Turk. J. Math. Comput. Sci.
16(1)(2024) 1–5
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DOI : 10.47000/tjmcs.1403706

Differential Equations of Spacelike Parametrized Curves in the Lorentz Plane

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Received: 12-12-2023 • Accepted: 14-05-2024

Abstract. We introduce four ordinary differential equations for a fixed natural parametrization of a spacelike curve C in the Lorentz plane. The relationships between these differential equations is studied through the curvature of C.

2020 AMS Classification: 53A04, 53A45, 53A55

Keywords: Lorentz plane, spacelike curve, curvature, differential equations.

Introduction

The interplay between differential geometry and differential equations is very natural and well-known. The purpose of this short note is to revisit this very interesting subject in the classical setting of smooth curves. More precisely, our framework is that of the Lorentz (or Minkowski) plane and the considered curve is a spacelike one. A natural motivation of our choice is the potential application of this type of curves in physical theories.

An important aspect of our study is that the given curve is naturally parametrized. With these considerations the components x, y of our curve satisfy two ordinary differential equations, denoted $\mathcal{E}^2$ and $\mathcal{E}^3$ respectively. Then, our concrete problem is to discuss the relationship between $\mathcal{E}^2$ and $\mathcal{E}^3$ into the theory of differential operators. Hence, four natural problems arise and we treat in detail all these questions.

1. From One Curve to Four Differential Equations

The setting of this paper is the Lorentz plane $\mathbb{L}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$:

$$\langle u, v\rangle_L = -u^1v^1 + u^2v^2, \quad u = (u^1, u^2) \in \mathbb{R}^2, \quad v = (v^1, v^2) \in \mathbb{R}^2, 0 \leq \|u\|_L^2 = |\langle u, u\rangle_L|.$$

Fix an open interval $I \subseteq \mathbb{R}$ and consider $C \subset \mathbb{R}^2$ a smmoth spacelike parametrized curve of equation:

$$C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \bar{i} = (1, 0), \bar{j} = (0, 1), \langle r'(t), r'(t)\rangle_L > 0, \quad t \in I.$$

The appropriate algebraic structure of the Lorentz plane is the two-dimensional paracomplex algebra $(\mathbb{R}^2, j)$, $j^2 = 1$, [4]. So, the Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|_L}, N(t) = j \cdot T(t) = \frac{1}{\|r'(t)\|_L}(y'(t), x'(t)), \langle T(t), T(t)\rangle_L = 1 = -\langle N(t), N(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'(t)\|_L}\langle T'(t), N(t)\rangle_L = \frac{1}{\|r'(t)\|_L^3}\langle r''(t), jr'(t)\rangle_L = \frac{1}{\|r'(t)\|_L^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases}$$

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Hence, T is an unit spacelike vector field along C while N is an unit timelike vector field along C. The curvature function can be expressed using a 2×2 determinant:

$$k_L(t) = \frac{1}{\|r'(t)\|_L^3}\det\begin{pmatrix} x'(t) & y'(t) \\ x''(t) & y''(t)\end{pmatrix} \tag{1.1}$$

and the difference to the Euclidean curvature consists in the ratio in front of this determinant; in the Euclidean case is the Euclidean norm $\|r'(t)\|^{-3}$. The Lorentz rotated curve $jC : r_j(t) := j\cdot r(t) = (y(t), x(t))$ is a timelike curve since $< r'_j(t), r'_j(t) >_L = - < r'(t), r'(t) >_L$.

The key observation concerning the formula (1.1) is that the given determinant is exactly the Wronskian of the smooth functions (x', y') and we denote it as $W(r')$. Our main assumptions from now are:
H1) the Wronskian $W(r)$ is non-zero; hence r is not a line through the origin O of $\mathbb{R}^2$,
H2) the Wronskian $W(r')$ is non-zero; hence k_L is not zero and C is a not a general line,
H3) r is a natural parametrization of C, i.e. $\|r'(t)\|_L = +1$,
and then a similar result to the fundamental theorem of Euclidean plane curves gives the expression of the derivative:

$$r'(t) = (\sinh(-K_L(t)), \cosh(-K_L(t))), \quad K_L(t) := \int_{t_0}^{t} k_L(s)ds. \tag{1.2}$$

This short note introduces four linear ordinary differential equations for C. A main motivation is that the curves in the Lorentzian plane know a recent increasing interest as the papers [2, 3, 8] show. The first differential equation is provided by the following result:

