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Research article

On Wintgen Ideal Submanifolds Satisfying Some Pseudo-symmetry Type Curvature Conditions

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

Abstract

Let M be a Wintgen ideal submanifold of dimension n in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$ of dimension $(n+m)$ and of constant curvature $\widetilde{k}$, $n \geq 4$, $m \geq 1$. We determine necessary and sufficient conditions for M to be a submanifold satisfying pseudo-symmetry type curvature conditions of the form: the derivation-commutator $R \cdot C - C \cdot R$ (resp., the tensors $R \cdot C$ and $C \cdot R$) formed by the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of M, and some Tachibana tensors formed by the metric tensor g, the Ricci tensor Ricc and the tensors R and C of M are linearly dependent.

Keywords: *Tachibana tensor, pseudo-symmetry type curvature condition, generalized Einstein metric condition, submanifold, Wintgen ideal submanifold, DDVV conjecture.*

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1. Preliminaries

1.1. *Pseudo-symmetry type curvature conditions*

Let $M = M^n$ be a connected n-dimensional, $n \geq 4$, Riemannian manifold of class C^∞. Further, let ∇ be its Levi-Civita connection and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\mathfrak{X}(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$
$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively, where $X, Y, Z \in \mathfrak{X}(M)$ and A is a symmetric $(0,2)$-tensor on M. The Ricci tensor Ricc, the Ricci operator $\mathcal{S}$, the scalar curvature κ and the endomorphism $\mathcal{C}(X,Y)$ of (M,g) are defined by

$$\mathrm{Ricc}(X,Y) = \mathrm{tr}\{Z \to \mathcal{R}(Z,X)Y\},$$
$$g(\mathcal{S}U,V) = \mathrm{Ricc}(U,V) = \sum_{i=1}^{n} g(\mathcal{R}(E_i,U)V,E_i), \quad \kappa = \mathrm{tr}\,\mathcal{S},$$
$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z,$$

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respectively, where $X, Y, Z, U, V \in \mathfrak{X}(M)$ and $\{E_k\}_{k\in\{1,\ldots,n\}}$ is a local orthonormal tangent framefield. The *Riemann-Christoffel tensor* R and the *Weyl conformal curvature tensor* C of M are defined by

$$R(X,Y,Z,W) = g\left(\mathcal{R}(X,Y)Z,W\right),$$
$$C(X,Y,Z,W) = g\left(\mathcal{C}(X,Y)Z,W\right),$$

respectively, where $X, Y, Z, W \in \mathfrak{X}(M)$. Let A and B be the $(0,2)$-tensors; their *Kulkarni-Nomizu product* $A \wedge B$ is defined by

$$\begin{aligned}(A\wedge B)(X_1,X_2\,;X,Y) &= A(X_1,Y)B(X_2,X) + A(X_2,X)B(X_1,Y)\\ &\quad - A(X_1,X)B(X_2,Y) - A(X_2,Y)B(X_1,X),\end{aligned}$$

where $X, Y, X_1, X_2 \in \mathfrak{X}(M)$. Now the tensor C can be expressed in the form

$$C = R - \frac{1}{n-2} g \wedge \mathrm{Ricc} + \frac{\kappa}{2(n-1)(n-2)} g \wedge g.$$

Let A be a symmetric $(0,2)$-tensor and T a $(0,k)$-tensor, $k \geq 1$; we define the $(0,k+2)$*-tensors* $R\cdot T$ and $Q(A,T)$ by

$$\begin{aligned}(R\cdot T)(X_1,X_2,\ldots,X_k\,;X,Y) &= (R(X,Y)\cdot T)(X_1,\ldots,X_k)\\ &= -T\left(\mathcal{R}(X,Y)X_1,X_2,\ldots,X_k\right)\\ &\quad - \cdots - T\left(X_1,X_2,\ldots,X_{k-1},\mathcal{R}(X,Y)X_k\right),\\ Q(A\cdot T)(X_1,X_2,\ldots,X_k\,;X,Y) &= ((X\wedge_A Y)\cdot T)(X_1,\ldots,X_k)\\ &= -T\left((X\wedge_A Y)X_1,X_2,\ldots,X_k\right)\\ &\quad - \cdots - T\left(X_1,X_2,\ldots,X_{k-1},(X\wedge_A Y)X_k\right),\end{aligned}$$

respectively, where $X, Y, X_1, X_2, \ldots, X_n \in \mathfrak{X}(M)$. If we set in the above formulas $T = R$, $T = \mathrm{Ricc}$, $T = C$, $A = g$ or $A = \mathrm{Ricc}$, then we obtain the tensors: $R\cdot R$, $R\cdot\mathrm{Ricc}$, $R\cdot C$, $Q(g,R)$, $Q(g,\mathrm{Ricc})$, $Q(g,C)$, $Q(g, g\wedge\mathrm{Ricc})$, $Q(\mathrm{Ricc},R)$ and $Q(\mathrm{Ricc},C)$. Similarly we define the tensors: $C\cdot R$, $C\cdot C$ and $C\cdot\mathrm{Ricc}$. The derivation-commutator $R\cdot C - C\cdot R$ reads (see, e.g., [20, 28])

$$(n-2)(R\cdot C - C\cdot R) = Q\left(\mathrm{Ricc} - \frac{\kappa}{n-1}g, R\right) - g\wedge(R\cdot\mathrm{Ricc}) + P,$$

where the $(0,6)$-tensor P is defined by

$$\begin{aligned}P(X_1,X_2,X_3,X_4\,;X,Y) &= g(X,X_1)R(\mathcal{S}(Y),X_2,X_3,X_4) - g(Y,X_1)R(\mathcal{S}(X),X_2,X_3,X_4)\\ &\quad + g(X,X_2)R(X_1,\mathcal{S}(Y),X_3,X_4) - g(Y,X_2)R(X_1,\mathcal{S}(X),,X_3,X_4)\\ &\quad + g(X,X_3)R(X_1,X_2,\mathcal{S}(Y),X_4) - g(Y,X_3)R(X_1,X_2,\mathcal{S}(Y),X_4)\\ &\quad + g(X,X_4)R(X_1,X_2,X_3,\mathcal{S}(Y)) - g(Y,X_4)R(X_1,X_2,X_3,\mathcal{S}(X)),\end{aligned}$$

$X, Y, X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$.

A Riemannian manifold (M,g) is said to be *semi-symmetric* (see [47]) if $R\cdot R = 0$ on M. An extension form for semi-symmetric manifolds are the pseudo-symmetric manifolds. A Riemannian manifold (M,g) is said to be *pseudo-symmetric* (see [16, 20]) if at every point of M the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent, which means that there exists a function L_R on the set $U_R = \left\{x\in M \;\middle|\; R - (\kappa/(2n(n-1)))g\wedge g \neq 0 \text{ at } x\right\}$ such that

$$R\cdot R = L_R Q(g,R) \tag{1.1}$$

on U_R. Every semi-symmetric manifold is pseudo-symmetric. The converse is not true (see, e.g., [14, Corollary 3.2 (i)]). Pseudo-symmetric manifolds also are called *pseudo-symmetric (in the sense of Deszcz)* or *Deszcz symmetric spaces* (see, e.g., [10, 13, 48, 50, 51, 52]).

A Riemannian manifold (M,g) is said to be *Weyl-semi-symmetric* (see [16, 20]) if

$$R\cdot C = 0$$

on M. An extension form for Weyl-semi-symmetric manifolds are the Weyl pseudo-symmetric manifolds. A Riemannian manifold (M,g) is said to be *Weyl-pseudo-symmetric* (see [16, 20]) if at every point of M the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent, which means that there exists a function L_C on the set $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$ such that

$$R \cdot C = L_C Q(g, C) \tag{1.2}$$

on U_C. It is obvious that every pseudo-symmetric manifold is Weyl-pseudo-symmetric. The converse is not true.

A Riemannian manifold (M, g) is said to be a *manifold with pseudo-symmetric Weyl tensor* (see [16, 17, 20]) if at every point of M the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent, which means that there exists a function L on the set U_C such that

$$C \cdot C = LQ(g, C) \tag{1.3}$$

on U_C. Every Chen ideal submanifold in a space of constant curvature is a manifold with pseudo-symmetric Weyl tensor [22, 27, 28, 32].

A Riemannian manifold (M, g) is said to be *Ricci-pseudo-symmetric* (see [16, 20, 30]) if at every point of M the tensors $R \cdot \mathrm{Ricc}$ and $Q(g, \mathrm{Ricc})$ are linearly dependent, which means that there exists a function L_{Ricc} on the set $U_{\mathrm{Ricc}} = \{x \in M \mid \mathrm{Ricc} - (\kappa/n)g \neq 0 \text{ at } x\}$ such that

$$R \cdot \mathrm{Ricc} = L_{\mathrm{Ricc}} Q(g, \mathrm{Ricc}) \tag{1.4}$$

on U_{Ricc}. Every Cartan hypersurface M in the sphere S^{n+1}, $n = 6,\ 12$ or 24, is a non-pseudo-symmetric, Ricci-pseudo-symmetric with non-pseudo-symmetric Weyl tensor hypersurface [33] (see also [31]). We also mention that Ricci-pseudo-symmetric manifolds also are called *Ricci pseudo-symmetric in the sense of Deszcz*, or simply *Deszcz Ricci-symmetric* (see, e.g., [10]).

A Riemannian manifold (M, g) is said to be *Ricci-Weyl-pseudo-symmetric* (see [16, 20]) if at every point of M the tensors $R \cdot C$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent, which means that there exists a function L' on the set $U = \{x \in M \mid Q(\mathrm{Ricc}, C) \neq 0 \text{ at } x\}$ such that

$$R \cdot C = L' Q(\mathrm{Ricc}, C) \tag{1.5}$$

on U. It is obvious that every semi-symmetric manifold ($R \cdot R = 0$) satisfies (1.5) with $L' = 0$.

