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RESEARCH ARTICLE

# Twisted Surfaces in Semi-Euclidean 4-Space with Index 2

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

**ABSTRACT**

**In this paper, we consider the twisted surfaces in semi-Euclidean 4-space with index 2. We classify the twisted surface with respect to their spine curve which are non-null or null curves. So, we study the geometric properties of these surfaces. Also we obtain the family of some special surfaces such as flat surfaces, marginally trapped surfaces.**

***Keywords:*** *Twisted surface, normal curvature tensor, parallel mean curvature vector, minimal surface, flat surface.*

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## 1. Introduction

Twisted surfaces, a generalization of the process used to construct the Möbius strip and the twisted Klein bottle, were first introduced by A. Gray in Euclidean 3-space [7]. A twisted surface arises when a planar curve is rotated in its supporting plane while simultaneously this supporting plane rotates about a containing axis, possibly at a different rotation speed [3], [4]. Hence twisted surfaces represent generalizations of the surfaces of the revolution (or a rotational surface) and also can be called a double rotational surfaces.

In Minkowski 3-space, twisted surfaces were defined and studied by Goemans and Woestyne [3], [5]. Also they classified those with constant Gaussian curvature or constant mean curvature. They also showed that there exists no minimal (regular) twisted surface in Euclidean or in Minkowski 3-space when excluding the surfaces of revolution. In another study by the same authors, the curvature properties of twisted surfaces with null rotation axis in Minkowski 3-space were studied [6].

The notion of the second kind twisted surfaces in Minkowski 3-space was introduced by Grbović et al. [8]. They classified all non-degenerate second kind twisted surfaces in terms of flat, minimal, constant Gaussian and constant mean curvature surfaces, with respect to a chosen lightlike transversal bundle. They also proved that a lightlike second kind twisted surfaces, with respect to a chosen lightlike transversal vector bundle, are the lightcones, the lightlike binormal surfaces over pseudo null base curve and the lightlike ruled surfaces with null rulings whose base curve lies on lightcone.

Twisted surfaces in Galilean 3-space were described by Dede et al. in [1]. The existence of Euclidean rotations and isotropic rotations leads to the existence of three different types of bent surfaces in Galilean 3-space. Then they classified twisted surfaces in Galilean 3-space with zero Gaussian curvature or zero mean curvature.

Similar to the generalization of rotational surfaces and their related properties from 3-dimensional Euclidean space to 4-dimensional space[13], Twisted surfaces are defined in 4-dimensional Euclidean space and 4-dimensional Lorentz space and their related properties are studied in detail[6].

In this paper, we consider the twisted surfaces in semi-Euclidean 4-space with index 2. We classify the twisted surface with respect to their spine curve which are non-null or null curves. So, we study the geometric properties of these surfaces. Also we obtain the family of some special surfaces such as flat surfaces, marginally trapped surfaces.

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## 2. Preliminaries

Let $\mathbb{R}_2^n$ be the $n$-dimensional semi-Euclidean space of index 2 with inner product $\langle\ ,\ \rangle$ and flat connection $D$. Let $M$ be a semi-Riemannian submanifold in $\mathbb{R}_2^n$. According to decomposition

$$\mathbb{R}_2^n|_M = TM \perp TM^{\perp},$$

we have

$$D_X Y = \nabla_X Y + h(X, Y),$$

and

$$D_X \xi = -A_\xi X + {}^{\perp}\nabla_X\, \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$. Then $\nabla$ is the Levi-Civita connection of $M$, $h$ is the second fundamental form, $A_\xi$ is the shape operator, and ${}^{\perp}\nabla$ is the normal connection. We note that

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

The normal curvature tensor ${}^{\perp}R$ is defined by

$${}^{\perp}R\,(X, Y)\,\xi = {}^{\perp}\nabla_X\ {}^{\perp}\nabla_Y\ \xi - {}^{\perp}\nabla_Y\ {}^{\perp}\nabla_X\ \xi - {}^{\perp}\nabla_{[X,Y]}\xi\ ,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$. Taking the normal part of the following equation

$$D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi = 0$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$, we get the Ricci equation

$$\left\langle {}^{\perp}R\,(X, Y)\,\xi, \eta\right\rangle = \langle A_\eta X, A_\xi Y\rangle - \langle A_\xi X, A_\eta Y\rangle$$

where $\eta \in \Gamma(TM^{\perp})$.

Let $\mathbb{R}_2^4$ be the semi-Euclidean 4-space with index 2 with standard coordinate system $\{x_1, x_2, x_3, x_4\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2.$$

In the following, we assume that $M$ is a surface in $\mathbb{R}_2^4$. We use the following convention on the ranges of indices:

$$1 \le A, B \ldots \le 4, \quad 1 \le i, j \ldots \le 2, \quad 3 \le \alpha, \beta \ldots \le 4.$$

Let $\{e_i\}$ be a local orthonormal frame field on $M$ and $\{e_\alpha\}$ be a normal orthonormal frame field to $M$. Let $\varepsilon_A = \langle e_A, e_A\rangle = \pm 1$. Set

$$h_{ij}^{\alpha} = \varepsilon_\alpha \langle h(e_i, e_j), e_\alpha\rangle$$

and

$$R_{\beta ij}^{\alpha} = \varepsilon_\alpha \left\langle {}^{\perp}R\,(e_i, e_j)\, e_\beta, e_\alpha\right\rangle,$$

which are the components of the second fundamental form $h$ and the normal curvature tensor ${}^{\perp}R$, respectively.

