \right)^{\frac{1}{q}}. \quad (24)$$

Lemma 1. *(Fubini's Theorem, [24]) Let (Φ, N, πΔ) and (Ψ, M, γΔ) be measure spaces in time scales. If Λ: Φ × Ψ → ℝ is a* $\pi_\Delta \times \gamma_\Delta -$ *integrable function, then* $\varphi_1(x_2) = \int_\Phi \Lambda(x_1, x_2)\Delta x_1$ *exists for any* $x_1 \in \Psi$ *and* $\varphi_2(x_1) = \int_\Psi \Lambda(x_1, x_2)\Delta x_2$ *exists for* $x_2 \in \Phi$,

$$\int_\Phi \Delta x_1 \int_\Psi \Lambda(x_1, x_2)\Delta x_2 = \int_\Psi \Delta x_2 \int_\Phi \Lambda(x_1, x_2)\Delta x_1. \quad (25)$$

The following Lemma 2 and Lemma 3 express the new version of Jensen's inequality. For details, see [6].

Lemma 2. *Let α > 0 and a, b, j ∈* $\mathbb{T}$ *be such that* $0 \le a < b \le j$.

D1) If α > 1, then

$$\int_b^j (t-a)^{\propto-1}\Delta t \le \propto^{-1} [(j-a)^{\propto} - (b-a)^{\propto}] \le \int_b^j (\sigma(t)-a)^{\propto-1}\Delta t. \quad (26)$$

D2) If α < 1, then

$$\int_b^j (t-a)^{\propto-1}\Delta t \ge \propto^{-1} [(j-a)^{\propto} - (b-a)^{\propto}] \ge \int_b^j (\sigma(t)-a)^{\propto-1}\Delta t. \quad (27)$$

Lemma 3. *Let d ∈* N. *If* $0 \le x_k \le y_k$ *for k ∈ [1, d], then*

$$\prod_{k=1}^d (y_k - x_k) \le \prod_{k=1}^d y_k - \prod_{k=1}^d x_k. \quad (28)$$

Definition 1 (Jensen's inequality [23, Theorem 6.17]). Let $a, b \in \mathbb{T}$ with $a < b$, and suppose $I \subset \mathbb{R}$ is an interval. If $\Phi \in C(I, \mathbb{R})$ is convex and $f \in C_{rd}([a, b], I)$, then

$$\Phi = \left(\frac{\int_a^b f(t)\Delta t}{b-a} \right) \le \frac{\int_a^b \Phi\big(f(t)\big)\Delta t}{b-a}. \quad (29)$$

Moreover, M. Anwar et al. [28] demonstrated some results of Jensen's inequality for several variables.

Theorem 1. *[6]Let* (Φ, N, π_Δ) *and* (Ψ, M, γ_Δ) *be measure spaces in time scales. Assume that* $V \subset \mathbb{R}^d$ *is a closed convex set and* $\Omega \in C(V, \mathbb{R})$ *is convex. Furthermore, let* m: $\Phi \times \Psi \to R$ *be non-negative function such that* $m(x1, .)$ *is* γ_Δ*- integrable function. Then*

$$\Omega \left(\frac{\int_\Psi m(x_1, x_2)g(x_2)\Delta x_2}{\int_\Psi m(x_1, x_2)\Delta x_2} \right) \le \frac{\int_\Psi m(x_1, x_2)\Omega\big(g(x_2)\big)\Delta x_2}{\int_\Psi m(x_1, x_2)\Delta x_2} \quad (30)$$

holds for all functions g: $\Psi \to V$, *where* $g_k(x_2)$ *is* π_{Δ_2}*-integrable for all k ∈ {1, … , d} and*

$\int_\Psi m(x_1, x_2)g(x_2)\Delta x_2$ *denote the d-tuple*

$$\left(\int_\Psi m(x_1, x_2)g_1(x_2)\Delta x_2, \dots, \int_\Psi m(x_1, x_2)g_d(x_2)\Delta x_2 \right). \quad (31)$$

Subsequently, we use the following notations.

(K1) $\Phi = \Psi = [a, J) = [a_1, j_1) \cap \mathbb{T} \times \dots \times [a_d, j_d) \cap \mathbb{T}$, where $a_k < j_k$ for $a_k, j_k \in [0, \infty]$.

(K2) $a < b$ if componentwise $a_k < b_k$, $k \in \{1, \dots, d\}$.