Proposition 1.1. *Under the hypothesis H1-H3 the components functions x, y of r satisfy the third-order linear ordinary differential equation:*

$$\mathcal{E}^3 : U''' - \frac{k'_L}{k_L}U'' - k_L^2 U' = 0. \tag{1.3}$$

Proof. The two derivatives in (1.2) gives:

$$\begin{cases} r''(t) = (-k_L(t)\cosh(-K_L(t)), -k_L(t)\sinh(-K_L(t))), \\ r'''(t) = \left(-k'_L\cosh(-K_L(t)) + k_L^2\sinh(-K_L(t)), -k'_L\sinh(-K_L(t)) + k_L^2\cosh(-K_L(t))\right), \end{cases}$$

and these relations yield the conclusion. □

We point out that the components functions x, y are also solutions of the Wronskian linear differential equation:

$$\begin{cases} W(x, y, u = u(\cdot)) := \begin{vmatrix} x & y & u \\ x' & y' & u' \\ x'' & y'' & u'' \end{vmatrix} = 0 \to \mathcal{E}^2 : u''(t) + p(t)u'(t) + q(t)u(t) = 0, \\ p := -\frac{[W(r)]'}{W(r)}, \quad q := \frac{W(r')}{W(r)} = \frac{k_L}{W(r)}, \quad \mathcal{E}^2 : \frac{d}{dt}\left(\frac{u'}{W(r)}\right) + \frac{W(r')u}{(W(r))^2} = 0. \end{cases} \tag{1.4}$$

It is well-known that the general solution of (1.4) is provided by two real constants C_1, C_2 through the formula:

$$u(t) = C_1 x(t) + C_2 y(t), \quad C_1 = \frac{W(u, y)}{W(r)}, \quad C_2 = \frac{W(x, u)}{W(r)}.$$

The second order ordinary differential equation (SODE) $\mathcal{E}^2$ expresses u as belonging to the kernel of the differential operator:

$$\mathcal{D} := \frac{d^2}{dt^2} + p\frac{d}{dt} + q = \sum_{i=0}^{2}\mu_i\frac{d^i}{dt^i} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$

and we recall that any n-differential operator $\mathcal{D} := \sum_{i=0}^{n}\mu_i\frac{d^i}{dt^i}$ has an *adjoint operator* ([7, p. 218]):

$$\mathcal{D}_a := \sum_{i=0}^{n}(-1)^i\frac{d^i}{dt^i}(\mu_i\cdot).$$

Our aim is to study the corresponding four transformations:
Problem 1: when $d\mathcal{E}^2 = \mathcal{E}^3$?
Problem 2: when $d\mathcal{E}^2 = \mathcal{E}^3_a$?

Problem 3: when $d\mathcal{E}^2_a = \mathcal{E}^3$?
Problem 4: when $d\mathcal{E}^2_a = \mathcal{E}^3_a$?
The Euclidean variant of the first problem was studied in [5]. In the Euclidean plane the equation (1.4) is the same while in the equation (1.3) the coefficient of U is $+k^2_E$, i.e. the square of the Euclidean curvature of Euclidean naturally parametrized C.

2. The Problem 1

Comparing $d\mathcal{E}^2 : u''' + pu'' + (p' + q)u' + q'u = 0$ with (1.3) it results the system:

$$p = -\frac{k'_L}{k_L}, \quad p' + q = -k^2_L, \quad q = constant.$$

The first equation gives:

$$W(r) = C_1 k_L, \quad C_1 \in \mathbb{R}^*$$

while the following two equations yield:

$$\left(\frac{k'_L}{k_L}\right)' - k^2_L = constant = \frac{1}{C_1} \tag{2.1}$$

and we choose this last constant to be (-1). WolframAplha provides only an implicit solution:

$$C_2 + t = \int_1^{k_L} \frac{\pm du}{\sqrt{C_2 + u^2 - 2\ln u}}.$$

It remains an open problem to find the Lagrangian $L = L(t, y(t), y'(t))$ whose Euler-Lagrange equation is given by (2.1) i.e.:

$$\left(\frac{y'}{y}\right)' - y^2 = -1.$$

Example 2.1 In the paper [4] we have introduced a Lorentzian version of the Grim-Reaper curve as being the graph $r(t) = (t, -ln(\sinh t))$ for $t \in (0, +\infty)$. Its data is then:

$$\begin{cases} k_L(t) = \sinh t > 0, \quad k_E(t) = \frac{\sinh t}{(\cosh t)^{\frac{3}{2}}} > 0, \\ W(r)(t) = \ln(\sinh t) - t\tanh t, \quad W(r')(t) = \frac{1}{\sinh^2 t} > 0. \end{cases}$$

It is not naturally parametrized but the SODE (1.4) can be written explicitly. □

3. The Problem 2

The adjoint equation of $\mathcal{E}^3$ is:

$$\mathcal{E}^3_a : U'''_a + \frac{k'_L}{k_L}U''_a + \left[2\left(\frac{k'_L}{k_L}\right)' - k^2_L\right]U'_a + \left[\left(\frac{k'_L}{k_L}\right)'' - 2k'_L k_l\right]U_a = 0 \tag{3.1}$$

and a comparison with $d\mathcal{E}^2$ gives:

$$\begin{cases} p = \frac{k'_L}{k_L} \to W(r) = \frac{C_1}{k_L}, \quad C_2 \in \mathbb{R}^* \\ q = \left(\frac{k'_L}{k_L}\right)' - k^2_L = \frac{k^2_L}{C_1}. \end{cases} \tag{3.2}$$