We refer to [18] (see also [19, 21]) for a recent survey on manifolds satisfying (1.1)–(1.5) and other conditions of this kind. Such conditions are called pseudo-symmetry type curvature conditions. It seems that (1.1) is the most important condition of that family of curvature conditions (see [17, 23, 24, 25, 28, 38, 39, 48, 49, 50, 51, 52]).

We also refer to [19] (see also [20]) for a survey of results on semi-Riemannian manifolds (M, g) and in particular, hypersurfaces in spaces of constant curvature or Chen ideal submanifolds in Euclidean spaces, satisfying pseudo-symmetry type curvature conditions of the form: the derivation-commutator $R \cdot C - C \cdot R$, formed by the tensors R and C, and a finite sum of the Tachibana tensors of the form $Q(A, T)$ are linearly dependent, where A is a symmetric $(0, 2)$-tensor and T a generalized curvature tensor. Conditions of this kind belong to the family of curvature conditions called *generalized Einstein metric conditions* [19, 20, 26, 30].

1.2. *Submanifolds in space forms*

Throughout all sections of the present paper, let $M = M^n$ be a connected Riemannian manifold of class C^∞ of dimension n in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$ of dimension $(n + m)$ and of constant curvature $\widetilde{k}$, $n \geq 4$, $m \geq 1$.

On $\mathbb{R}^{n+m}(\widetilde{k})$, we denote by $\widetilde{g}$ and $\widetilde{\nabla}$ respectively the *Riemannian metric* and the corresponding *Levi-Civita connection*. On the submanifold M, the *induced Riemannian metric* and the corresponding *Levi-Civita connection* on M will be denoted by g, ∇. We will write as X, Y, ... the *tangent vector fields* on M, and as ξ, η, ... the *normal vector fields* on M.

The well-known *formulae of Gauss* and *Weingarten* are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\widetilde{\nabla}_X \xi = \nabla^{\perp}_X \xi - A_\xi Y,$$

respectively, whereby $\nabla^{\perp}$ is the *normal connection* induced in the *normal bundle* of M in $\mathbb{R}^{n+m}(\widetilde{k})$, h is the *second fundamental form* of the submanifold M and A_ξ is the *shape operator* or the *Weingarten map* on M with respect to the normal vector field ξ. We have

$$g(A_\xi(X), Y) = g\left(\xi, h(X, Y)\right),$$

or still

$$h(X,Y) = \sum_{\alpha=1}^{m} g(A_\alpha(X),Y)\xi_\alpha,$$

whereby $\{\xi_\alpha\}_{\alpha\in\{1,\ldots,m\}}$ is any *local orthonormal normal framefield* on M in $\mathbb{R}^{n+m}(\widetilde{k})$ and $A_\alpha = A_{\xi_\alpha}$.

The *mean curvature vector field* of M in $\mathbb{R}^{n+m}(\widetilde{k})$ is defined by

$$\vec{H} = \frac{1}{n}\sum_{i=1}^{n} h(E_i,E_i) = \frac{1}{n}\sum_{\alpha=1}^{m} (\mathrm{tr}\ A_\alpha)\,\xi_\alpha,$$

whereby $\{E_s\}_{s\in\{1,\ldots,n\}}$ is any *local orthonormal tangent framefield* on M, and $\{\xi_\alpha\}_{\alpha\in\{1,\ldots,m\}}$ is any *local orthonormal normal framefield* on M in $\mathbb{R}^{n+m}(\widetilde{k})$ and its length $H = \left\|\vec{H}\right\|$ is the *mean curvature* of M in $\mathbb{R}^{n+m}(\widetilde{k})$.

The submanifold M in $\mathbb{R}^{n+m}(\widetilde{k})$ is *totally geodesic* when $h = 0$. M is *totally umbilical* when $h = g\,\vec{H}$. It is *minimal* when $\vec{H} = \vec{0}$, or equivalently, when its *squared mean curvature function* $H^2 = \widetilde{g}\left(\vec{H},\vec{H}\right)$ vanishes identically. M is *pseudo-umbilical* when the mean curvature vector field $\vec{H}$ determines an *umbilical normal direction* on M in $\mathbb{R}^{n+m}(\widetilde{k})$, i.e. when $A_{\vec{H}} = \chi\mathrm{Id}$, whereby Id stands for the identity operator on TM and χ is some real function on M.

Let R denote the *induced Riemann-Christoffel curvature* corresponding to the induced Levi-Civita connection ∇ on M. Then according to the *equation of Gauss*

$$R(X,Y,Z,W) = \widetilde{g}\left(h(X,W)h(Y,Z) - h(X,Z)h(Y,W)\right) + \widetilde{k}\left(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\right),$$

where X,Y,Z,W are tangent vector fields in M.

Let κ be the *scalar curvature function* of M in $\mathbb{R}^{n+m}(\widetilde{k})$; we have

$$\kappa(p) = \sum_{i<j} K\left(p, E_i(p)\wedge E_j(p)\right)$$

where $K\left(p, E_i(p)\wedge E_j(p)\right)$ is the sectional curvature of M at a point $p\in M$ for the plane section $\phi = E_i(p)\wedge E_j(p)$ in T_pM (where $\{E_i(p), E_j(p)\}$ is independent). By $\inf(K)$ we will further denote the *function* $\inf(K) : M\to\mathbb{R}$ which attains to $p\in M$ the minimal value $\inf(K)(p)$ of all sectional curvatures of M at p. The *normalized scalar curvature* of (M,g) is defined as follows

$$\rho = \frac{2}{n(n-1)}\sum_{i<j} R\left(E_i,E_j,E_j,E_i\right),$$

whereby $\{E_s\}_{s\in\{1,\ldots,n\}}$ is any *local orthonormal tangent framefield* on M.

By the *equation of Ricci*, the *normal curvature tensor* $R^\perp$ of M in $\mathbb{R}^{n+m}(\widetilde{k})$, i.e., the *curvature tensor of the normal connection* $\nabla^\perp$ of M in $\mathbb{R}^{n+m}(\widetilde{k})$, is defined as follows

$$R^\perp(X,Y;\xi,\eta) = \widetilde{g}\left(\nabla^\perp_X\nabla^\perp_Y\xi - \nabla^\perp_Y\nabla^\perp_X\xi - \nabla^\perp_{[X,Y]}\xi,\eta\right) = g\left(\left[A_\xi,A_\eta\right]X,Y\right), \tag{1.6}$$

whereby $[A_\xi,A_\eta] = A_\xi A_\eta - A_\eta A_\xi$.

The *normalized scalar normal curvature* $\rho^\perp$ of (M,g) in $\mathbb{R}^{n+m}(\widetilde{k})$ is given as follows

$$\rho^\perp = \frac{2}{n(n-1)}\sqrt{\sum_{i<j}\sum_{\alpha<\beta} R^\perp\left(E_i,E_j,\xi_\alpha,\xi_\beta\right)},$$

whereby $\{E_s\}_{s\in\{1,\ldots,n\}}$ is any *local orthonormal tangent framefield* on M, and $\{\xi_\alpha\}_{\alpha\in\{1,\ldots,m\}}$ is any *local orthonormal normal framefield* on M in $\mathbb{R}^{n+m}(\widetilde{k})$. One can remark that $\rho^\perp = 0$ if and only if $R^\perp = 0$, which means that *the normal connection is flat*. This follows from (1.6) and as it was already observed by E. Cartan (see [1]), is equivalent to the simultaneous diagonalisability of all shape operators A_ξ of M in $\mathbb{R}^{n+m}(\widetilde{k})$.

1.3. Wintgen ideal submanifolds satisfying some pseudo-symmetry type curvature conditions

In classical differential geometry, for a surface M^2 in a Euclidean 3-space $\mathbb{E}^3$, the well-known Euler inequality is given by

$$K \leq H^2,$$

whereby K is the intrinsic *Gauss curvature* of M^2 and H^2 is the extrinsic *squared mean curvature* of M^2 in $\mathbb{E}^3$, at once follows from the fact that $K = k_1k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ whereby k_1 and k_2 denote the *principal curvatures* of M^2 in $\mathbb{E}^3$. Obviously, $K = H^2$ everywhere on M^2 if and only the surface M^2 is totally umbilical in $\mathbb{E}^3$, i.e. $k_1 = k_2$ at all points of M^2, or still, by a *theorem of Meunier*, if and only M^2 is a part of a *plane* $\mathbb{E}^2$ or of a *round sphere* S^2 in $\mathbb{E}^3$. P. Wintgen (see [54]), in the late 19 seventies, proved that the *Gauss curvature* K and the *squared mean curvature* H^2 and the *normal curvature* $K^\perp$ of any surface M^2 in $\mathbb{E}^4$ always satisfy the inequality

$$K \leq H^2 - K^\perp$$

and that actually *the equality* if and only if the curvature ellipse of M^2 in $\mathbb{E}^4$ is a circle. We recall that the *ellipse of curvature* at a point p of M^2 is defined as

$$\Sigma_p = \left\{ h(X,X) \;\middle|\; X \in T_pM \text{ and } \|X\| = 1 \right\}.$$

The *ellipse of curvature* is the analogue of the *Dupin curvature* of an ordinary surface in $\mathbb{E}^3$.