(K3) m: $[a, J) \times [a, J) \to R_+$ is such that

$$m(x_1, x_2) = \begin{cases} 1 & \text{if } a \le x_2 < \sigma(x_1) \le J, \\ 0 & \text{otherwise,} \end{cases}$$

that is

$$m(x_{11}, \dots, x_{1d}, x_{21}, \dots, x_{2d}) = \begin{cases} 1 & \text{if } a_k \le x_{2k} < \sigma(x_k) \le J_k, \quad k = 1, \dots, d \\ 0 & \text{otherwise,} \end{cases}$$

where $x_1 = x_{11}, \dots, x_{1d}$ and $x_2 = x_{21}, \dots, x_{2d}$.

(K4) $\Omega(f) = f^q$, $q > 1$.

Remark 1. For $d = 1$, Theorem 1 yields the inequality

$$\left(\frac{1}{\prod_{k=1}^d (\sigma(x_k) - a_k)} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g(x_{21}, \dots, x_{2d})\Delta x_{21}, \dots, \Delta x_{2d} \right)^q$$
$$\le \frac{1}{\prod_{k=1}^d (\sigma(x_k) - a_k)} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g^q(x_{21}, \dots, x_{2d})\Delta x_{21}, \dots, \Delta x_{2d}. \quad (32)$$

NON-LINEAR MULTIVARIATE FRACTIONAL HARDY-TYPE INTEGRAL INEQUALITIES ON TIME SCALES

In this section, we will state and prove our main theorems.

Theorem 2. *Let two functions* $q, \propto: [a, b] \cap \mathbb{T} \to \mathrm{R}$ *be defined by*

$$q(x) = \begin{cases} q_0, & x \in [0, b], \\ q_1, & x > b, \end{cases} \quad and \quad \propto(x) = \begin{cases} \propto_0, & x \in [0, b], \\ \propto_1, & x > b, \end{cases} \tag{33}$$

where $0 \leq a < b < \infty$ *and let fractional Hardy-type integral operator* $H_\beta g(t) = \frac{1}{t^{1-\beta}} \int_0^t g(s) ds$ *for* $\beta \in [0,1)$. *Furthermore, suppose that* $0 \neq q_0$, $q_1 \in \mathrm{R}$ *are such that* $q_0, q_1 < 0$ *or* $q_0 < 0$, $q_1 \geq 1$ *or* $q_0 \geq 1$, $q_1 < 0$ *or* $q_0, q_1 \geq 1$. *If* $g: [a, \varepsilon] \to \mathrm{R}$ *is non-negative delta (Δ, Hilger) integrable and* $g \in C_{rd}([a, \varepsilon], \mathbb{R}$ *for which*

$$\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q(x)}(y_1, \dots, y_d) \prod_{i=1}^{d} \frac{1}{\propto(x)(y_i - a_i)^{\propto(x)}} \left[1 - \prod_{i=1}^{d} \left(\frac{\varepsilon_i - a_i}{y_i - a_i}\right)^{-\propto(x)}\right] \Delta y_1 \dots \Delta y_d < \infty, \tag{34}$$

then

$$\begin{aligned} &\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right)^{q(x)} \\ &\qquad \times \prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-\propto(x)} \Delta x_1 \dots \Delta x_d \\ &\leq \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q(x)}(y_1, \dots, y_d) \prod_{i=1}^{d} \frac{1}{\propto(x)(y_i - a_i)^{\propto(x)}} \\ &\qquad \times \left[1 - \prod_{i=1}^{d} \left(\frac{\varepsilon_i - a_i}{y_i - a_i}\right)^{-\propto(x)}\right] \Delta y_1 \dots \Delta y_d + J_0, \end{aligned} \tag{35}$$

where $J_0 = 0$ *if* $b \geq \varepsilon$ *(so that* $\propto(x) = \propto_0, q(x) = q_0$*) and*

$$\begin{aligned} J_0 = &\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_1}(y_1, \dots, y_d) \prod_{i=1}^{d} \frac{1}{\propto_1 [(y_i - a_i)^{\propto_1} - (\varepsilon_i - a_i)^{\propto_1}]} \Delta y_1 \dots \Delta y_d \\ -&\prod_{i=0}^{d} (b_i)^{\beta-1} \int_{a_0}^{b_0} \dots \int_{a_d}^{b_d} g^{q_0}(y_1, \dots, y_d) \prod_{i=1}^{d} \frac{1}{\propto_0 [(y_i - a_i)^{\propto_0} - (\varepsilon_i - a_i)^{\propto_0}]} \Delta y_1 \dots \Delta y_d. \end{aligned} \tag{36}$$