Again we choose $C_1 = -1$ and the second relation (3.2) gives:

$$k_L(t) = \exp(C_2 t), \quad C_2 \in \mathbb{R}^* \to p = C_2 \neq 0. \tag{3.3}$$

In conclusion, the searched curve is:

$$r_{C_2}(t) = \left(\int_{t_0}^t \sinh\left(-\frac{\exp(C_2 s)}{C_2}\right)ds, \int_{t_0}^t \cosh\left(-\frac{\exp(C_2 s)}{C_2}\right)ds\right).$$

4. The Problem 3

We have $\mathcal{E}^2_a : u''_a - pu'_a + (q - p')u = 0$ and then:

$$d\mathcal{E}^2_a : u'''_a - pu''_a + (q - 2p')u'_a + (q' - p'')u = 0. \tag{4.1}$$

Comparing with (1.3) we obtain:

$$\begin{cases} p = \frac{k'_L}{k_L} \to W(r) = \frac{C_1}{k_L}, \quad C_2 \in \mathbb{R}^* \\ q = 2\left(\frac{k'_L}{k_L}\right)' - k_L^2 = \frac{k_L^2}{C_1}. \end{cases}$$

The same option $C_1 = -1$ yields again the curvature (3.3).

At the end of this section, we comment about these first three problems. We remark that in the first one the coefficient q of $\mathcal{E}^2$ is constant while p is variable while in the next two problems the coefficient p is constant and q is variable. The constancy of both coefficients is concerned with the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{E}^3 & \stackrel{a}{\to} & \mathcal{E}^3_a \\ \uparrow d & ? & \uparrow d \\ \mathcal{E}^2 & \stackrel{a}{\to} & \mathcal{E}^2_a \end{array}.$$

More precisely, we have the commutativity $d(\mathcal{E}^2_a) = -(d\mathcal{E}^2)_a$ if and only if p and q are simultaneous constant and this will be the case of the last problem.

5. The Problem 4

Comparing (3.1) and (4.1) gives:

$$p = -\frac{k'_L}{k_L} \to W(r) = C_1 k_L, \quad C_1 \in \mathbb{R}^*$$

as well as:

$$q = -k_L^2 = \frac{1}{C_1}$$

and hence, we choose $C_1 = -\frac{1}{C_2^2} < 0$. The final expression of the curvature is then:

$$k_L = C_2 \neq 0 \to p = 0.$$

The significance of the vanishing of p is that the differential operator $\mathcal{D}$ is a self-adjoint one: $\mathcal{D} = \mathcal{D}_a$.

Example 5.1 The corresponding of the circles $C(O, R > 0)$ of Euclidean plane geometry is provided by the equilateral hyperbola as integral curve of ξ_L:

$$\begin{cases} H_e(R) : x^2 - y^2 = R^2, \quad r_e(t) = R\left(\cosh \frac{t}{R}, \sinh \frac{t}{R}\right), \quad t \in \mathbb{R} \\ k_L = constant = -\frac{1}{R} < 0. \end{cases}$$

We note that, $H_e(R)$ is called *pseudo-circle* in [6, p. 110] and is denoted $H^1(-R)$. Recall that the infinitesimal generator of the Lorentz rotations in $\mathbb{R}^2_1$ is the linear vector field:

$$\xi_L(u) := u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2}, \quad \xi_L(u) = j \cdot u = j \cdot (u^1 + iu^2).$$

The Euclidean curvature of H_e is still negative but non-constant:

$$k_E(t) = -\frac{1}{R(\cosh(2t))^{\frac{3}{2}}} < 0.$$

Remark 5.2 This example suggests a suitable generalization for a semi-Riemannian 2-dimensional manifold (M^2, g) for which ∇ is the Levi-Civita connection. Fix $r : I \to M^2$ a smooth curve and let k be its geodesic curvature which we suppose to be non-zero i.e. r is not a geodesic of g. Then we call r a *pseudo-circle* on the Lorentzian surface (M^2, g) if it is a spacelike curve parametrized by the arc-length and the following differential equation holds:

$$\nabla_{r'}\nabla_{r'}r' - \frac{k'}{k}\nabla_{r'}r' - k^2 r' = 0. \tag{5.1}$$

The adaptation of the Nomizu-Yano notion of circle from the Riemannnian to semi-Riemannian geometry is considered in [1]. From (5.1)it follows immediately that $\nabla_{r'} r'$ is a timelike vector field along the curve with:

$$\langle \nabla_{r'} r', \nabla_{r'} r' \rangle_L = -k^2.$$

6. Conclusion and Recommendations

The classical theory of differential curves can be enriched by using new points of view. Our study proposes such a new interplay between geometry and differential equations in the framework of two-dimensional Lorentz geometry. We hope our ideas to be useful into some physical settings. A possible future research is the three-dimensional Lorentz geometry, with a natural distinction between spacelike and timelike curves.

Acknowledgements

I am grateful to three anonymous referees for some suggestions to an initial version of this work.

Funding

This research received no external funding.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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