By the Ricci equation, the normal curvature tensor satisfies

$$R_{\beta ij}^{\alpha} = \varepsilon_\alpha \left(\left\langle A_{e_\alpha} e_i, A_{e_\beta} e_j\right\rangle - \left\langle A_{e_\beta} e_i, A_{e_\alpha} e_j\right\rangle\right).$$

Noting that

$$A_{e_\alpha} e_i = \varepsilon_\alpha \sum_k \varepsilon_k h_{ik}^{\alpha} e_k,$$

we obtain

$$R_{\beta ij}^{\alpha} = \varepsilon_\beta \sum_k \varepsilon_k \left(h_{ik}^{\alpha} h_{jk}^{\beta} - h_{jk}^{\alpha} h_{ik}^{\beta}\right).$$

The mean curvature vector $H$ of $M$ is given by

$$H = \frac{1}{2} \sum_\alpha \left(\varepsilon_1 h_{11}^{\alpha} + \varepsilon_2 h_{22}^{\alpha}\right) e_\alpha.$$

A surface is called minimal if $H = 0$ identically. We say that a surface has parallel mean curvature vector if $^{\perp}\nabla H = 0$. A surface is called marginally trapped (quasi-minimal) if $H$ is lightlike at each point.
The Gauss curvature $K$ of $M$ in $\mathbb{R}^4_2$ is given by

$$K = \varepsilon_1\varepsilon_2\sum_{\alpha}\varepsilon_\alpha\left(h^\alpha_{11}h^\alpha_{22} - (h^\alpha_{12})^2\right). \tag{2.1}$$

A surface $M$ is called flat if $K = 0$ identically.

## 3. Twisted surface in $\mathbb{R}^4_2$

$$\begin{aligned} F(t,u) &= \begin{bmatrix} \cos u & -\sin u & 0 & 0 \\ \sin u & \cos u & 0 & 0 \\ 0 & 0 & \cos u & -\sin u \\ 0 & 0 & \sin u & \cos u \end{bmatrix}\begin{bmatrix} \cosh u & 0 & 0 & \sinh u \\ 0 & \cosh u & \sinh u & 0 \\ 0 & \sinh u & \cosh u & 0 \\ \sinh u & 0 & 0 & \cosh u \end{bmatrix}\begin{bmatrix} 0 \\ x_1(t) \\ x_2(t) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -x_1(t)\cosh u\sin u - x_2(t)\sinh u\sin u \\ x_1(t)\cosh u\cos u + x_2(t)\sinh u\cos u \\ x_1(t)\sinh u\cos u + x_2(t)\cosh u\sin u \\ x_1(t)\sinh u\sin u + x_2(t)\cosh u\sin u \end{bmatrix}. \end{aligned}$$

So we get

$$F_t = \begin{bmatrix} -x_1'(t)\cosh u\sin u - x_2'(t)\sinh u\sin u \\ x_1'(t)\cosh u\cos u + x_2'(t)\sinh u\cos u \\ x_1'(t)\sinh u\cos u + x_2'(t)\cosh u\cos u \\ x_1'(t)\sinh u\sin u + x_2'(t)\cosh u\sin u \end{bmatrix}$$

$$F_u = \begin{bmatrix} -(\cos u\cosh u + \sin u\sinh u)x_1(t) - (\cosh u\sin u + \cos u\sinh u)x_2(t) \\ (-\cosh u\sin u + \cos u\sinh u)x_1(t) + (\cos u\cosh u - \sin u\sinh u)x_2(t) \\ (\cos u\cosh u - \sin u\sinh u)x_1(t) + (-\cosh u\sin u + \cos u\sinh u)x_2(t) \\ (\cosh u\sin u + \cos u\sinh u)x_1(t) + (\cos u\cosh u + \sin u\sinh u)x_2(t) \end{bmatrix}$$

and

$$E = \langle F_t, F_t\rangle = x_1'^2 - x_2'^2, \qquad F = \langle F_t, F_u\rangle = x_2x_1' - x_1x_2', \qquad G = \langle F_u, F_u\rangle = 0.$$

### 3.1. *Twisted surface with non-null spine curve*

In this subsection, we assume that $x_1'^2 - x_2'^2 \neq 0$. So we construct the orthonormal basis as follows

$$e_1 = \frac{1}{\sqrt{m_1(x_1'^2 - x_2'^2)}}F_t = \frac{1}{\sqrt{m_1(x_1'^2 - x_2'^2)}}\left(-a_2\sin u, a_2\cos u, a_1\cos u, a_1\sin u\right),$$

$$\begin{aligned} e_2 &= \frac{m_1\sqrt{m_1(x_1'^2 - x_2'^2)}}{x_2x_1' - x_1x_2'}F_u - \frac{1}{\sqrt{m_1(x_1'^2 - x_2'^2)}}F_t \\ &= \left(\frac{-a_1(x_1x_1' - x_2x_2')\sin u - b_2\left(x_1'^2 - x_2'^2\right)\cos u}{(x_2x_1' - x_1x_2')\sqrt{m_1(x_1'^2 - x_2'^2)}}, \frac{a_1(x_1x_1' - x_2x_2')\cos u - b_2\left(x_1'^2 - x_2'^2\right)\sin u}{(x_2x_1' - x_1x_2')\sqrt{m_1(x_1'^2 - x_2'^2)}}\right. \\ &\quad \left.\frac{a_2(x_1x_1' - x_2x_2')\cos u - b_1\left(x_1'^2 - x_2'^2\right)\sin u}{(x_2x_1' - x_1x_2')\sqrt{m_1(x_1'^2 - x_2'^2)}}, \frac{a_2(x_1x_1' - x_2x_2')\sin u + b_1\left(x_1'^2 - x_2'^2\right)\cos u}{(x_2x_1' - x_1x_2')\sqrt{m_1(x_1'^2 - x_2'^2)}}\right), \end{aligned}$$