If $q(x) \in (0,1]$, *then (35) holds in the reverse direction.*

Proof. **Case 1.** Let $b \geq \varepsilon$. If we apply Jensen's inequality with Lemma 1 and Lemma 2, then we have

$$\begin{aligned} &\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right)^{q(x)} \\ &\qquad \times \prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-\propto(x)} \Delta x_1 \dots \Delta x_d \\ &\leq \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} \left[\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g^{q_0}(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right] \\ &\qquad \times \prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-\propto_0} \Delta x_1 \dots \Delta x_d \\ &\leq \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} g^{q_0}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} \prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-\propto_0 - 1} \Delta x_1 \dots \Delta x_d \right] \Delta y_1 \dots \Delta y_d \\ &\leq \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} g^{q_0}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} \frac{1}{\propto(x)(y_i - a_i)^{\propto(x)}} \right] \left[1 - \prod_{i=1}^{d} \left(\frac{\varepsilon_i - a_i}{y_i - a_i}\right)^{-\propto(x)}\right] \Delta y_1 \dots \Delta y_d. \end{aligned} \tag{37}$$

Herewith, (35) is proved.

Case 2. Let $b \leq \varepsilon$. If we apply Jensen's inequality with Lemma 1, then we have

$$\begin{aligned} &\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{\varepsilon_1} \dots \int_{a_d}^{\varepsilon_d} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right)^{q(x)} \\ &\qquad \times \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto(x)}} \Delta x_1 \dots \Delta x_d \\ &\leq \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_d}^{\sigma(x_d)} g(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right)^{q_0} \\ &\qquad \times \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto_0}} \Delta x_1 \dots \Delta x_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{b_1}^{\varepsilon_1} \dots \int_{b_d}^{\varepsilon_d} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right)^{q_1} \\ &\qquad \times \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto_1}} \Delta x_1 \dots \Delta x_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{b_1}^{\varepsilon_1} \dots \int_{b_d}^{\varepsilon_d} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i)^{-1} \int_{b_1}^{\sigma(x_1)} \dots \int_{b_d}^{\sigma(x_d)} g(y_1, \dots, y_d) \Delta y_1 \dots \Delta y_d \right)^{q_1} \\ &\qquad \times \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto_1}} \Delta x_1 \dots \Delta x_d \\ &\leq \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_0}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{y_1}^{b_1} \dots \int_{y_d}^{b_d} \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto_0}} \Delta x_1 \dots \Delta x_d \right] \Delta y_1 \dots \Delta y_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_1}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{b_1}^{\varepsilon_1} \dots \int_{b_d}^{\varepsilon_d} \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto_0}} \Delta x_1 \dots \Delta x_d \right] \Delta y_1 \dots \Delta y_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{b_1}^{\varepsilon_1} \dots \int_{b_d}^{\varepsilon_d} g^{q_1}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{y_1}^{\varepsilon_1} \dots \int_{y_d}^{\varepsilon_d} \prod_{i=1}^{d} \frac{1}{(\sigma(x_i) - a_i)^{\propto_0}} \Delta x_1 \dots \Delta x_d \right] \Delta y_1 \dots \Delta y_d \\ &= J. \end{aligned} \tag{38}$$

