Turk. J. Math. Comput. Sci.
16(2)(2024) 411–418
© MatDer
DOI : 10.47000/tjmcs.1405022

Split (Pre)crossed Modules over R-Algebroids

Osman Avcioğlu

Department of Mathematics, Faculty of Engineering and Natural Sciences, Usak University, 64200, Uşak, Turkey.

Received: 14-12-2023 • Accepted: 07-10-2024

Abstract. In this paper, we introduce (d-)split and (d-)split⁺ epimorphisms and (d-)split and d-split* (pre)crossed modules in the context of algebroids. Moreover, we examine their categorical properties, and in particular, we give a necessary and sufficient condition for a morphism of pre-R-algebroids to be a d-split precrossed module and a necessary and sufficient condition for a d-split* precrossed module to be a crossed module. In addition, we examine the hierarchical relations between the categories obtained and look over some results for split (pre)crossed modules over associative R-algebras, as a reduced case.

2020 AMS Classification: 18A20, 18E05, 18G45

Keywords: Algebroids, crossed modules, split epimorphisms.

1. Introduction

Crossed modules were introduced by Whitehead in his studies [33, 34] on homotopy groups and also studied by Peiffer and by Reidemeister respectively in their studies [26] and [27] on identities among relations. Then, they have become an important concept in both homological and homotopical algebra. In particular, crossed modules over groups and groupoids have been examined over the years in many studies such as, for instance, [10, 12, 30], and their higher dimensional versions in such studies as [11, 14, 16, 32].

The notion of crossed modules over associative algebras goes back to Dedecker and Lue [15] and to the coeval studies [17, 20, 21], in essence. Then, over the time, they have been studied in many works. In particular, crossed modules over associative algebras were basically examined in such studies, for example, as [19, 31], and those over commutative algebras in [5, 6, 28, 29].

R-algebroids, which can be regarded as a generalisation of associative algebras, were initially studied by Mitchell in [22–24] and also by Amgott in [4], and crossed modules over R-algebroids were introduced as a generalisation of crossed modules over associative algebras by Mosa in his thesis [25]. Later on, they have been examined in various studies such as [1–3, 7–9].

Split epimorphisms, on the other hand, have begun to be studied in categorical algebra more occasionally in recent years. Janalidze, for example, used in [18] the equivalence between split epimorphisms and object actions in a semiabelian category, in order eventually to construct an equivalence between internal categories and internal crossed modules, and Böhm obtained in [13], for a fixed class of spans in a monoidal category, an equivalent description of a split epimorphism in terms of a distributive law to get further equivalences.

The basic purpose of this study is to explore the relations between split epimorphisms and (pre)crossed modules in the context of algebroids. To this end, after giving some preliminary data in Sect. 2, we define split epimorphisms

Email address: osman.avcioglu@usak.edu.tr (O. Avcıoğlu)

between arbitrary pre-R-algebroids and introduce the notion of a *split*$^+$ *epimorphism* between pre-R-algebroids having the same object set in Sect. 3. In the same section, we also introduce the notions of *definite split* and *definite split*$^+$ *epimorphisms* of pre-R-algebroids and show that each definite split$^+$ epimorphism with an R-algebroid codomain is a precrossed module. Then, in Sect. 4, we introduce the notions of *split* and *definite split (pre)crossed modules over R-algebroids* and give a necessary and sufficient condition for a morphism $\eta : \mathrm{V} \to \mathrm{A}$ of pre-R-algebroids to be a definite split precrossed module (cf. Theorem 4.4). Next, in the same section, we introduce the notion of a *definite split** *(pre)crossed module over R-algebroids* and examine their properties. Moreover, we examine the hierarchical relations between the categories obtained and illustrate the resultant hierarchical structure with a schematic diagram. Then, in Sect. 5, we explore some further properties of split (pre)crossed modules over R-algebroids and give a necessary and sufficient condition for a definite split* precrossed module $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ to be a definite split crossed module (cf. Theorem 5.5). Finally, in Sect. 6, we give a brief explanation on adjusting the findings to associative R-algebras and state some important results.

Throughout this paper, R will be a fixed commutative ring. Moreover, for convenience, given a map f and an element x in its domain we shall write fx instead of $f(x)$ if there exists no ambiguity.