$$\begin{aligned} e_3 &= \left(\frac{a_1\left(x_1^2 - x_2^2\right)\sin u + b_2(x_1x_1' - x_2x_2')\cos u}{\sqrt{m_2(x_1^2 - x_2^2)}(-x_2x_1' + x_1x_2')}, \frac{a_1\left(x_1^2 - x_2^2\right)\cos u - b_2(x_1x_1' - x_2x_2')\sin u}{\sqrt{m_2(x_1^2 - x_2^2)}(x_2x_1' - x_1x_2')}\right. \\ &\quad \left.\frac{a_2\left(x_1^2 - x_2^2\right)\cos u - b_1(x_1x_1' - x_2x_2')\sin u}{\sqrt{m_2(x_1^2 - x_2^2)}(x_2x_1' - x_1x_2')}, \frac{a_2\left(x_1^2 - x_2^2\right)\sin u + b_1(x_1x_1' - x_2x_2')\cos u}{\sqrt{m_2(x_1^2 - x_2^2)}(x_2x_1' - x_1x_2')}\right), \end{aligned}$$

$$e_4 = \left( -\frac{b_1 \cos u}{\sqrt{m_2\left(x_1^2 - x_2^2\right)}}, -\frac{b_1 \sin u}{\sqrt{m_2\left(x_1^2 - x_2^2\right)}}, -\frac{b_2 \sin u}{\sqrt{m_2\left(x_1^2 - x_2^2\right)}}, \frac{b_2 \cos u}{\sqrt{m_2\left(x_1^2 - x_2^2\right)}} \right)$$

where $m_1 = sign\left(x_1'^2 - x_2'^2\right)$, $m_2 = sign\left(x_1^2 - x_2^2\right)$ and

$$\begin{aligned} a_1 &= x_1'(t)\sinh u + x_2'(t)\cosh u, \\ a_2 &= x_1'(t)\cosh u + x_2'(t)\sinh u, \\ b_1 &= x_1(t)\sinh u + x_2(t)\cosh u, \\ b_2 &= x_1(t)\cosh u + x_2(t)\sinh u. \end{aligned}$$

Here $\{e_1, e_2\}$ is an orthonormal frame field on $F(t,u)$ of signature $(m_1, -m_1)$ and $\{e_3, e_4\}$ is a normal orthonormal frame field to $F(t,u)$ of signature $(+,-)$.

Since the results for the case $x_1^2(t) - x_2^2(t) < 0$ are similar to the results for the case $x_1^2(t) - x_2^2(t) > 0$, in this paper we consider $x_1^2(t) - x_2^2(t) > 0$. So we can choose

$$x_1(t) = P_1(t)\cosh P_2(t), \qquad x_2(t) = P_1(t)\sinh P_2(t)$$

where $P_1(t)$ and $P_2(t)$ are smooth functions. Here we denote the surface $M_1$ as

$$M_1: \qquad F(t,u) = \begin{bmatrix} -x_1(t)\cosh u \sin u - x_2(t)\sinh u \sin u \\ x_1(t)\cosh u \cos u + x_2(t)\sinh u \cos u \\ x_1(t)\sinh u \cos u + x_2(t)\cosh u \sin u \\ x_1(t)\sinh u \sin u + x_2(t)\cosh u \sin u \end{bmatrix} \tag{3.1}$$

where $x_1(t) = P_1(t)\cosh P_2(t)$ and $x_2(t) = P_1(t)\sinh P_2(t)$. Thus we can easily obtain that

$$D_{e_1}e_1 = \frac{-P_2'\left(-2P_1'^2 + P_1\left(P_1 P_2'^2 + P_1''\right)\right) + P_1 P_1' P_2''}{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)^2}\left(-A_1 \sin u, A_1 \cos u, A_2 \cos u, A_2 \sin u\right),$$

$$D_{e_2}e_1 = \frac{1}{m_1 P_1^2 P_2'\left(P_1'^2 - P_1^2 P_2'^2\right)^2}\begin{bmatrix} A_2\left(P_1'^2 - P_1^2 P_2'^2\right)^2 \cos u + A_1\left(P_1'^4 - P_1^3 P_2'^2 P_1''^3 P_1' P_2' P_2''\right)\sin u \\ A_2\left(P_1'^2 - P_1^2 P_2'^2\right)^2 \sin u - A_1\left(P_1'^4 - P_1^3 P_2'^2 P_1''^3 P_1' P_2' P_2''\right)\cos u \\ A_1\left(P_1'^2 - P_1^2 P_2'^2\right)^2 \sin u - A_2\left(P_1'^4 - P_1^3 P_2'^2 P_1''^3 P_1' P_2' P_2''\right)\cos u \\ -A_1\left(P_1'^2 - P_1^2 P_2'^2\right)^2 \cos u - A_2\left(P_1'^4 - P_1^3 P_2'^2 P_1''^3 P_1' P_2' P_2''\right)\sin u \end{bmatrix},$$