B. Rouxel in [43] and V. Guadalupe and L. Rodriguez in [36] extended Wintgen's inequality to surfaces of arbitrary codimension in real space forms $\mathbb{R}^{2+m}(c)$ with $m \geq 2$. Also, B.-Y. Chen extended Wintgen's inequality in [6, 8] to surfaces in a pseudo-Euclidean 4-space $\mathbb{E}^4_2(c)$ with a neutral metric.

In 1999, P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken proved in [15] the Wintgen inequality

$$\rho \leq H^2 - \rho^\perp + \widetilde{k}$$

for all submanifolds M^n of codimension 2 in all real space forms, whereby ρ is the normalised scalar curvature of the Riemannian manifold M, and whereby H^2 and $\rho^\perp$, are the squared mean curvature and the normalized normal scalar curvature of M in the ambient space, respectively, characterizing the equality in terms of the shape operators of M^n in $\mathbb{R}^{n+2}(\widetilde{k})$. And in [15] they proposed a conjecture of Wintgen inequality for general Riemannian submanifolds in real space forms, which was later well-known as the DDVV conjecture. This conjecture was proven to be true by Z. Lu (see [40]), and by J. Ge and Z. Tang (see [34, 35]) independently. In [10], B.-Y. Chen provided a comprehensive survey on developments in Wintgen inequality and Wintgen ideal submanifolds, we also refer to the recent article of G.-E. Vilcu [53, Chapter 7].

The main purpose of the present article is to study the so-called *Wintgen ideal submanifolds*. The following theorem of J. Ge and Z. Tang (see [34, 35]) and Lu (see [40]) states *the Wintgen inequality* for submanifolds $M = M^n$ of any codimension $m \geq 1$ in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, and *characterizes its equality case*. As it is stated in [34], it concerns *a basic general optimal inequality between* likely *the most primitive scalar valued geometric quantities* that can be defined *on submanifolds* as *intrinsic invariant* it involves *the scalar curvature* and as *extrinsic invariants* it involves *the scalar normal curvature* and *the squared mean curvature*.

Theorem A. (see [11, 34, 35, 40]) *Let $M = M^n$ be a submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. Then*

$$(*) \quad \rho \leq H^2 - \rho^\perp + \widetilde{k},$$

and in $(*)$ actually the equality holds if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by

$$A_1 = \begin{pmatrix} a & \mu & 0 & \cdots & 0 \\ \mu & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}, \; A_2 = \begin{pmatrix} b+\mu & 0 & 0 & \cdots & 0 \\ 0 & b-\mu & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix}, \; A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \tag{1.7}$$

$$A_4 = \ldots = A_m = 0.$$

where a, b, c and μ are real functions on M. □

Definition 1.1. The submanifold M in $\mathbb{R}^{n+m}(\widetilde{k})$ which satisfy the equality

$$(\widetilde{*}) \qquad \rho = H^2 - \rho^{\perp} + \widetilde{k}$$

in the *Wintgen's general inequality* $(*)$ is called a *Wintgen ideal submanifold*. In such case, the frames $\{E_i, \xi_\alpha\}$ in which the shape operators assume the forms of (1.7) will further be called the *Choi-Lu frames*, and the corresponding tangent E_1E_2-planes will be called the *Choi-Lu planes* of M in $\mathbb{R}^{n+m}(\widetilde{k})$.

From [13] (see also [29, 42]) we recall the following results.

Theorem B**.** *A Wintgen ideal submanifold $M = M^n$ of dimension $n \geq 4$ and of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$ ($m \geq 2$) is a Deszcz symmetrical Riemannian manifold if and only if M is totally umbilical (with Deszcz sectional curvature $L_R = 0$), or, M is a minimal or pseudo-umbilical submanifold (with $L_R = \widetilde{k} + H^2$) of this space form $\mathbb{R}^{n+m}(\widetilde{k})$.* □

Theorem C**.** *A Wintgen ideal submanifold $M = M^n$ of dimension $n \geq 4$ and of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$ ($m \geq 2$) is Deszcz symmetric if and only if M is Deszcz Ricci-symmetric.* □

Theorem D**.** *Let $M = M^n$ be a Wintgen ideal submanifold of dimension $n \geq 4$ and of codimension $m \geq 2$ in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$. Then M is a Riemannian manifold with a pseudo-symmetric conformal Weyl tensor C.* □

Theorem E**.** *Let $M = M^n$ be a Wintgen ideal submanifold of dimension $n \geq 4$ and of codimension $m \geq 2$ in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$.*
(i) M is conformally flat if and only if M is a totally umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$ (and, hence, M is a real space form).
(ii) If M is not a conformally flat submanifold, then M has a pseudo-symmetric conformal Weyl tensor C and the corresponding function is given by

$$L_C = \frac{n-3}{(n-1)(n-2)}\Big(\kappa - n(n-1)\inf K\Big).$$

where κ is the scalar curvature of M. □

We also refer to [12, 41, 44, 45, 46] for further results on Wintgen ideal submanifolds.

2. Main results on Wintgen ideal submanifolds

2.1. Main results

In the paper we investigate Wintgen ideal submanifolds $M = M^n$ in real space forms $\mathbb{R}^{n+m}(\widetilde{k})$, $n > 3$, $m \geq 1$, satisfying the following pseudo-symmetry type curvature conditions:

(i) the tensor $R \cdot C$ and the Tachibana tensor $Q(g, R)$ (resp., the tensor $Q(g, C)$, $Q(g, g \wedge \mathrm{Ricc})$, $Q(\mathrm{Ricc}, R)$, or $Q(\mathrm{Ricc}, C)$) are linearly dependent;

(ii) the tensor $C \cdot R$ and the Tachibana tensor $Q(g, R)$ (resp., the tensor $Q(g, C)$, $Q(g, g \wedge \mathrm{Ricc})$, $Q(\mathrm{Ricc}, R)$, or $Q(\mathrm{Ricc}, C)$) are linearly dependent;

(iii) the derivation-commutator $R \cdot C - C \cdot R$, formed by the tensors R and C, and the Tachibana tensor $Q(g, R)$ (resp., the tensor $Q(g, C)$, $Q(g, g \wedge \mathrm{Ricc})$, $Q(\mathrm{Ricc}, R)$, or $Q(\mathrm{Ricc}, C)$) are linearly dependent.

In this subsection we present our main results.

Theorem 2.1. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} > 0$, then M is Weyl-semi-symmetric if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.2. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} \leq 0$, then M is Weyl-semi-symmetric if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence*

M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad (2.1)$$
$$A_4 = \ldots = A_m = 0,$$

where c, μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$. In this second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. □

Corollary 2.1. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} = 0$, then M is Weyl-semi-symmetric if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (2.2)$$
$$A_3 = A_4 = \ldots = A_m = 0,$$

where μ is a real function on M such that $\mu \neq 0$. In this second case, M is minimal in $\mathbb{R}^{n+m}$. □

Theorem 2.3. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. Then*
$$C \cdot R = 0$$
if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat). □

Theorem 2.4. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. Then*
$$R \cdot C - C \cdot R = 0$$
if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat). □

Theorem 2.5. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} > 0$, then the tensors $R \cdot C$ and $Q(g, R)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.6. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} \leq 0$, then the tensors $R \cdot C$ and $Q(g, R)$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad (2.3)$$
$$A_4 = \ldots = A_m = 0,$$

where c, μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$. Moreover, M is Weyl-semi-symmetric, and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. □

Corollary 2.2. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} = 0$, then the tensors $R \cdot C$ and $Q(g, R)$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}$, the shape operators are given by*

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\ A_3 &= A_4 = \ldots = A_m = 0, \end{aligned} \tag{2.4}$$

where μ is a real function on M such that $\mu \neq 0$. In this second case, M is minimal in $\mathbb{R}^{n+m}$. $\square$

Theorem 2.7. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(g, R)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* $\square$

Theorem 2.8. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} > 0$, then the tensors $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* $\square$

Theorem 2.9. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} \leq 0$, then the tensors $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \\ A_4 &= \ldots = A_m = 0, \end{aligned} \tag{2.5}$$

where c, μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$. Moreover,

$$R \cdot C - C \cdot R = -\frac{2(n-3)\mu^2}{(n-1)(n-2)} Q(g, R) \,;$$

and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. $\square$

Corollary 2.3. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 2$. If $\widetilde{k} = 0$, then the tensors $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}$, the shape operators are given by*

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\ A_3 &= A_4 = \ldots = A_m = 0, \end{aligned} \tag{2.6}$$

where μ is a real function on M such that $\mu \neq 0$. In this second case, M is minimal in $\mathbb{R}^{n+m}$. $\square$

Theorem 2.10. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad (2.7)$$
$$A_4 = \ldots = A_m = 0,$$

where c and μ are real functions on M such that $\mu \neq 0$. Moreover,

$$R \cdot C = (\widetilde{k} + c^2) Q(g, C)\ ;$$

and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. □

Theorem 2.11. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(g, C)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$. In this case, M is conformally flat.* □

Theorem 2.12. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(g, C)$ are linearly dependent if and if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$. In this case, M is conformally flat.* □

Theorem 2.13. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} > 0$, then the tensors $R \cdot C$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.14. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} \leq 0$, then the tensors $R \cdot C$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad (2.8)$$
$$A_4 = \ldots = A_m = 0,$$

where c and μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$. Moreover, M is Weyl-semi-symmetric ; and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. □

Corollary 2.4. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} = 0$, then the tensors $R \cdot C$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (2.9)$$
$$A_3 = A_4 \ldots = A_m = 0,$$

where μ is a real function on M such that $\mu \neq 0$. In this second case, M is minimal in $\mathbb{R}^{n+m}$. □