If we use Lemma 2 and Lemma 3, then we obtain

$$\begin{aligned} J \leq &\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_0}(y_1, \dots, y_d) \prod_{i=1}^{d} \left[\frac{((y_i - a_i)^{-\propto_0} - (b_i - a_i)^{-\propto_0})}{\propto_0} \right] \Delta y_1 \dots \Delta y_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_1}(y_1, \dots, y_d) \prod_{i=1}^{d} \left[\frac{((b_i - a_i)^{-\propto_1} - (\varepsilon_i - a_i)^{-\propto_1})}{\propto_1} \right] \Delta y_1 \dots \Delta y_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{b_1}^{\varepsilon_1} \dots \int_{b_d}^{\varepsilon_d} g^{q_1}(y_1, \dots, y_d) \prod_{i=1}^{d} \left[\frac{((y_i - a_i)^{-\propto_1} - (\varepsilon_i - a_i)^{-\propto_1})}{\propto_0} \right] \Delta y_1 \dots \Delta y_d \\ \leq &\prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_0}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} \frac{((y_i - a_i)^{-\propto_0})}{\propto_0} \right] \left[1 - \prod_{i=1}^{d} \left(\frac{\varepsilon_i - a_i}{y_i - a_i}\right)^{-\propto_0}\right] \Delta y_1 \dots \Delta y_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{b_1}^{\varepsilon_1} \dots \int_{b_d}^{\varepsilon_d} g^{q_1}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} \frac{((y_i - a_i)^{-\propto_1})}{\propto_1} \right] \left[1 - \prod_{i=1}^{d} \left(\frac{\varepsilon_i - a_i}{y_i - a_i}\right)^{-\propto_1}\right] \Delta y_1 \dots \Delta y_d \\ &+ \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q_1}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} \frac{((y_i - a_i)^{-\propto_1} - (\varepsilon_i - a_i)^{-\propto_1})}{\propto_1} \right] \Delta y_1 \dots \Delta y_d \\ &- \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_0} \dots \int_{a_d}^{b_d} g^{q_0}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} \frac{((y_i - a_i)^{-\propto_1} - (\varepsilon_i - a_i)^{-\propto_0})}{\propto_0} \right] \Delta y_1 \dots \Delta y_d \\ &= \prod_{i=1}^{d} (b_i)^{\beta-1} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g^{q(x)}(y_1, \dots, y_d) \left[\prod_{i=1}^{d} \frac{((y_i - a_i)^{-\propto(x)})}{\propto_0} \right] \\ &\qquad \times \left[1 - \prod_{i=1}^{d} \left(\frac{\varepsilon_i - a_i}{y_i - a_i}\right)^{-\propto(x)}\right] \Delta y_1 \dots \Delta y_d + J_0. \end{aligned} \tag{39}$$

Herewith, if we combine the two cases, then we complete the proof of Theorem 2.

Remark 2. Let two functions $q, \propto: [a, b] \cap \mathbb{T} \to \mathbb{R}$ be defined by

$$q(x) = \begin{cases} q_0, & x \in [0, b], \\ q_1, & x > b, \end{cases} \quad \text{and} \quad \propto(x) = \begin{cases} \propto_0, & x \in [0, b], \\ \propto_1, & x > b, \end{cases}$$

and $\beta = 1$ where $0 \le a < b < \infty$, then inequality (35) in Theorem 2 is reduced to the multivariate Hardy-type integral inequality.

Example 1. Let two functions $q, \propto : [0,1] \cap \mathbb{T} \to \mathbb{R}$ be defined by

$$q(x) = \begin{cases} q_0, & x \in [0,1], \\ q_1, & x > 1, \end{cases} \quad \text{and} \quad \propto(x) = \begin{cases} \propto_0, & x \in [0,1], \\ \propto_1, & x > 1. \end{cases}$$

If we take $d = 1$, $\beta = 1$, then inequality (11) in Theorem 2 is reduced to Hardy-type integral inequality.

$$\int_0^{\varepsilon} \left(\frac{1}{\sigma(x)} \int_a^{\sigma(x)} g(y)\Delta y \right)^{q(x)} (\sigma(x))^{-\propto(x)} \Delta x \le \int_0^1 g^{q(x)}(y) \frac{\varepsilon^{\propto(x)} - y^{\propto(x)}}{\propto(x)(\varepsilon y)^{\propto(x)}} \Delta y + J_0 \quad (40)$$

where, $J_0 = 0$ if $1 \ge \varepsilon$ and

$$J_0 = \int_0^1 \frac{g^{q_1}(y)}{\propto_1 [y^{\propto_1} - \varepsilon^{\propto_1}]} \Delta y - \int_0^1 \frac{g^{q_0}(y)}{\propto_0 [y^{\propto_0} - \varepsilon^{\propto_0}]} \Delta y.$$

Our next theorem deals with the adjoint of the fractional Hardy-type integral operator $\widetilde{H}_\beta$.