$$D_{e_2}e_2 = \frac{1}{m_1 P_1^3 P_2'^2\left(P_1'^2 - P_1^2 P_2'^2\right)^2}\begin{bmatrix} -\left(P_1'^2 - P_1^2 P_2'^2\right) D_1 \cos u + P_1^2\left(C_1 + P_1^2 P_1' P_2'^3 B_1 + C_3 + P_1'^4 B_3\right)\sin u \\ -\left(P_1'^2 - P_1^2 P_2'^2\right) D_1 \sin u - P_1^2\left(C_1 + P_1^2 P_1' P_2'^3 B_1 + C_3 + P_1'^4 B_3\right)\cos u \\ -\left(P_1'^2 - P_1^2 P_2'^2\right) D_2 \sin u + P_1^2\left(C_2 + P_1^2 P_1' P_2'^3 B_2 + C_4 + P_1'^4 B_4\right)\cos u \\ \left(P_1'^2 - P_1^2 P_2'^2\right) D_2 \cos u + P_1^2\left(C_2 + P_1^2 P_1' P_2'^3 B_2 + C_4 + P_1'^4 B_4\right)\sin u \end{bmatrix}$$

where

$$\begin{aligned} A_1 &= P_1' \sinh(u + P_2) + P_1 P_2' \cosh(u + P_2), \\ A_2 &= P_1' \cosh(u + P_2) + P_1 P_2' \sinh(u + P_2), \end{aligned}$$

$$\begin{aligned} B_1 &= 2P_1'' \sinh(u + P_2) - P_1 P_2'' \cosh(u + P_2), \\ B_2 &= -2P_1'' \cosh(u + P_2) + P_1 P_2'' \sinh(u + P_2), \\ B_3 &= -2P_2'^2 \cosh(u + P_2) + P_2'' \sinh(u + P_2), \\ B_4 &= 2P_2'^2 \sinh(u + P_2) - P_2'' \cosh(u + P_2), \end{aligned}$$

$$\begin{aligned} C_1 &= P_1^3 P_2'^4 \left(-P_1 P_2'^2 + P_1''\right) \cosh\left(u+P_2\right) - P_1'^3 P_2' \left(P_1 P_2'^2 + P_1''\right) \sinh\left(u+P_2\right), \\ C_2 &= P_1^3 P_2'^4 \left(P_1 P_2'^2 - P_1''\right) \sinh\left(u+P_2\right) + P_1'^3 P_2' \left(P_1 P_2'^2 + P_1''\right) \cosh\left(u+P_2\right), \\ C_3 &= 2P_1^2 P_1'^2 P_2'^2 \left(P_2'^2 \cosh\left(u+P_2\right) - P_2'' \sinh\left(u+P_2\right)\right), \\ C_4 &= 2P_1^2 P_1'^2 P_2'^2 \left(-P_2'^2 \sinh\left(u+P_2\right) + P_2'' \cosh\left(u+P_2\right)\right), \end{aligned}$$

$$\begin{aligned} D_1 &= P_1' \left(2P_1'^3 \sinh\left(u+P_2\right) + P_1^2 P_2' \left(-P_1 P_2'^2 + P_1''\right) \cosh\left(u+P_2\right) - P_1^2 P_1' \left(2P_2'^2 \sinh\left(u+P_2\right) + P_2'' \cosh\left(u+P_2\right)\right)\right), \\ D_2 &= P_1' \left(2P_1'^3 \cosh\left(u+P_2\right) + P_1^2 P_2' \left(-P_1 P_2'^2 + P_1''\right) \sinh\left(u+P_2\right) - P_1^2 P_1' \left(2P_2'^2 \cosh\left(u+P_2\right) + P_2'' \sinh\left(u+P_2\right)\right)\right), \end{aligned}$$

The components of the second fundamental form $h$ are given as follows

$$\begin{aligned} h_{11}^3 &= \langle D_{e_1} e_1, e_3 \rangle = \frac{-2P_1'^2 P_2' + P_1 P_2' \left(P_1 P_2'^2 + P_1''\right) - P_1 P_1' P_2''}{m_1 m_3 P_1 P_2' \left(-P_1'^2 + P_1^2 P_2'^2\right)}, \\ h_{12}^3 &= \langle D_{e_1} e_2, e_3 \rangle = \frac{P_1'^2 P_2' - P_1 P_2' P_1'' + P_1 P_1' P_2''}{m_1 m_3 P_1 P_2' \left(-P_1'^2 + P_1^2 P_2'^2\right)}, \\ h_{22}^3 &= \langle D_{e_2} e_2, e_3 \rangle = -\frac{P_1 \left(P_1 P_2'^3 - P_2' P_1'' + P_1' P_2''\right)}{m_1 m_3 P_1 P_2' \left(-P_1'^2 + P_1^2 P_2'^2\right)}, \\ h_{11}^4 &= -\langle D_{e_1} e_2, e_4 \rangle = 0, \\ h_{12}^4 &= -\langle D_{e_1} e_2, e_4 \rangle = -\frac{1}{m_1 m_3 P_1}, \\ h_{22}^4 &= -\langle D_{e_2} e_2, e_4 \rangle = \frac{2P_1'^2}{m_1 m_3 P_1^3 P_2'^2} \end{aligned}$$

where $m_3 = sign\left(P_1\right)$.