Theorem 2.15. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then $C \cdot R$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.16. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.17. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} > 0$, then the tensors $R \cdot C$ and $Q(\mathrm{Ricc}, R)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.18. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} \leq 0$, then the tensors $R \cdot C$ and Q(Ricc,R) are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \\ A_4 &= \ldots = A_m = 0, \end{aligned} \tag{2.10}$$

where c and μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$. Moreover, M is Weyl-semi-symmetric ; and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. □

Corollary 2.5. *Let M be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} = 0$, then the tensors $R \cdot C$ and Q(Ricc,R) are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}$, the shape operators are given by*

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\ A_3 &= A_4 = \ldots = A_m = 0, \end{aligned} \tag{2.11}$$

where μ is a real function on M such that $\mu \neq 0$. In this second case, M is minimal in $\mathbb{R}^{n+m}$. □

Theorem 2.19. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(\mathrm{Ricc}, R)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.20. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(\mathrm{Ricc}, R)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.21. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} > 0$, then the tensors $R \cdot C$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* □

Theorem 2.22. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} \leq 0$, then the tensors $R \cdot C$ and Q(Ricc,C) are linearly dependent if and only if (i) either M is totally* *umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad A_4 = \ldots = A_m = 0, \tag{2.12}$$

where c and μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$. Moreover, M is Weyl-semi-symmetric in $\mathbb{R}^{n+m}(\widetilde{k})$; and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. $\square$

Corollary 2.6. *Let M be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. If $\widetilde{k} = 0$, then the tensors $R \cdot C$ and Q(Ricc,C) are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_3 = A_4 = \ldots = A_m = 0, \tag{2.13}$$

where μ is a real function on M such that $\mu \neq 0$. In this second case, M is minimal in $\mathbb{R}^{n+m}$. $\square$

Theorem 2.23. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* $\square$

Theorem 2.24. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* $\square$

Theorem 2.25. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C$ and $Q(\mathrm{Ricc}, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad A_4 = \ldots = A_m = 0, \tag{2.14}$$

where c and μ are real functions on M such that $\mu \neq 0$ and $c^2 + \widetilde{k} \neq 0$. Moreover,

$$R \cdot C = \frac{\widetilde{k} + c^2}{2\mu^2} Q(\mathrm{Ricc}, g \wedge \mathrm{Ricc}) \; ;$$

and, in the second case, M is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\widetilde{k})$. $\square$

Theorem 2.26. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(\mathrm{Ricc}, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat).* $\square$

Theorem 2.27. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(\mathrm{Ricc}, g \wedge \mathrm{Ricc})$ are linearly dependent if and only if (i) either M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by*

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 2\mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \tag{2.15}$$
$$A_4 = \cdots = A_m = 0,$$

where c and μ are real functions on M such that $\mu \neq 0$; in this second case,

$$H^2 + \widetilde{k} = \frac{2}{n-1}\mu^2 \ \text{ and } \ R \cdot C - C \cdot R = -\frac{n-3}{(n-1)(n-2)} Q(\mathrm{Ricc}, g \wedge \mathrm{Ricc}). \ \square$$

Corollary 2.7. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Suppose $\widetilde{k} \leq 0$. The following conditions are both equivalent.*

(I) With respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \tag{2.16}$$
$$A_4 = \ldots = A_m = 0,$$

where c and μ are real functions on M such that $\mu \neq 0$ and $c^2 = -\widetilde{k}$.

(II) M is Weyl-pseudo-symmetric, i.e., $R \cdot C$ and $Q(g, R)$ are linearly dependent.

(III) $R \cdot C$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent.

(IV) $R \cdot C$ and $Q(\mathrm{Ricc}, R)$ are linearly dependent.

(V) $R \cdot C$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent.

(VI) $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent.

In these cases (I) to (V), M is Weyl-semi-symmetric in $\mathbb{R}^{n+m}(\widetilde{k})$. In the case (VI),

$$R \cdot C - C \cdot R = -\frac{2(n-3)\mu^2}{(n-1)(n-2)} Q(g, R). \ \square$$

Corollary 2.8. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Suppose $\widetilde{k} = 0$. The following conditions are both equivalent.*

(I) M is minimal and with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on M in $\mathbb{R}^{n+m}(\widetilde{k})$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.17}$$
$$A_3 = A_4 = \ldots = A_m = 0,$$

where μ is a real function on M such that $\mu \neq 0$.

(II) M is Weyl-pseudo-symmetric, i.e. $R \cdot C$ and $Q(g, R)$ are linearly dependent.

(III) $R \cdot C$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent.

(IV) $R \cdot C$ and $Q(\mathrm{Ricc}, R)$ are linearly dependent.

(V) $R \cdot C$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent.

(VI) $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent. □

Corollary 2.9. *Let $M = M^n$ be a Wintgen ideal submanifold of codimension m in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$, $n \geq 4$ and $m \geq 1$. Suppose $\widetilde{k} > 0$. Then the following conditions are both equivalent.*

(I) $R \cdot C$ and $Q(g, R)$ are linearly dependent.

(II) $R \cdot C$ and $Q(g, g \wedge \mathrm{Ricc})$ are linearly dependent.

(III) $R \cdot C$ and $Q(\mathrm{Ricc}, R)$ are linearly dependent.

(IV) $R \cdot C$ and $Q(\mathrm{Ricc}, C)$ are linearly dependent.

(V) M is totally umbilical in $\mathbb{R}^{n+m}(\widetilde{k})$ (and hence M is conformally flat). □

2.2. *Proofs of main results*

2.2.1. Introduction We consider mainly the Wintgen ideal submanifold $M = M^n$ of dimension $n > 3$ and of codimension $m \geq 2$ in a real space form $\mathbb{R}^{n+m}(\widetilde{k})$. We note that the second fundamental tensor H satisfies:

$$H^2 = a^2 + b^2 + c^2.$$

As in Theorem A, the shape operator A satisfies the following relations, for all $\beta \geq 4$ and $3 \leq i < j \leq n$:

$$\begin{cases} A_1(E_1) \wedge_g A_1(E_2) = \left(a^2 - \mu^2\right) E_1 \wedge_g E_2, \\ A_2(E_1) \wedge_g A_2(E_2) = (b^2 - \mu^2) E_1 \wedge_g E_2, \\ A_3(E_1) \wedge_g A_3(E_2) = c^2 E_1 \wedge_g E_2, \\ A_\beta(E_1) \wedge_g A_\beta(E_2) = 0, \end{cases} \tag{2.18}$$

$$\begin{cases} A_1(E_1) \wedge_g A_1(E_i) = a^2 E_1 \wedge_g E_i + a\mu E_2 \wedge_g E_i, \\ A_2(E_1) \wedge_g A_2(E_i) = (b^2 + b\mu) E_1 \wedge_g E_i, \\ A_3(E_1) \wedge_g A_3(E_i) = c^2 E_1 \wedge_g E_i, \\ A_\beta(E_1) \wedge_g A_\beta(E_i) = 0, \end{cases} \tag{2.19}$$

$$\begin{cases} A_1(E_2) \wedge_g A_1(E_i) = a\mu E_1 \wedge_g E_i + a^2 E_2 \wedge_g E_i, \\ A_2(E_2) \wedge_g A_2(E_i) = (b^2 - b\mu) E_2 \wedge_g E_i, \\ A_3(E_2) \wedge_g A_3(E_i) = c^2 E_2 \wedge_g E_i, \\ A_\beta(E_2) \wedge_g A_\beta(E_i) = 0, \end{cases} \tag{2.20}$$

$$\begin{cases} A_1(E_i) \wedge_g A_1(E_j) = a^2 E_i \wedge_g E_j, \\ A_2(E_i) \wedge_g A_2(E_j) = b^2 E_i \wedge_g E_j, \\ A_3(E_i) \wedge_g A_3(E_j) = c^2 E_i \wedge_g E_j, \\ A_\beta(E_i) \wedge_g A_\beta(E_j) = 0. \end{cases} \tag{2.21}$$

Due to the relations (2.18) to (2.21), the Riemannian-Christoffel curvature operator $\mathcal{R}$ satisfies the following equalities:

$$\begin{cases} \mathcal{R}(E_1,E_2) = \left(H^2+\widetilde{k}-2\mu^2\right) E_1 \wedge_g E_2, \\ \mathcal{R}(E_1,E_i) = \left(H^2+\widetilde{k}+b\mu\right) E_1 \wedge_g E_i + a\mu E_2 \wedge_g E_i, \\ \mathcal{R}(E_2,E_i) = a\mu E_1 \wedge_g E_i + \left(H^2+\widetilde{k}-b\mu\right) E_2 \wedge_g E_i, \\ \mathcal{R}(E_i,E_j) = (H^2+\widetilde{k}) E_i \wedge_g E_j. \end{cases} \tag{2.22}$$

The local components $R_{uvwt} = R\left(E_u,E_v,E_w,E_t\right)$ of the Riemannian-Christoffel curvature $(0,4)$-tensor R are given by:

$$\begin{cases} R_{1221} = H^2+\widetilde{k}-2\mu^2, \\ R_{1ii1} = H^2+\widetilde{k}+b\mu \quad \text{for} \quad i\geq 3, \\ R_{2ii2} = H^2+\widetilde{k}-b\mu \quad \text{for} \quad i\geq 3, \\ R_{ijji} = H^2+\widetilde{k} \quad \text{for} \quad 3\leq i<j\leq n, \\ R_{1ii2} = a\mu \quad \text{for} \quad 3\leq i\leq n, \end{cases} \tag{2.23}$$

the other values of R_{uvwt} being null.