2. Preliminaries

A great deal of the data below can originally be found in [22, 23, 25], as indicated wherever needed, and also in [4, 24]. Moreover, some further readings, in this context, can be found in [1–3, 7–9] and in related papers.

Definition 2.1 ([22, 23]). A small category in which each homset is an R-module and the composition is R-bilinear is called an *R-algebroid*. Moreover, if the identity-existence axiom is omitted from an R-algebroid structure then the remaining structure is called a *pre-R-algebroid*.

Remark 2.2. Throughout this paper, for a (pre-)R-algebroid A
1. A_0 will denote the object set of A and s and t the source and target functions, respectively;
2. $\mathrm{A}(x, y)$ will denote the homset from x to y;
3. the identity morphism on any $x \in \mathrm{A}_0$, when exists, will be denoted by $1_{\mathrm{A}(x)}$, or just by 1 if there is no ambiguity;
4. $a \in \mathrm{A}$ will mean that a is a morphism in A and aa' will stand for the composition of any $a, a' \in \mathrm{A}$ with $ta = sa'$.

Note from the definition that if A is a (pre-)R-algebroid then its composition is R-bilinear. That is, for all $r \in R$ and $a, a', a'', a''' \in A$ with $ta = sa' = sa''$, $ta' = ta'' = sa'''$ the following equations hold:

$$r \cdot (aa') = (r \cdot a)a' = a(r \cdot a'), \quad a(a' + a'') = aa' + aa'', \quad (a' + a'')a''' = a'a''' + a''a'''.$$

Definition 2.3 ([22, 23]). An R-linear functor between two R-algebroids is called an *R-algebroid morphism* and an assignment between two pre-R-algebroids which satisfies all axioms of an R-linear functor except for the identity-preservation axiom is called a *pre-R-algebroid morphism*.

Then, the following result is clear:

Corollary 2.4. *Each R-algebroid is a pre-R-algebroid and each R-algebroid morphism is a pre-R-algebroid morphism.*

Definition 2.5 ([25]). Given two pre-R-algebroids A and V with $\mathrm{A}_0 = \mathrm{V}_0$, a family of maps defined for all $x, y, z \in \mathrm{A}_0$ as

$$\begin{array}{rcl} \mathrm{V}(x, y) \times \mathrm{A}(y, z) & \to & \mathrm{V}(x, z) \\ (v, a) & \mapsto & v^a \end{array}$$

is called a *right action of* A *on* V if

1.	$v^{a_1 + a_2} = v^{a_1} + v^{a_2}$,	4.	$(v'v)^a = v'v^a$,
2.	$(v_1 + v_2)^a = v_1^a + v_2^a$,	5.	$r \cdot v^a = (r \cdot v)^a = v^{r \cdot a}$,
3.	$(v^a)^{a'} = v^{aa'}$,		

and if $v^{1_{\mathrm{A}(tv)}} = v$, whenever $1_{\mathrm{A}(tv)}$ exists, for all $r \in R$, $a, a', a_1, a_2 \in \mathrm{A}$ and $v, v', v_1, v_2 \in \mathrm{V}$ with $tv' = sv$, $sv_1 = sv_2$, $tv = tv_1 = tv_2 = sa = sa_1 = sa_2$, $ta_1 = ta_2$ and $ta = sa'$. In this case, it can also be said that A *acts on* V *from right*. A *left action of* A *on* V is defined in the same way.

Definition 2.6 ([25])**.** Given two pre-R-algebroids A and V with $A_0 = V_0$, A is said *to act on* V *associatively* or to have an *associative action on* V, or there is said to be an *associative* A*-action on* V if A has a left and a right action on V and if $({}^{a}v)^{a'} = {}^{a}(v^{a'})$ for all $a, a' \in A$ and $v \in V$ with $ta = sv$, $tv = sa'$.

Definition 2.7 ([25])**.** Let V be a pre-R-algebroid and A be an R-algebroid with $A_0 = V_0$ and let A act on V associatively. A pre-R-algebroid morphism $\eta : V \to A$ is called a *precrossed module over R-algebroids* if it satisfies the condition

$$\text{CM1)}\quad \eta\,({}^{a}v) = a\,(\eta v) \quad \text{and} \quad \eta(v^{a'}) = (\eta v)\,a'$$

for all $a, a' \in A$ and $v \in V$ with $ta = sv$ and $tv = sa'$, and $\eta : V \to A$ is called a *crossed module over R-algebroids* if it satisfies both CM1 and the condition

$$\text{CM2)}\quad v^{\eta v'} = vv' = {}^{\eta v}v'$$

for all $v, v' \in V$ with $tv = sv'$.