**Theorem 3.1.** *The Gaussian curvature of $M_1$ given by (3.1) is identically zero.*

*Proof.* By straight calculations in (2.1) , we obtain $K = 0$. □

So we have the following corollary.

**Corollary 3.1.** *The surface $M_1$ given by (3.1) is a flat surface.*

In the following theorem, we consider the normal curvature tensor of the twisted surface $M_1$.

**Theorem 3.2.** *The normal curvature tensor of the surface $M_1$ is identically zero if and only if*

$$P_1'^2 P_2' - P_1 P_2' P_1'' + P_1 P_1' P_2'' = 0$$

*or equivalently*

$$P_2\left(t\right) = c_1 \ln\left(P_1\left(t\right)\right) + c_2$$

*where $c_1, c_2$ are real constant.*

*Proof.* We have

$$R_{412}^3 = -h_{11}^3 h_{21}^4 + h_{21}^3 h_{11}^4 + h_{12}^3 h_{22}^4 - h_{22}^3 h_{12}^4 = -\frac{2\left(P_1'^2 P_2' - P_1 P_2' P_1'' + P_1 P_1' P_2''\right)}{P_1^4 P_2'^3}$$

Assume that ${}^{\perp}R = 0$. Then we have $P_1'^2 P_2' - P_1 P_2' P_1'' + P_1 P_1' P_2'' = 0$, which implies that $P_2\left(t\right) = c_1 \ln\left(P_1\left(t\right)\right) + c_2$, where $c_1, c_2$ are real constant. □

**Theorem 3.3.** *The mean curvature vector $H$ of the rotation surface $M_1$ is given by*

$$\begin{aligned} H &= \left(\frac{A_1\left(P_1' \cos u + P_1 P_2' \sin u\right)}{P_1^3 P_2'^2}, \frac{\left(P_1' \sin u - P_1 P_2' \cos u\right) A_1}{P_1^3 P_2'^2}, \right. \\ &\quad \left. \frac{\left(P_1' \sin u - P_1 P_2' \cos u\right) A_2}{P_1^3 P_2'^2}, -\frac{\left(P_1' \cos u + P_1 P_2' \sin u\right) A_2}{P_1^3 P_2'^2}\right). \end{aligned} \tag{3.2}$$

**Corollary 3.2.** *The surface $M_1$ can not be minimal.*

Let us consider that

$$\langle H, H\rangle = \frac{-P_1'^4 + P_1^4 P_2'^4}{P_1^6 P_2'^4} = \frac{\left(P_1^2 P_2'^2 - P_1'^2\right)\left(P_1^2 P_2'^2 + P_1'^2\right)}{P_1^6 P_2'^4} \neq 0. \tag{3.3}$$

So we have the following corollary.

**Corollary 3.3.** *The surface $M_1$ cannot be a marginally trapped surface.*

By covariant differentiation with respect to $e_1$ and $e_2$, a straightforward calculation gives

$$D_{e_1} e_3 = \frac{P_2'\left(-2P_1'^2 + P_1\left(P_1 P_2'^2 + P_1''\right)\right) - P_1 P_1' P_2''}{m_3 P_1 P_2'\left(P_1'^2 - P_1^2 P_2'^2\right)} e_1 + \frac{P_2'\left(P_1'^2 - P_1 P_1''\right) + P_1 P_1' P_2''}{m_3 P_1 P_2'\left(-P_1'^2 + P_1^2 P_2'^2\right)} e_2 - \frac{P_1'}{P_1\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}} e_4,$$

$$D_{e_1} e_4 = \frac{1}{m_3 P_1} e_2 - \frac{P_1'}{P_1\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}} e_3$$

and

$$D_{e_2} e_3 = \frac{P_1'^2 P_2' - P_1 P_2' P_1'' + P_1 P_1' P_2''}{m_3 P_1 P_2'\left(P_1'^2 - P_1^2 P_2'^2\right)} e_1 + \frac{P_1\left(P_1 P_2'^3 - P_2' P_1'' + P_1' P_2''\right)}{m_3 P_1 P_2'\left(P_1'^2 - P_1^2 P_2'^2\right)} e_2 + \frac{2P_1'^3 - P_1^2 P_1' P_2'^2}{P_1^3 P_2'^2\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}} e_4,$$

$$D_{e_2} e_4 = -\frac{1}{m_3 P_1} e_1 - \frac{2P_1'^2}{m_3 P_1^3 P_2'^2} e_2 + \frac{2P_1'^3 - P_1^2 P_1' P_2'^2}{P_1^3 P_2'^2\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}} e_3$$