We denote Ricc the Ricci tensor. We set $S_{uv} = \mathrm{Ricc}\left(E_u,E_v\right)$. Then

$$\begin{cases} S_{11} = (n-1)\left(H^2+\widetilde{k}\right) - 2\mu^2 + (n-2)b\mu, \\ S_{12} = (n-2)a\mu, \\ S_{22} = (n-1)\left(H^2+\widetilde{k}\right) - 2\mu^2 - (n-2)b\mu \quad \text{for} \quad i\geq 3, \\ S_{ii} = (n-1)\left(H^2+\widetilde{k}\right) \quad \text{for} \quad 3\leq i\leq n, \end{cases} \tag{2.24}$$

the other values of S_{uv} being null. Next, computing the Ricci operator $\mathcal{S}$ (associated to the Ricci tensor Ricc) by using the above equalities, we obtain:

$$\begin{cases} \mathcal{S}(E_1) = \mathrm{Ricc}\left(E_1,E_1\right)E_1 + \mathrm{Ricc}\left(E_1,E_2\right)E_2 \\ \qquad\quad = \left[(n-1)\left(H^2+\widetilde{k}\right) - 2\mu^2 + (n-2)b\mu\right] E_1 + \left[(n-2)a\mu\right] E_2, \\ \mathcal{S}(E_2) = \mathrm{Ricc}\left(E_2,E_1\right)E_1 + \mathrm{Ricc}\left(E_2,E_2\right)E_2 \\ \qquad\quad = \left[(n-2)a\mu\right] E_1 + \left[(n-1)\left(H^2+\widetilde{k}\right) - 2\mu^2 - (n-2)b\mu\right] E_2, \\ \mathcal{S}(E_i) = \mathrm{Ricc}\left(E_i,E_1\right)E_1 + \mathrm{Ricc}\left(E_i,E_2\right)E_2 \\ \qquad\quad = \left[(n-1)\left(H^2+\widetilde{k}\right)\right] E_i \quad \text{for} \quad i\geq 3. \end{cases} \tag{2.25}$$

The scalar curvature τ is given by

$$\tau = \sum_{i=1}^{n} \mathrm{Ricc}(E_i,E_i) = n(n-1)\left(H^2+\widetilde{k}\right) - 4\mu^2, \tag{2.26}$$

so that the normalized scalar of M is

$$\rho = \frac{\tau}{n(n-1)} = \left(H^2+\widetilde{k}\right) - \frac{4\mu^2}{n(n-1)}. \tag{2.27}$$

We recall that for any tangent vector fields X, Y, Z, W we have

$$\begin{aligned} (g\wedge \mathrm{Ricc})(X,Y;Z,W) &= g(X,W)\mathrm{Ricc}(Y,Z) + \mathrm{Ricc}(X,W)g(Y,Z) \\ &\quad - g(X,Z)\mathrm{Ricc}(Y,W) - \mathrm{Ricc}(X,Z)g(Y,W) \\ &= g\left(\left(X\wedge_g \mathcal{S}(Y) + \mathcal{S}(X)\wedge_g Y\right)(Z),W\right). \end{aligned}$$

If we set $(g \wedge \mathrm{Ricc})_{uvst} = (g \wedge \mathrm{Ricc})\,(E_u, E_v, E_s, E_t)$, for any $u \geq 1$, $v \geq 1$, $s \geq 1$ and $t \geq 1$, then

$$\begin{cases} (g \wedge \mathrm{Ricc})_{1221} = 2(n-1)\left(H^2 + \widetilde{k}\right) - 4\mu^2, \\ (g \wedge \mathrm{Ricc})_{1ii1} = 2(n-1)\left(H^2 + \widetilde{k}\right) - 2\mu^2 + (n-2)b\mu, \\ (g \wedge \mathrm{Ricc})_{2ii2} = 2(n-1)(\widetilde{k} + H^2) - 2\mu^2 - (n-2)b\mu \quad \text{for} \quad i \geq 3, \\ (g \wedge \mathrm{Ricc})_{ijji} = 2(n-1)\left(H^2 + \widetilde{k}\right) \quad \text{for} \quad 3 \leq i \leq n, \end{cases} \tag{2.28}$$

the other values of $(g \wedge \mathrm{Ricc})_{uvst}$ being null.

Let C be the Weyl conformal curvature tensor. Setting $C_{uvwt} = C\,(E_u, E_v, E_w, E_t)$, we get

$$\begin{cases} C_{1221} = -\dfrac{2(n-3)\mu^2}{n-1}, \\ C_{1ii1} = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)} \quad \text{for} \quad i \geq 3, \\ C_{2ii2} = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)} \quad \text{for} \quad i \geq 3, \\ C_{ijji} = -\dfrac{4\mu^2}{(n-1)(n-2)} \quad \text{for} \quad 3 \leq i < j \leq n, \end{cases} \tag{2.29}$$

the other values of C_{uvwt} being null.

2.2.2. *Proofs of theorems* **2.1, 2.2, 2.3, 2.4** *and corollary* **2.1** Now we compute the local components of the tensors $R \cdot C$, $C \cdot R$ and $R \cdot C - C \cdot R$ of a Wintgen ideal submanifold. Let Z, W be tangent vector fields.

Firstly, we compute the local components of the tensor $R \cdot C$. For any index $i \in \{3, \ldots, n\}$,

$$\begin{cases} (R \cdot C)\,(E_1, E_2, Z, W; E_1, E_i) = \dfrac{2(n-3)\mu^3 a}{n-2} \left\langle (E_1 \wedge_g E_i)\,(Z), W \right\rangle \\ -\dfrac{2(n-3)\mu^2}{n-2}\left(H^2 + \widetilde{k} + \mu b\right) \left\langle (E_2 \wedge_g E_i)\,(Z), W \right\rangle, \\ (R \cdot C)\,(E_1, E_2, Z, W; E_2, E_i) = \dfrac{2(n-3)\mu^2}{n-2}\left(H^2 + \widetilde{k} - \mu b\right) \left\langle (E_1 \wedge_g E_i)\,(Z), W \right\rangle \\ -\dfrac{2(n-3)\mu^3 a}{n-2} \left\langle (E_2 \wedge_g E_i)\,(Z), W \right\rangle. \end{cases} \tag{2.30}$$

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,

$$\begin{cases} (R \cdot C)\,(E_1, E_i, Z, W; E_1, E_2) = 0, \\ (R \cdot C)\,(E_1, E_i, Z, W; E_1, E_j) = \dfrac{2\mu^2}{n-2}\left(H^2 + \widetilde{k} + \mu b\right) \left\langle (E_i \wedge_g E_j)\,(Z), W \right\rangle, \\ (R \cdot C)\,(E_1, E_i, Z, W; E_2, E_i) = \dfrac{2(n-3)\mu^2}{n-2}\left(H^2 + \widetilde{k} - \mu b\right) \left\langle (E_1 \wedge_g E_2)\,(Z), W \right\rangle, \\ (R \cdot C)\,(E_1, E_i, Z, W; E_i, E_j) = 0. \end{cases} \tag{2.31}$$

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,

$$\begin{cases} (R \cdot C)\,(E_2, E_i, Z, W; E_2, E_1) = 0, \\ (R \cdot C)\,(E_2, E_i, Z, W; E_2, E_j) = \dfrac{2\mu^2}{n-2}\left(H^2 + \widetilde{k} - \mu b\right) \left\langle (E_i \wedge_g E_j)\,(Z), W \right\rangle, \\ (R \cdot C)\,(E_2, E_i, Z, W; E_i, E_1) = \dfrac{2(n-3)\mu^2}{n-2}\left[H^2 + \widetilde{k} + \mu b\right] \left\langle (E_1 \wedge_g E_2)\,(Z), W \right\rangle, \\ (R \cdot C)\,(E_2, E_i, Z, W; E_i, E_j) = 0. \end{cases} \tag{2.32}$$

For any indexes i, j, $k \in \{3, \ldots, n\}$ such that $i \neq j$, $i \neq k$ and $j \neq k$,

$$\begin{cases} (R \cdot C)\,(E_i, E_j, Z, W; E_i, E_1) = \dfrac{2\mu^2}{n-2}\left(H^2 + \widetilde{k} + \mu b\right)\langle (E_1 \wedge_g E_j)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{2\mu^2 a}{n-2}\langle (E_2 \wedge_g E_j)\,(Z), W\rangle\,, \\ (R \cdot C)\,(E_i, E_j, Z, W; E_i, E_2) = \dfrac{2\mu^3 a}{n-2}\langle (E_1 \wedge_g E_j)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{2\mu^2}{n-2}\left(H^2 + \widetilde{k} - \mu b\right)\langle (E_2 \wedge_g E_j)\,(Z), W\rangle\,, \\ (R \cdot C)\,(E_i, E_j, Z, W; E_i, E_k) = 0, \\ (R \cdot C)\,(E_i, E_j, Z, W; E_j, E_k) = 0. \end{cases} \tag{2.33}$$

Secondly, we compute the local components of the tensor $C \cdot R$. For any index $i \in \{3, \ldots, n\}$,

$$\begin{cases} (C \cdot R)\,(E_1, E_2, Z, W; E_1, E_i) = \dfrac{2(n-3)\mu^3 a}{(n-1)(n-2)}\langle (E_1 \wedge_g E_i)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left(2\mu^2 - \mu b\right)\langle (E_2 \wedge_g E_i)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_1, E_2, Z, W; E_2, E_i) = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left(2\mu^2 + \mu b\right)\langle (E_1 \wedge_g E_i)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{2(n-3)\mu^3 a}{(n-1)(n-2)}\langle (E_2 \wedge_g E_i)\,(Z), W\rangle\,. \end{cases} \tag{2.34}$$