Theorem 3. *Let two functions* $q, \propto : [a, b] \cap \mathbb{T} \to \mathbb{R}$ *be defined by*

$$q(x) = \begin{cases} q_0, & x \in [0,b], \\ q_1, & x > b, \end{cases} \quad \text{and} \quad \propto(x) = \begin{cases} \propto_0, & x \in [0,b], \\ \propto_1, & x > b, \end{cases} \quad (41)$$

where $0 \le a < b < \infty$ *and let fractional adjoint Hardy-type integral operator* $\widetilde{H}_\beta g(t) = \int_t^\infty \frac{1}{t^{1-\beta}} g(s) ds$ *for* $\beta \in [0,1)$. *Furthermore, suppose that* $0 \ne q_0, q_1 \in \mathbb{R}$ *are such that* $q_0, q_1 < 0$ *or* $q_0 < 0$, $q_1 \ge 1$ *or* $q_0 \ge 1$, $q_1 < 0$ *or* $q_0, q_1 \ge 1$. *If* $g: [a, \varepsilon] \to \mathbb{R}$ *is non-negative delta* (Δ, *Hilger*) *integrable and* $g \in C_{rd}([a, \varepsilon], \mathbb{R}$ *for which*

$$\int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=1}^{d} (b_i)^{\beta-1} g^{q_1}(y_1, \ldots, y_d) \prod_{i=1}^{d} \frac{1}{\propto(y)(y_i - a_i)^{-\propto(y)}} \times \left[\frac{1 - \prod_{i=1}^{d} \left(\frac{y_i - a_i}{\varepsilon_i - a_i} \right)^{-\propto(y)}}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \right] \Delta y_1 \ldots \Delta y_d < \infty, \quad (42)$$

Then

$$\begin{aligned} &\int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=1}^{d} (b_i)^{\beta-1} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i) \int_{\sigma(x_1)}^{\infty} \cdots \int_{\sigma(x_d)}^{\infty} \left(\frac{g(y_1, \ldots, y_d)}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \right) \Delta y_1 \ldots \Delta y_d \right)^{q(x)} \\ &\times \left(\prod_{i=1}^{d} (x_i - a_i) \right)^{\propto(x)-1} \frac{1}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)} \Delta y_1 \ldots \Delta y_d \\ &\le \int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=1}^{d} (b_i)^{\beta-1} g^{q_1}(y_1, \ldots, y_d) \left(\prod_{i=1}^{d} \frac{1}{\propto(y)(y_i - a_i)^{-\propto(y)}} \right) \\ &\times \left[\frac{1 - \prod_{i=1}^{d} \left(\frac{y_i - a_i}{\varepsilon_i - a_i} \right)^{-\propto(y)}}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \right] \Delta y_1 \ldots \Delta y_d + J_0. \end{aligned} \quad (43)$$

where $J_0 = 0$ if $b \ge \varepsilon$ (*so that* $\propto(x) = \propto_0, q(x) = q_0$) *and*

$$\begin{aligned} J_0 = &\int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=1}^{d} (b_i)^{\beta-1} \frac{g^{q_1}(y_1, \ldots, y_d)}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \prod_{i=1}^{d} \left[\frac{(y_i - a_i)^{\propto_1} - (\varepsilon_i - a_i)^{\propto_1}}{\propto_1} \right] \Delta y_1 \ldots \Delta y_d \\ &- \int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=0}^{d} (b_i)^{\beta-1} \frac{g^{q_0}(y_1, \ldots, y_d)}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \prod_{i=1}^{d} \left[\frac{(y_i - a_i)^{\propto_0} - (\varepsilon_i - a_i)^{\propto_0}}{\propto_0} \right] \Delta y_1 \ldots \Delta y_d. \end{aligned} \quad (44)$$

If $q(x) \in (0,1]$, *then (43) holds in the reverse direction.*

Proof. **Case 1.** Let $b \ge \varepsilon$. If we apply Jensen's inequality with Lemma 1, then we find that