We note that ${}^{\perp}\nabla_X H = \left(D_X H\right)^{\perp}$, which is the normal part of $D_X H$. Then we can calculate that

$$H = \frac{1}{m_3 P_1} e_3 - \frac{P_1'^2}{m_3 P_1^3 P_2'^2} e_4,$$

$$\begin{aligned} D_{e_1} H &= \frac{1}{\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}\left(\frac{1}{m_3 P_1}\right)' e_3 + \frac{1}{m_3 P_1} D_{e_1} e_3 \\ &\quad - \frac{1}{\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}\left(\frac{P_1'^2}{m_3 P_1^3 P_2'^2}\right)' e_4 - \frac{P_1'^2}{m_3 P_1^3 P_2'^2} D_{e_1} e_4, \end{aligned}$$

$$\begin{aligned} D_{e_2} H &= -\frac{1}{\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}\left(\frac{1}{m_3 P_1}\right)' e_3 + \frac{1}{m_3 P_1} D_{e_2} e_3 \\ &\quad + \frac{1}{\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}\left(\frac{P_1'^2}{m_3 P_1^3 P_2'^2}\right)' e_4 - \frac{P_1'^2}{m_3 P_1^3 P_2'^2} D_{e_2} e_4, \end{aligned}$$

which implies that

$$\begin{aligned} \langle D_{e_1} H, e_3\rangle &= \frac{m_1 m_3 P_1'\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}{P_1^4 P_2'^2}, \\ \langle D_{e_1} H, e_4\rangle &= \frac{m_3 P_1'\left(-3P_1'^2 P_2' + P_1 P_2'\left(P_1 P_2'^2 + 2P_1''\right) - 2P_1 P_1' P_2''\right)}{P_1^4 P_2'^3\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}, \\ \langle D_{e_2} H, e_3\rangle &= -\frac{m_1 m_3 P_1'\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}\left(2P_1'^2 + P_1^2 P_2'^2\right)}{P_1^6 P_2'^4}, \\ \langle D_{e_2} H, e_4\rangle &= \frac{m_3 P_1'\left(P_1'^2 P_2' + P_1 P_2'\left(P_1 P_2'^2 - 2P_1''\right) + 2P_1 P_1' P_2''\right)}{P_1^4 P_2'^3\sqrt{m_1\left(P_1'^2 - P_1^2 P_2'^2\right)}}. \end{aligned}$$

Assume that ${}^{\perp}\nabla_X H = 0$. Then we find that $P_1' = 0$. So we can give the following corollary.

**Corollary 3.4.** *The surface $M_1$ has the parallel mean curvature vector if and only if $P_1(t)$ is a nonzero constant.*

*3.2. Twisted surface with null spine curve*

In this subsection, we assume that $x_1'^2(t) - x_2'^2(t) = 0$. So we can choose

$$x_1(t) = Q_1(t)\cosh(c_1 + m_4 \ln Q_1(t)), \qquad x_2(t) = Q_1(t)\sinh(c_1 + m_4 \ln Q_1(t))$$

where $c_1$ is a real constant, $Q_1(t)$ is a nonconstant smooth function and $m_4 = \pm 1$. Here we denote the surface $M_2$ as

$$M_2 : \qquad F(t,u) = \begin{bmatrix} -x_1(t)\cosh u \sin u - x_2(t)\sinh u \sin u \\ x_1(t)\cosh u \cos u + x_2(t)\sinh u \cos u \\ x_1(t)\sinh u \cos u + x_2(t)\cosh u \sin u \\ x_1(t)\sinh u \sin u + x_2(t)\cosh u \sin u \end{bmatrix} \tag{3.4}$$

where $x_1(t) = Q_1(t)\cosh(c_1 + m_4 \ln Q_1(t))$ and $x_2(t) = Q_1(t)\sinh(c_1 + m_4 \ln Q_1(t))$. Then we construct the orthonormal basis as follows

$$e_1 = \frac{1}{\sqrt{2m_4 m_5 Q_1 Q_1'}}(F_t + F_u) = \frac{1}{\sqrt{2m_4 m_5 Q_1 Q_1'}} \begin{bmatrix} -Q_1 V_1 - m_4 W_1 Q_1' \sin u \\ Q_1 V_4 + m_4 W_1 Q_1' \cos u \\ Q_1 V_2 + W_1 Q_1' \cos u \\ Q_1 V_3 + W_1 Q_1' \sin u \end{bmatrix},$$

$$e_2 = \frac{1}{\sqrt{2m_4 m_5 Q_1 Q_1'}}(F_t - F_u) = \frac{1}{\sqrt{2m_4 m_5 Q_1 Q_1'}} \begin{bmatrix} Q_1 V_1 - m_4 W_1 Q_1' \sin u \\ -Q_1 V_4 + m_4 W_1 Q_1' \cos u \\ -Q_1 V_2 + W_1 Q_1' \cos u \\ -Q_1 V_3 + W_1 Q_1' \sin u \end{bmatrix},$$

$$e_3 = \frac{1}{U_1} \begin{bmatrix} -Q_1 W_2(\cos u \cosh(u+Q_2) + W_1 \sin u) + m_4 W_1 W_3 Q_1' \cos(2u) \\ Q_1 W_2(-\sin u \cosh(u+Q_2) + W_1 \cos u) + m_4 W_1 W_4 Q_1' \cos(2u) \\ Q_1 W_2(-\sin u \sinh(u+Q_2) + m_4 W_1 \cos u) + W_1 W_4 Q_1' \cos(2u) \\ Q_1 W_2(\cos u \sinh(u+Q_2) + m_4 W_1 \sin u) - W_1 W_3 Q_1' \cos(2u) \end{bmatrix},$$

$$e_4 = \frac{1}{U_1} \begin{bmatrix} -m_4 W_2 Q_1 \cos u \sinh(u+Q_2) + m_4 W_1 W_3 Q_1' \cos(2u) \\ -m_4 W_2 Q_1 \sin u \sinh(u+Q_2) + m_4 W_1 W_4 Q_1' \cos(2u) \\ -m_4 W_2 Q_1 \sin u \cosh(u+Q_2) + W_1 W_4 Q_1' \cos(2u) \\ m_4 W_2 Q_1 \cos u \cosh(u+Q_2) - W_1 W_3 Q_1' \cos(2u) \end{bmatrix}$$