For any indexes i, $j \in \{3, \ldots, n\}$ such that $i \neq j$,

$$\begin{cases} (C \cdot R)\,(E_1, E_i, Z, W; E_1, E_2) = -\dfrac{4(n-3)\mu^3 a}{n-1}\langle (E_1 \wedge_g E_i)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{4(n-3)\mu^3 b}{n-1}\langle (E_2 \wedge_g E_i)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_1, E_i, Z, W; E_1, E_j) = \dfrac{2(n-3)\mu^3 b}{(n-1)(n-2)}\langle (E_i \wedge_g E_j)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_1, E_i, Z, W; E_2, E_i) = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left(2\mu^2 + \mu b\right)\langle (E_1 \wedge_g E_2)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_1, E_i, Z, W; E_i, E_j) = 0. \end{cases} \tag{2.35}$$

For any indexes i, $j \in \{3, \ldots, n\}$ such that $i \neq j$,

$$\begin{cases} (C \cdot R)\,(E_2, E_i, Z, W; E_2, E_1) = -\dfrac{4(n-3)\mu^3 b}{n-1}\langle (E_1 \wedge_g E_i)\,(Z), W\rangle \\ \qquad\qquad - \dfrac{4(n-3)\mu^3 a}{n-1}\langle (E_2 \wedge_g E_i)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_2, E_i, Z, W; E_2, E_j) = -\dfrac{2(n-3)\mu^3 b}{(n-1)(n-2)}\langle (E_i \wedge_g E_j)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_2, E_i, Z, W; E_i, E_1) = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left(2\mu^2 - b\mu\right)\langle (E_1 \wedge_g E_2)\,(Z), W\rangle\,, \\ (C \cdot R)\,(E_2, E_i, Z, W; E_i, E_j) = 0. \end{cases} \tag{2.36}$$

For any indexes $i, j, k \in \{3, \ldots, n\}$ such that $i \neq j$, $i \neq k$ and $j \neq k$,

$$\begin{cases} (C \cdot R)(E_i, E_j, Z, W; E_i, E_1) = \dfrac{2(n-3)\mu^3 b}{(n-1)(n-2)} \left\langle (E_1 \wedge_g E_j)(Z), W \right\rangle \\ \qquad\qquad\qquad\qquad + \dfrac{2(n-3)\mu^3 a}{(n-1)(n-2)} \left\langle (E_2 \wedge_g E_j)(Z), W \right\rangle, \\ (C \cdot R)(E_i, E_j, Z, W; E_i, E_2) = \dfrac{2(n-3)\mu^3 a}{(n-1)(n-2)} \left\langle (E_1 \wedge_g E_i)(Z), W \right\rangle \\ \qquad\qquad\qquad\qquad - \dfrac{2(n-3)\mu^3 b}{(n-1)(n-2)} \left\langle (E_2 \wedge_g E_i)(Z), W \right\rangle, \\ (C \cdot R)(E_i, E_j, Z, W; E_i, E_k) = 0, \\ (C \cdot R)(E_i, E_j, Z, W; E_j, E_k) = 0. \end{cases} \tag{2.37}$$

Thirdly, we compute the local components of the tensor $R \cdot C - C \cdot R$ on a Wintgen ideal submanifold. For any index $i \in \{3, \ldots, n\}$,

$$\begin{cases} (R \cdot C - C \cdot R)(E_1, E_2, Z, W\,; E_1, E_i) = \dfrac{2(n-3)\mu^3 a}{n-1} \left\langle (E_1 \wedge_g E_i)(Z), W \right\rangle \\ - \dfrac{2(n-3)\mu^2}{(n-1)(n-2)} \left((n-1)\left(H^2 + \widetilde{k}\right) + (n-2)\mu b + 2\mu^2 \right) \left\langle (E_2 \wedge_g E_i)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_1, E_2, Z, W\,; E_2, E_i) = -\dfrac{2n(n-3)\mu^3 a}{(n-1)(n-2)} \left\langle (E_2 \wedge_g E_i)(Z), W \right\rangle \\ + \dfrac{2(n-3)\mu^2}{(n-1)(n-2)} \left((n-1)\left(H^2 + \widetilde{k}\right) - n\mu b - 2\mu^2 \right) \left\langle (E_1 \wedge_g E_i)(Z), W \right\rangle. \end{cases} \tag{2.38}$$

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$\begin{cases} (R \cdot C - C \cdot R)(E_1, E_i, Z, W\,; E_1, E_2) \\ = \dfrac{4(n-3)\mu^3 a}{n-1} \left\langle (E_1 \wedge_g E_i)(Z), W \right\rangle - \dfrac{4(n-3)\mu^3 b}{n-1} \left\langle (E_2 \wedge_g E_i)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_1, E_i, Z, W\,; E_1, E_j) \\ = \dfrac{2\mu^2}{(n-1)(n-2)} \left((n-1)\left(H^2 + \widetilde{k}\right) + 2\mu b \right) \left\langle (E_i \wedge_g E_j)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_1, E_i, Z, W\,; E_2, E_i) \\ = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)} \left((n-1)\left(H^2 + \widetilde{k}\right) - n\mu b - 2\mu^2 \right) \left\langle (E_1 \wedge_g E_2)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_1, E_i, Z, W\,; E_i, E_j) = 0. \end{cases} \tag{2.39}$$

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$\begin{cases} (R \cdot C - C \cdot R)(E_2, E_i, Z, W\,; E_2, E_1) \\ = \dfrac{4(n-3)\mu^3 b}{n-1} \left\langle (E_1 \wedge_g E_i)(Z), W \right\rangle + \dfrac{4(n-3)\mu^3 a}{n-1} \left\langle (E_2 \wedge_g E_i)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_2, E_i, Z, W\,; E_2, E_j) \\ = \dfrac{2\mu^2}{(n-1)(n-2)} \left((n-1)\left(H^2 + \widetilde{k}\right) - 2\mu b \right) \left\langle (E_i \wedge_g E_j)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_2, E_i, Z, W\,; E_i, E_1) \\ = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)} \left((n-1)(H^2 + \widetilde{k}) + n\mu b \right) \left\langle (E_1 \wedge_g E_2)(Z), W \right\rangle, \\ (R \cdot C - C \cdot R)(E_2, E_i, Z, W\,; E_i, E_j) = 0. \end{cases} \tag{2.40}$$

For any different indexes $i, j, k \in \{3, \dots, n\}$,

$$\begin{cases} (R\cdot C - C\cdot R)\,(E_i, E_j, Z, W\,; E_i, E_1) = \dfrac{2\mu^2}{(n-1)(n-2)}\left((n-1)\left(H^2+\widetilde{k}\right)+2\mu b\right)\langle (E_1\wedge_g E_i)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{4\mu^3 a}{(n-1)(n-2)}\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,, \\ (R\cdot C - C\cdot R)\,(E_i, E_j, Z, W\,; E_i, E_2) = \dfrac{4\mu^3 a}{(n-1)(n-2)}\langle (E_1\wedge_g E_i)\,(Z), W\rangle \\ \qquad\qquad + \dfrac{2\mu^2}{(n-1)(n-2)}\left((n-1)\left(H^2+\widetilde{k}\right)-2\mu b\right)\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,, \\ (R\cdot C - C\cdot R)\,(E_i, E_j, Z, W\,; E_i, E_k) = 0, \\ (R\cdot C - C\cdot R)\,(E_i, E_j, Z, W\,; E_j, E_k) = 0. \end{cases} \tag{2.41}$$

In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), and (2.38), (2.39), (2.40), (2.41), we prove theorems 2.1, 2.2, 2.3, 2.4 and corollary 2.1 above.

2.2.3. Proofs of theorems **2.5, 2.6, 2.7, 2.8, 2.9** *and corollaries* **2.2, 2.3** We compute the local components of the tensor $Q(g,R)$ of a Wintgen ideal submanifold. Let Z, W be tangent vector fields of M. For any index $i \in \{3, \dots, n\}$,

$$\begin{cases} Q\,(g,R)\,(E_1, E_2, Z, W; E_1, E_i) = \left(b\mu - 2\mu^2\right)\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_1, E_2, Z, W; E_2, E_i) = \left(b\mu + 2\mu^2\right)\langle (E_1\wedge_g E_i)\,(Z), W\rangle - a\mu\,\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,. \end{cases} \tag{2.42}$$

For any indexes $i, j \in \{3, \dots, n\}$ such that $i \neq j$,

$$\begin{cases} Q\,(g,R)\,(E_1, E_i, Z, W; E_1, E_2) = a\mu\,\langle (E_1\wedge_g E_i)\,(Z), W\rangle - 2b\mu\,\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_1, E_i, Z, W; E_1, E_j) = b\mu\,\langle (E_i\wedge_g E_j)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_1, E_i, Z, W; E_2, E_i) = \left(b\mu + 2\mu^2\right)\langle (E_1\wedge_g E_2)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_1, E_i, Z, W; E_i, E_j) = 0. \end{cases} \tag{2.43}$$

For any indexes $i, j \in \{3, \dots, n\}$ such that $i \neq j$,

$$\begin{cases} Q\,(g,R)\,(E_2, E_i, Z, W; E_1, E_2) = 2b\mu\,\langle (E_1\wedge_g E_i)\,(Z), W\rangle - a\mu\,\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_2, E_i, Z, W; E_2, E_j) = -b\mu\,\langle (E_i\wedge_g E_j)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_2, E_i, Z, W; E_i, E_1) = \left(-b\mu + 2\mu^2\right)\langle (E_1\wedge_g E_2)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_2, E_i, Z, W; E_i, E_j) = 0. \end{cases} \tag{2.44}$$

For any indexes $i, j, k \in \{3, \dots, n\}$ such that $i \neq j$, $i \neq k$ and $j \neq k$,

$$\begin{cases} Q\,(g,R)\,(E_i, E_j, Z, W; E_i, E_1) = b\mu\,\langle (E_1\wedge_g E_j)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_i, E_j, Z, W; E_i, E_2) = a\mu\,\langle (E_1\wedge_g E_j)\,(Z), W\rangle - b\mu\,\langle (E_2\wedge_g E_j)\,(Z), W\rangle\,, \\ Q\,(g,R)\,(E_i, E_j, Z, W; E_i, E_k) = 0, \\ Q\,(g,R)\,(E_i, E_j, Z, W; E_j, E_k) = 0. \end{cases} \tag{2.45}$$

In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) , (2.41) and (2.42), (2.43), (2.44), (2.45), we prove theorems 2.5, 2.6, 2.7, 2.8, 2.9 and corollaries 2.2, 2.3 above.