Remark 2.8. An immediate observation from Definition 2.7 is that each crossed module is in fact a precrossed module. Note also that if $\eta : V \to A$ is a (pre)crossed module over R-algebroids then none of the homsets of V and A is empty, since they are all R-modules. Therefore, for each object $x \in A_0$ there exist a morphism $a \in A$ and a morphism $v \in V$ with $ta = x$ and $tv = sa$, and thus, $\eta x = \eta\,(ta) = \eta\,(t\,(v^a)) = t\,(\eta\,(v^a)) = t\,((\eta v)\,a) = ta = x$, meaning that η is equal to the identity on $A_0\,(= V_0)$.

Definition 2.9 ([25])**.** Given two (pre)crossed modules $\eta : V \to A$ and $\eta' : V' \to A'$ over R-algebroids, a pre-R-algebroid morphism $f : V \to V'$ and an R-algebroid morphism $g : A \to A'$, if the conditions

$$\text{CMM1)}\quad f\,({}^{a}v) = {}^{ga}\,(fv) \quad \text{and} \quad f(v^{a'}) = (fv)^{ga'}$$

$$\text{CMM2)}\quad \eta' f = g\eta$$

are satisfied for all $v \in V$, $a, a' \in A$ with $ta = sv$ and $tv = sa'$, then the pair (f, g) is called a *(pre)crossed module morphism*, *over R-algebroids*, from $\eta : V \to A$ to $\eta' : V' \to A'$.

Remark 2.10. If (f, g) is a (pre)crossed module morphism from $\eta : V \to A$ to $\eta' : V' \to A'$ then $fx = \eta' fx = g\eta x = gx$ for all $x \in A_0$, since η and η' are equal to the identities on object sets by Remark 2.8, meaning that $f = g$ on A_0.

Proposition 2.11. *All precrossed modules over R-algebroids form with their morphisms a category, denoted by* PXAlg$\,(R)$*, in which the composition is defined pointwisely by $(f', g')\,(f, g) = (f'f, g'g)$, where $f'f$ and $g'g$ are the composite pre-R-algebroid and R-algebroid morphisms, respectively. Likewise, with the same composition all crossed modules over R-algebroids and their morphisms form the full subcategory* XAlg$\,(R)$ *of* PXAlg$\,(R)$*.*

3. Split Epimorphisms of Pre-R-Algebroids

In this section, we first define *split epimorphisms* between arbitrary pre-R-algebroids and then introduce the notion of a *split$^+$ epimorphism* between pre-R-algebroids having the same object set. Next, we introduce the notions of *definite split* and *definite split$^+$ epimorphism of pre-R-algebroids* and show that a definite split$^+$ epimorphism with an R-algebroid codomain is a precrossed module:

Definition 3.1. Let A and V be two pre-R-algebroids, and $\alpha : V \to A$ and $e : A \to V$ be two pre-R-algebroid morphisms. If $\alpha e = id_A$ then α is called a *split epimorphism of pre-R-algebroids* and e is called a *splitting* of α.

As stated in the introduction, our ultimate aim in this paper is to explore the relations between split epimorphisms and (pre)crossed modules in the context of algebroids. But note on the one hand, from Definition 2.7 and Remark 2.8, that if $\eta : V \to A$ is a (pre)crossed module over R-algebroids then V and A have the same object set and η is equal to the identity on $A_0\,(= V_0)$, and on the other hand, from Definition 3.1, that if $\alpha : V \to A$ is a split epimorphism of pre-R-algebroids then V and A do not need to have the same object set and even if their object sets are equal α does not need to be the identity on the common object set. Therefore, in this study, we shall focus on the following special type of split epimorphisms:

Definition 3.2. Let $\alpha : V \to A$ be a split epimorphism of pre-R-algebroids. If $V_0 = A_0$ and if α is equal to the identity on A_0, then we shall call α a *split$^+$ epimorphism* of pre-R-algebroids.