where $m_5 = sign(m_4 Q_1 Q_1')$,

$$U_1 = \sqrt{Q_1 W_2 (W_2 Q_1 - 2m_4 Q_1' \cos(2u))},$$

$$\begin{aligned} V_1 &= \cos u \cosh(u+Q_2) + \sin u \sinh(u+Q_2), \\ V_2 &= \cos u \cosh(u+Q_2) - \sin u \sinh(u+Q_2), \\ V_3 &= \sin u \cosh(u+Q_2) + \cos u \sinh(u+Q_2), \\ V_4 &= \cos u \sinh(u+Q_2) - \sin u \cosh(u+Q_2), \end{aligned}$$

$$\begin{aligned} W_1 &= \sinh(u+Q_2) + m_4 \cosh(u+Q_2), \\ W_2 &= \cos(2u) - m_4 \sin(2u), \\ W_3 &= \sin u + m_4 \cos u, \\ W_4 &= m_4 \sin u - \cos u, \end{aligned}$$

$$Q_2(t) = c_1 + m_4 \ln Q_1(t).$$

Here $\{e_1, e_2\}$ is an orthonormal frame field on $F(t,u)$ of signature $(-m_5, m_5)$ and $\{e_3, e_4\}$ is a normal orthonormal frame field to $F(t,u)$ of signature $(+,-)$. Thus we can easily obtain that

$$D_{e_1}e_1 = \frac{1}{U_2}\begin{bmatrix} -m_4W_1Q_1'^3\sin u + Q_1^2\left(-4Q_1'\cos u\sinh(u+Q_2) + V_1Q_1''\right) - Q_1Q_1'\left((3V_1+4m_4V_3)Q_1' + m_4W_1Q_1''\sin u\right) \\ m_4W_1Q_1'^3\cos u - Q_1^2\left(4Q_1'\sin u\sinh(u+Q_2) + V_4Q_1''\right) + Q_1Q_1'\left((-3V_4+4m_4V_2)Q_1' + m_4W_1Q_1''\cos u\right) \\ W_1Q_1'^3\cos u - Q_1^2\left(4Q_1'\cosh(u+Q_2)\sin u + V_2Q_1''\right) + Q_1Q_1'\left((3V_2+4m_4V_4)Q_1' + W_1Q_1''\cos u\right) \\ W_1Q_1'^3\sin u + Q_1^2\left(4Q_1'\cos u\cosh(u+Q_2) - V_3Q_1''\right) + Q_1Q_1'\left((3V_3+4m_4V_1)Q_1' + W_1Q_1''\sin u\right) \end{bmatrix},$$

$$D_{e_2}e_1 = \frac{1}{U_2}\begin{bmatrix} -m_4W_1Q_1'^3\sin u + {Q_1}^2\left(4Q_1'\cos u\sinh(u+Q_2) + V_1Q_1''\right) + Q_1Q_1'\left(V_1Q_1' - m_4W_1Q_1''\sin u\right) \\ m_4W_1Q_1'^3\cos u + Q_1^2\left(4Q_1'\sin u\sinh(u+Q_2) - V_4Q_1''\right) - Q_1Q_1'\left(V_4Q_1' - m_4W_1Q_1''\cos u\right) \\ W_1Q_1'^3\cos u + Q_1^2\left(4Q_1'\cosh(u+Q_2)\sin u - V_2Q_1''\right) - Q_1Q_1'\left(V_2Q_1' - W_1Q_1''\cos u\right) \\ W_1Q_1'^3\sin u - Q_1^2\left(4Q_1'\cos u\cosh(u+Q_2) + V_3Q_1''\right) - Q_1Q_1'\left(V_3Q_1' - W_1Q_1''\sin u\right) \end{bmatrix},$$

$$D_{e_2}e_2 = \frac{1}{U_2}\begin{bmatrix} -m_4W_1Q_1'^3\sin u - Q_1^2\left(4Q_1'\cos u\sinh(u+Q_2) + V_1Q_1''\right) + Q_1Q_1'\left((3V_1+4m_4V_3)Q_1' - m_4W_1Q_1''\sin u\right) \\ m_4W_1Q_1'^3\cos u + Q_1^2\left(-4Q_1'\sin u\sinh(u+Q_2) + V_4Q_1''\right) + Q_1Q_1'\left(-(3V_4+4m_4V_2)Q_1' + m_4W_1Q_1''\cos u\right) \\ W_1Q_1'^3\cos u + Q_1^2\left(-4Q_1'\cosh(u+Q_2)\sin u + V_2Q_1''\right) + Q_1Q_1'\left(-(3V_2+4m_4V_4)Q_1' + W_1Q_1''\cos u\right) \\ W_1Q_1'^3\sin u + Q_1^2\left(4Q_1'\cos u\cosh(u+Q_2) + V_3Q_1''\right) + Q_1Q_1'\left(-(3V_3+4m_4V_1)Q_1' + W_1Q_1''\sin u\right) \end{bmatrix}$$

where $U_2 = 4m_4m_5Q^2Q'^2$. The components of the second fundamental form $h$ are given as follows