2.2.4. Proofs of theorems **2.10, 2.11, 2.12** Now we compute the local components of the tensor $Q(g,C)$ for the considered Wintgen ideal submanifold. Let X, Y, Z, W be tangent vector fields. For any index $i \in \{3, \dots, n\}$,

$$\begin{cases} Q\,(g,C)\,(E_1, E_2, Z, W; E_1, E_i) = -\dfrac{2(n-3)\mu^2}{n-2}\langle (E_2\wedge_g E_i)\,(Z), W\rangle\,, \\ Q\,(g,C)\,(E_1, E_2, Z, W; E_2, E_i) = \dfrac{2(n-3)\mu^2}{n-2}\langle (E_1\wedge_g E_i)\,(Z), W\rangle\,. \end{cases} \tag{2.46}$$

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,

$$\begin{cases} Q(g,C)(E_1,E_i,Z,W;E_1,E_2) = 0, \\ Q(g,C)(E_1,E_i,Z,W;E_1,E_j) = \dfrac{2\mu^2}{n-2}\left\langle (E_i \wedge_g E_j)(Z), W\right\rangle, \\ Q(g,C)(E_1,E_i,Z,W;E_2,E_i) = \dfrac{2(n-3)\mu^2}{n-2}\left\langle (E_1 \wedge_g E_2)(Z), W\right\rangle, \\ Q(g,C)(E_1,E_i,Z,W;E_i,E_j) = 0. \end{cases} \tag{2.47}$$

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,

$$\begin{cases} Q(g,C)(E_2,E_i,Z,W;E_2,E_1) = 0, \\ Q(g,C)(E_2,E_i,Z,W;E_2,E_j) = \dfrac{2\mu^2}{n-2}\left\langle (E_i \wedge_g E_j)(Z), W\right\rangle, \\ Q(g,C)(E_2,E_i,Z,W;E_i,E_1) = \dfrac{2(n-3)\mu^2}{n-2}\left\langle (E_1 \wedge_g E_2)(Z), W\right\rangle, \\ Q(g,C)(E_2,E_i,Z,W;E_i,E_1) = 0. \end{cases} \tag{2.48}$$

For any indexes $i, j, k \in \{3, \ldots, n\}$ such that $i \neq j$, $i \neq k$ and $j \neq k$,

$$\begin{cases} Q(g,C)(E_i,E_j,Z,W;E_i,E_1) = \dfrac{2\mu^2}{n-2}\left\langle (E_1 \wedge_g E_j)(Z), W\right\rangle, \\ Q(g,C)(E_i,E_j,Z,W;E_i,E_2) = \dfrac{2\mu^2}{n-2}\left\langle (E_2 \wedge_g E_j)(Z), W\right\rangle, \\ Q(g,C)(X,Y,Z,W;E_i,E_k) = 0, \\ Q(g,C)(X,Y,Z,W;E_j,E_k) = 0. \end{cases} \tag{2.49}$$

In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) , (2.41) and (2.46), (2.47), (2.48), (2.49), we prove theorems 2.10, 2.11, 2.12.

2.2.5. Proofs of theorems **2.13, 2.14, 2.15, 2.16** *and corollary* **2.4** We compute the local components of the tensor $Q(g, g \wedge \mathrm{Ricc})$ of a Wintgen ideal submanifold. Let X, Y, Z, W be tangent vector fields of M. For any index $i \in \{3, \ldots, n\}$,

$$\begin{cases} Q(g, g \wedge \mathrm{Ricc})(E_1,E_2,Z,W;E_1,E_i) = \left((n-2)b\mu - 2\mu^2\right)\left\langle (E_2 \wedge_g E_i)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_1,E_2,Z,W;E_2,E_i) = \left((n-2)b\mu + 2\mu^2\right)\left\langle (E_1 \wedge_g E_i)(Z), W\right\rangle. \end{cases} \tag{2.50}$$

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$\begin{cases} Q(g, g \wedge \mathrm{Ricc})(E_1,E_i,Z,W;E_1,E_2) = -2(n-2)b\mu\left\langle (E_2 \wedge_g E_i)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_1,E_i,Z,W;E_1,E_j) = \left(-(n-2)b\mu + 2\mu^2\right)\left\langle (E_i \wedge_g E_j)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_1,E_i,Z,W;E_2,E_i) = \left((n-2)b\mu + 2\mu^2\right)\left\langle (E_1 \wedge_g E_2)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_1,E_i,Z,W;E_i,E_j) = 0. \end{cases} \tag{2.51}$$

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$\begin{cases} Q(g, g \wedge \mathrm{Ricc})(E_2,E_i,Z,W;E_2,E_1) = 2(n-2)b\mu\left\langle (E_1 \wedge_g E_i)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_2,E_i,Z,W;E_2,E_j) = \left(-(n-2)b\mu - 2\mu^2\right)\left\langle (E_i \wedge_g E_j)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_2,E_i,Z,W;E_i,E_1) = \left(-(n-2)b\mu + 2\mu^2\right)\left\langle (E_1 \wedge_g E_2)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_2,E_i,Z,W;E_i,E_j) = 0. \end{cases} \tag{2.52}$$

For any different indexes $i, j, k \in \{3, \ldots, n\}$,

$$\begin{cases} Q(g, g \wedge \mathrm{Ricc})(E_i,E_j,Z,W;E_i,E_1) = \left((n-2)b\mu - 2\mu^2\right)\left\langle (E_1 \wedge_g E_j)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_i,E_j,Z,W;E_i,E_2) = \left(-(n-2)b\mu - 2\mu^2\right)\left\langle (E_2 \wedge_g E_i)(Z), W\right\rangle, \\ Q(g, g \wedge \mathrm{Ricc})(E_i,E_j,Z,W;E_i,E_j) = 0, \\ Q(g, g \wedge \mathrm{Ricc})(E_i,E_j,Z,W;E_j,E_k) = 0. \end{cases} \tag{2.53}$$

In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) , (2.41) and (2.50), (2.51), (2.52), (2.53), we proved theorems 2.13, 2.14, 2.15, 2.16 and corollary 2.4 above.

2.2.6. Proofs of theorems **2.17, 2.18, 2.19, 2.20** *and corollary* **2.5** We compute the local components of the tensor $Q(\mathrm{Ricc}, R)$ for a Wintgen ideal submanifold. Let X, Y, Z, W be the tangent vector fields of M. For any index $i \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, R\right)\left(E_1, E_2, Z, W; E_1, E_i\right) = a\mu\left(-\left(H^2+\widetilde{k}\right)+2\mu^2\right)\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ +\left((n-2)(a^2+b^2)\mu^2 - 2b\mu^3 - (2n-4)\,\mu^2\left(H^2+\widetilde{k}\right)+b\mu\left(H^2+\widetilde{k}\right)\right)\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_1, E_2, Z, W; E_2, E_i\right) = \left[-2b\mu^3 - (n-2)(a^2+b^2)\mu^2\right. \\ \left.+(2n-4)\mu^2\left(H^2+\widetilde{k}\right)+b\mu\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ +\left(-2a\mu^3 + a\mu\left(H^2+\widetilde{k}\right)\right)\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle. \end{cases} \tag{2.54}$$

For any different indexes i, $j \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, R\right)\left(E_1, E_i, Z, W; E_1, E_2\right) = 2a\mu\left(\left(H^2+\widetilde{k}\right)-2\mu^2\right)\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ \qquad + 2b\mu\left(-\left(H^2+\widetilde{k}\right)+2\mu^2\right)\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_1, E_i, Z, W; E_1, E_j\right) = \left(H^2+\widetilde{k}\right)\left(b\mu+2\mu^2\right)\left\langle\left(E_i \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_1, E_i, Z, W; E_2, E_i\right) = \left[-(n-2)\left(a^2+b^2\right)\mu^2 - 2b\mu^3\right. \\ \qquad \left.+(2n-4)\mu^2(H^2+\widetilde{k})+b\mu(H^2+\widetilde{k})\right]\left\langle\left(E_1 \wedge_g E_2\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_1, E_i, Z, W; E_i, E_j\right) = 0. \end{cases} \tag{2.55}$$

For any different indexes i, $j \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, R\right)\left(E_2, E_i, Z, W; E_2, E_1\right) = \left(2b\mu\left(H^2+\widetilde{k}\right)-4b\mu^3\right)\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ \qquad + \left(2a\mu\left(H^2+\widetilde{k}\right)-4a\mu^3\right)\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_2, E_i, Z, W; E_2, E_j\right) = \left(-b\mu+2\mu^2\right)\left(H^2+\widetilde{k}\right)\left\langle\left(E_i \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_2, E_i, Z, W; E_i, E_1\right) = \left[2b\mu^3 - (n-2)(a^2+b^2)\mu^2\right. \\ \qquad \left.+(2n-4)\mu^2\left(H^2+\widetilde{k}\right)-b\mu\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_1 \wedge_g E_2\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_2, E_i, Z, W; E_i, E_j\right) = 0. \end{cases} \tag{2.56}$$

For any different indexes i, j, $k \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, R\right)\left(E_i, E_j, Z, W; E_i, E_1\right) = \left(b\mu+2\mu^2\right)\left(H^2+\widetilde{k}\right)\left\langle\left(E_1 \wedge_g E_j\right)(Z), W\right\rangle \\ \qquad + a\mu\left(H^2+\widetilde{k}\right)\left\langle\left(E_2 \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_i, E_j, Z, W; E_i, E_2\right) = a\mu\left(H^2+\widetilde{k}\right)\left\langle\left(E_1 \wedge_g E_j\right)(Z), W\right\rangle \\ \qquad + \left(-b\mu+2\mu^2\right)\left(H^2+\widetilde{k}\right)\left\langle\left(E_2 \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_i, E_j, Z, W; E_i, E_k\right) = 0, \\ Q\left(\mathrm{Ricc}, R\right)\left(E_i, E_j, Z, W; E_j, E_k\right) = 0. \end{cases} \tag{2.57}$$

From the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) , (2.41) and (2.54), (2.55), (2.56), (2.57), we proved theorems 2.17, 2.18, 2.19, 2.20 and corollary 2.5 above.