$$\begin{aligned} &\int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=1}^{d} (b_i)^{\beta-1} \left(\prod_{i=1}^{d} (\sigma(x_i) - a_i) \int_{\sigma(x_1)}^{\infty} \cdots \int_{\sigma(x_d)}^{\infty} \left(\frac{g(y_1, \ldots, y_d)}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \right) \Delta y_1 \ldots \Delta y_d \right)^{q(x)} \\ &\times \frac{\left(\prod_{i=1}^{d} (x_i - a_i) \right)^{\propto(x)-1}}{\prod_{i=1}^{d} (\sigma(x_i) - a_i)} \Delta x_1 \ldots \Delta x_d \\ &\le \int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=0}^{d} (b_i)^{\beta-1} \int_{\sigma(x_1)}^{\infty} \cdots \int_{\sigma(x_d)}^{\infty} \left(\frac{g^{q_1}(y_1, \ldots, y_d)}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \Delta y_1 \ldots \Delta y_d \right) \left(\prod_{i=1}^{d} (x_i - a_i) \right)^{\propto_1 - 1} \Delta x_1 \ldots \Delta x_d \\ &\le \int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=0}^{d} (b_i)^{\beta-1} \frac{g^{q_1}(y_1, \ldots, y_d)}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \left[\int_{\varepsilon_1}^{y_1} \cdots \int_{\varepsilon_d}^{y_d} \left(\prod_{i=1}^{d} (x_i - a_i) \right)^{\propto_1 - 1} \Delta x_1 \ldots \Delta x_d \right] \Delta y_1 \ldots \Delta y_d \\ &\le \int_{\varepsilon_1}^{\infty} \cdots \int_{\varepsilon_d}^{\infty} \prod_{i=0}^{d} (b_i)^{\beta-1} g^{q_1}(y_1, \ldots, y_d) \left[\prod_{i=1}^{d} \frac{1}{\propto(y)((y_i - a_i))^{-\propto(y)}} \right] \left[1 - \prod_{i=1}^{d} \left(\frac{y_i - a_i}{\varepsilon_i - a_i} \right)^{-\propto(y)} \right] \\ &\times \frac{1}{\prod_{i=1}^{d} (\sigma(y_i) - a_i)(y_i - a_i)} \Delta y_1 \ldots \Delta y_d. \end{aligned} \quad (45)$$

Case 2. Assume that $b \le \varepsilon$. The proof of this case is completely similar to the proof of the second case of Theorem 2. The reader can easily show it. Hereby, if we combine the two cases, then we complete the proof of Theorem 3.

Remark 3. Let two functions $q, \propto : [a, b] \cap \mathbb{T} \to \mathbb{R}$ be defined by

$$q(x) = \begin{cases} q_0, & x \in [0,b], \\ q_1, & x > b, \end{cases} \quad \text{and} \quad \propto(x) = \begin{cases} \propto_0, & x \in [0,b], \\ \propto_1, & x > b, \end{cases}$$

and $\beta = 1$ where $0 \le a < b < \infty$, then inequality (43) in Theorem 3 is reduced to the adjoint multivariate Hardy-type integral inequality.

Example 2. Let two functions $q, \propto : [0,1] \cap \mathbb{T} \to \mathbb{R}$ be defined by

$$q(x) = \begin{cases} q_0, & x \in [0,1], \\ q_1, & x > 1, \end{cases} \quad \text{and} \quad \propto(x) = \begin{cases} \propto_0, & x \in [0,1], \\ \propto_1, & x > 1. \end{cases}$$

If we take $d = 1$, $\beta = 1$, then inequality (43) in Theorem 3 is reduced to the adjoint Hardy-type integral inequality.

$$\int_{\varepsilon}^{\infty} \left((\sigma(x)) \int_{\sigma(x)}^{\infty} \left(\frac{g(y)}{y(\sigma(y))} \right) \Delta y \right)^{q(x)} \frac{x^{\propto(x)-1}}{\sigma(x)} \Delta x \le \int_{\varepsilon}^{\infty} g^{q_1}(y) \propto(y) \frac{y^{\propto(y)} - \varepsilon^{\propto(y)}}{y\sigma(y)} \Delta y + J_0 \quad (46)$$

where $J_0 = 0$ if $1 \ge \varepsilon$ *and*

$$J_0 = \int_{\varepsilon}^{\infty} \frac{g^{q_1}(y)}{y\sigma(y)} \left[\frac{y^{\propto_1} - \varepsilon^{\propto_1}}{\propto_1} \right] \Delta y - \int_{\varepsilon}^{\infty} \frac{g^{q_0}(y)}{y\sigma(y)} \left[\frac{y^{\propto_0} - \varepsilon^{\propto_0}}{\propto_0} \right] \Delta y. \quad (47)$$

CONCLUSION

Inequalities and dynamic equations are one of the most studied topics by scientists in almost every discipline. Because they shed light on the solution of many problems in science branches other than mathematics. For example; quantum mechanics, physical problems, wave equations, heat transfer, optical problems and economic problems [29-40]. That is, they have multidisciplinary features. However, the most study area of integral inequalities and dynamic equations is undoubtedly mathematics. Many properties of fractional integral inequalities have been demonstrated by mathematicians. For more detailed information, we refer to the references. In this article, we obtained the multivariate fractional Hardy-type integral inequality using the new version Jensen's inequality in Lemma 2 and Lemma 3. In this study, we used the multi-variable Cartesian version. We are currently working on a multi-variable polar version. This method used can also be applied to different operators and inequalities.

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Authors equally contributed to this work.

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The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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