$$\begin{aligned} h_{11}^3 &= \langle D_{e_1}e_1, e_3\rangle = \frac{\cos(2u) + W_2}{m_4m_5U_1}, \\ h_{12}^3 &= \langle D_{e_1}e_2, e_3\rangle = -\frac{\cos(2u)}{m_4m_5U_1}, \\ h_{22}^3 &= \langle D_{e_2}e_2, e_3\rangle = \frac{\sin(2u)}{m_5U_1}, \\ h_{11}^4 &= -\langle D_{e_1}e_2, e_4\rangle = \frac{Q_1W_2 - Q_1'\sin(2u)}{m_5Q_1'U_1}, \\ h_{12}^4 &= -\langle D_{e_1}e_2, e_4\rangle = \frac{-m_4Q_1W_2 + Q_1'\cos(2u)}{m_4m_5Q_1'U_1}, \\ h_{22}^4 &= -\langle D_{e_2}e_2, e_4\rangle = \frac{m_4Q_1W_2 - (\cos(2u) + W_2)Q_1'}{m_4m_5Q_1'U_1}. \end{aligned}$$

**Theorem 3.4.** *The Gaussian curvature of $M_2$ given by (3.4) is identically zero.*

*Proof.* By straight calculations in (2.1) , we obtain $K = 0$. □

So we have the following corollary.

**Corollary 3.5.** *The surface $M_2$ given by (3.4) is a flat surface.*

In the following theorem, we consider the normal curvature tensor of the twisted surface $M_2$.

**Theorem 3.5.** *The normal curvature tensor of the surface $M_2$ is identically zero.*

*Proof.* We have
$$R^3_{412} = -h^3_{11}h^4_{21} + h^3_{21}h^4_{11} + h^3_{12}h^4_{22} - h^3_{22}h^4_{12} = 0$$
So the normal curvature tensor of the surface $M_2$ is identically zero. □

**Theorem 3.6.** *The mean curvature vector $H$ of the twisted surface $M_2$ is given by*
$$H = -\frac{m_4 W_2}{U_1}e_3 - \frac{m_4 W_2}{U_1}e_4.$$

**Corollary 3.6.** *The surface $M_2$ can not be minimal.*

**Corollary 3.7.** *The surface $M_2$ is a marginally trapped surface.*

*Proof.* We have $\langle H, H\rangle = 0$. So the surface $M_2$ is a marginally trapped surface. □


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## References


[1] Dede, M., Ekici, C., Goemans, W. and Ünlütürk, Y.: *Twisted surfaces with vanishing curvature in Galilean 3-space.* Int. J. Geom. Methods Mod. Phys. **15**, 1850001 (13 pages)(2018).
[2] Duggal, K. L. and Jin, D. H.: Null curves and Hypersurfaces of Semi-Riemannian Manifolds, World Scientific Publishing (2007).
[3] Goemans, W. and Woestyne, I. Van de: *Twisted surfaces in Euclidean and Minkowski 3-space*, Pure and Applied Differential Geometry: J. Van der Veken, I. Van de Woestyne, L. Verstraelen and L. Vrancken (Editors), Shaker Verlag Aachen, Germany, 143-151 (2013).
[4] Goemans, W. and Woestyne, I. Van de: *Constant curvature twisted surfaces in Euclidean and Minkowski 3-space*. In: Proceedings of the conference "Riemannian Geometry and Applications to Engineering and Economics-RIGA", Bucharest, Romania, 117-130 (2014).
[5] Goemans, W. and Woestyne, I. Van de: *Twisted surfaces with null rotation axis in Minkowski 3-space*. Results. Math. **70**, 81-93 (2016).
[6] Goemans, W.: *Flat double rotattional surfaces in Euclidean and Lorentz-Minkowski 4-space*. Publications de L'Institut Mathematique, Nouvelle série, tome **103**(117), 61–68(2018).
[7] Gray, A., Abbena, E., Salamon S. (eds.): Modern Differential Geometry of Curves and Surfaces with Mathematica. Chapman & Hall/CRC, Boca Raton (2006).
[8] Grbović, M., Nešović, E. and Pantić, A.: *On the second kind twisted surfaces in Minkowski 3-space*. Int. Electron. J. Geom. **8(2)**, 9–20(2015).
[9] Inoguchi, J. and Lee, S.: *Null curves in Minkowski 3-space* Int. Electron. J. Geom. **1 (2)**,40-83(2008).
[10] Kazan, A. and Karadağ, H.B.:*Twisted surfaces in the Pseudo-Galilean space*. New Trends Math. Sci. **5**, 72–79 (2017).
[11] Kuhnel, W.: Differential geometry: curves-surfaces-manifolds, Braunschweig, Wiesbaden, (1999).
[12] Lopez, R.: *Differential geometry of curves and surfaces in Lorentz-Minkowski space*. Int. Electron. J. Geom. **7 (1)**,44-107 (2014).
[13] Moore, C. L. E.: *Surfaces of rotation in a space of four dimensions*. Ann. of Math. **21(2)**, 81–93 (1919).
[14] O'Neill, B.: Semi-Riemannian Geometry. Academic Press, London (1983)
[15] Stanilov, G., and Slavova, G.: *Classification of some twisted surfaces and power series of such surfaces*. Comptes Rendus de l'Academie Bulgare des Sciences, **59(6)**,593-600 (2006).
[16] Walrave, J.: Curves and surfaces in Minkowski space, Doctoral thesis, K.U. Leuven, Faculty of Science, Leuven (1995).

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