2.2.7. Proofs of theorems **2.21, 2.22, 2.23, 2.24** *and corollary* **2.6** We compute the local components of the tensor $Q(\mathrm{Ricc}, C)$ for a Wintgen ideal submanifold. Let X, Y, Z, W be the tangent vector fields of M. For any index $i \in \{3, \ldots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, C\right)\left(E_1, E_2, Z, W; E_1, E_i\right) = \dfrac{2(n-3)a\mu^3}{n-1}\left\langle \left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ -\dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)+(n-2)b\mu-2\mu^2\right)\left\langle \left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_1, E_2, Z, W; E_2, E_i\right) = -\dfrac{2(n-3)a\mu^3}{n-1}\left\langle \left(E_2 \wedge_g E_i\right)(Z) W\right\rangle \\ +\dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)-(n-2)b\mu-2\mu^2\right)\left\langle \left(E_1 \wedge_g E_i\right)(Z), W\right\rangle. \end{cases} \tag{2.58}$$

For any different indexes i, $j \in \{3, \ldots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, C\right)\left(E_1, E_i, Z, W; E_1, E_2\right) \\ = -\dfrac{4(n-3)a\mu^3}{n-1}\left\langle \left(E_1 \wedge_g E_i\right)(Z), W\right\rangle + \dfrac{4(n-3)b\mu^3}{n-1}\left\langle \left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_1, E_i, Z, W; E_1, E_j\right) \\ = \dfrac{2\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)+2(n-2)b\mu-4\mu^2\right)\left\langle \left(E_i \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_1, E_i, Z, W; E_2, E_i\right) \\ = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)-(n-2)b\mu-2\mu^2\right)\left\langle \left(E_1 \wedge_g E_2\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_1, E_i, Z, W; E_i, E_j\right) = -\dfrac{2(n-3)a\mu^3}{n-1}\left\langle \left(E_1 \wedge_g E_2\right)(Z), W\right\rangle. \end{cases} \tag{2.59}$$

For any different indexes i, $j \in \{3, \ldots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, C\right)\left(E_2, E_i, Z, W; E_2, E_1\right) \\ = \dfrac{4(n-3)b\mu^3}{n-1}\left\langle \left(E_1 \wedge_g E_i\right)(Z), W\right\rangle - \dfrac{4(n-3)a\mu^3}{n-1}\left\langle \left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_2, E_i, Z, W; E_2, E_j\right) \\ = \dfrac{2\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)-2(n-2)b\mu-4\mu^2\right)\left\langle \left(E_i \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_2, E_i, Z, W; E_i, E_1\right) \\ = \dfrac{2(n-3)\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)+(n-2)b\mu-2\mu^2\right)\left\langle \left(E_1 \wedge_g E_2\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_2, E_i, Z, W; E_i, E_j\right) = 0. \end{cases} \tag{2.60}$$

For any different indexes i, j, $k \in \{3, \ldots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, C\right)\left(E_i, E_j, Z, W; E_i, E_1\right) = \dfrac{4a\mu^3}{n-1}\left\langle \left(E_2 \wedge_g E_j\right)(Z), W\right\rangle \\ +\dfrac{2\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)+2(n-2)b\mu-4\mu^2\right)\left\langle \left(E_1 \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_i, E_j, Z, W; E_i, E_2\right) = \dfrac{4a\mu^3}{n-1}\left\langle \left(E_1 \wedge_g E_j\right)(Z), W\right\rangle \\ +\dfrac{2\mu^2}{(n-1)(n-2)}\left((n-1)^2\left(H^2+\widetilde{k}\right)-2(n-2)b\mu-4\mu^2\right)\left\langle \left(E_2 \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_i, E_j, Z, W; E_i, E_k\right) = 0, \\ Q\left(\mathrm{Ricc}, C\right)\left(E_i, E_j, Z, W; E_j, E_k\right) = 0. \end{cases} \tag{2.61}$$

From the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) , (2.41) and (2.58), (2.59), (2.60), (2.61), we proved theorems 2.21, 2.22, 2.23, 2.24 and corollary 2.6 above.

2.2.8. Proofs of theorems **2.25, 2.26, 2.27** *and corollaries* **2.7, 2.8, 2.9** We compute the local components of the tensor $Q(\mathrm{Ricc}, g \wedge \mathrm{Ricc})$ for a Wintgen ideal submanifold. Let X, Y, Z, W be the tangent vector fields of M. For any index $i \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_1, E_2, Z, W; E_1, E_i\right) \\ =(n-1)a\mu\left[2(n-1)\left(H^2+\widetilde{k}\right)+(n-2)b\mu-2\mu^2\right]\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ \quad +\left[-4\mu^4+(n-2)^2b^2\mu^2+2(n-1)\mu^2\left(H^2+\widetilde{k}\right)-(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_1, E_2, Z, W; E_2, E_i\right) \\ =\left[4\mu^4-(n-2)^2b^2\mu^2-2(n-1)\mu^2\left(H^2+\widetilde{k}\right)-(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ \quad -(n-2)a\mu\left[2(n-1)\left(H^2+\widetilde{k}\right)-(n-2)b\mu-2\mu^2\right]\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle. \end{cases} \tag{2.62}$$

For any different indexes i, $j \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_1, E_i, Z, W; E_1, E_2\right) \\ =-2(n-2)a\mu\left[2(n-1)\left(H^2+\widetilde{k}\right)+(n-2)b\mu-2\mu^2\right]\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ \quad +2(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_1, E_i, Z, W; E_1, E_j\right) \\ =\left[-(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)+2(n-1)\mu^2\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_i \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_1, E_i, Z, W; E_2, E_j\right) \\ =\left[4\mu^4-(n-2)^2b^2\mu^2-2(n-1)\mu^2\left(H^2+\widetilde{k}\right)-(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_1 \wedge_g E_2\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_1, E_i, Z, W; E_i, E_j\right)=0. \end{cases} \tag{2.63}$$

For any different indexes i, $j \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_2, E_i, Z, W; E_2, E_1\right) \\ =-2(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)\left\langle\left(E_1 \wedge_g E_i\right)(Z), W\right\rangle \\ \quad -2(n-2)a\mu\left[2(n-1)\left(H^2+\widetilde{k}\right)-(n-2)b\mu-2\mu^2\right]\left\langle\left(E_2 \wedge_g E_i\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_2, E_i, Z, W; E_2, E_j\right) \\ =\left[(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)+2(n-1)\mu^2\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_i \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_2, E_i, Z, W; E_j, E_1\right) \\ =\left[4\mu^4-(n-2)b^2\mu^2+(n-1)(n-2)b\mu\left(H^2+\widetilde{k}\right)-2(n-1)\mu^2\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_1 \wedge_g E_2\right)(Z), W\right\rangle \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_2, E_i, Z, W; E_i, E_j\right)=0. \end{cases} \tag{2.64}$$

For any different indexes i, j, $k \in \{3, \dots, n\}$,

$$\begin{cases} Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_i, E_j, Z, W; E_i, E_1\right) \\ =\left[(n-1)\left(H^2+\widetilde{k}\right)\left(-(n-2)b\mu+2\mu^2\right)\right]\left\langle\left(E_1 \wedge_g E_j\right)(Z), W\right\rangle \\ \quad -2(n-1)(n-2)a\mu\left(H^2+\widetilde{k}\right)\left\langle\left(E_2 \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_i, E_j, Z, W; E_i, E_2\right) \\ =-2(n-1)(n-2)a\mu\left(H^2+\widetilde{k}\right)\left\langle\left(E_1 \wedge_g E_j\right)(Z), W\right\rangle \\ \quad +\left[(n-1)(n+2)b\mu\left(H^2+\widetilde{k}\right)+2(n-1)\mu^2\left(H^2+\widetilde{k}\right)\right]\left\langle\left(E_2 \wedge_g E_j\right)(Z), W\right\rangle, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_i, E_j, Z, W; E_i, E_k\right)=0, \\ Q\left(\mathrm{Ricc}, g \wedge \mathrm{Ricc}\right)\left(E_i, E_j, Z, W; E_j, E_k\right)=0. \end{cases} \tag{2.65}$$

Using (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) , (2.41) and (2.62), (2.63), (2.64), (2.65), we can prove theorems 2.25, 2.26, 2.27 and corollaries 2.7, 2.8, 2.9 above.

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