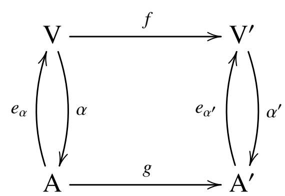
Corollary 3.3. *If $\alpha : \mathrm{V} \to \mathrm{A}$ is a split$^+$ epimorphism of pre-R-algebroids then any splitting of α is equal to the identity on* A_0.

Proof. If e is a splitting of α, then $e(x) = \alpha e(x) = id_{\mathrm{A}}(x) = x$ for all $x \in \mathrm{A}_0$. □

In line with our purposes in this paper, we shall show below in this section that a split$^+$ epimorphism with an R-algebroid codomain forms with one of its splitting morphisms a precrossed module, over R-algebroids, in which the associative action is determined by the splitting morphism. But, a potential problem here is that a split$^+$ epimorphism may have more than one splitting morphism, as it is clear from Definition 3.1 and Definition 3.2, and this may cause an ambiguity. That is, for instance, if $\alpha : \mathrm{V} \to \mathrm{A}$ is a split$^+$ epimorphism and if e and e' are two splittings of α then α determines two different precrossed module structures with e and e' separately. Therefore, in order to avoid such an ambiguity, we introduce the following customized definition of split epimorphisms:

Definition 3.4. A split (split$^+$) epimorphism, of pre-R-algebroids, which has just one splitting morphism or of which just one splitting morphism is taken into consideration will be called a *definite split* (*definite split*$^+$) *epimorphism* or shortly a *d-split* (*d-split*$^+$) *epimorphism* of pre-R-algebroids and we shall use the ordered quadruple $(\mathrm{V}, \mathrm{A}, \alpha; e_\alpha)$ to mean that $\alpha : \mathrm{V} \to \mathrm{A}$ is a d-split (d-split$^+$) epimorphism with the considered splitting morphism e_α.

Definition 3.5. Let $(\mathrm{V}, \mathrm{A}, \alpha; e_\alpha)$ and $(\mathrm{V}', \mathrm{A}', \alpha'; e_{\alpha'})$ be two d-split (d-split$^+$) epimorphisms of pre-R-algebroids. A pair (f, g) of pre-R-algebroid morphisms $f : \mathrm{V} \to \mathrm{V}'$ and $g : \mathrm{A} \to \mathrm{A}'$ is called a *morphism of d-split (d-split$^+$) epimorphisms* or shortly a *d-split-epi (d-split$^+$epi) morphism*, of pre-R-algebroids, from $(\mathrm{V}, \mathrm{A}, \alpha; e_\alpha)$ to $(\mathrm{V}', \mathrm{A}', \alpha'; e_{\alpha'})$ if $\alpha' f = g\alpha$ and $f e_\alpha = e_{\alpha'} g$:



We shall write $(f, g) : (\mathrm{V}, \mathrm{A}, \alpha; e_\alpha) \to (\mathrm{V}', \mathrm{A}', \alpha'; e_{\alpha'})$ to mean that (f, g) is a d-split-epi, or a d-split$^+$epi, morphism from $(\mathrm{V}, \mathrm{A}, \alpha; e_\alpha)$ to $(\mathrm{V}', \mathrm{A}', \alpha'; e_{\alpha'})$.

Proposition 3.6. *If the pair (f, g) is a d-split-epi (d-split$^+$epi) morphism, of pre-R-algebroids, from* $(\mathrm{V}, \mathrm{A}, \alpha; e_\alpha)$ *to* $(\mathrm{V}', \mathrm{A}', \alpha'; e_{\alpha'})$*, then* $g = \alpha' f e_\alpha$.

Proof. Using the equalities $f e_\alpha = e_{\alpha'} g$ and $\alpha' e_{\alpha'} = id_{\mathrm{A}'}$

$$g = id_{\mathrm{A}'} g = \alpha' e_{\alpha'} g = \alpha' f e_\alpha$$

as required. □

Proposition 3.7. *Like (pre)crossed module morphisms, two composible d-split-epi (d-split$^+$epi) morphisms (f, g) and (f', g') can be composed pointwisely and with this composition all d-split epimorphisms of pre-R-algebroids and their morphisms form a category, which we shall denote by* DSplitEpiPAlg (R)*, and all d-split$^+$ epimorphisms of pre-R-algebroids and their morphisms form its full subcategory* DSplit$^+$EpiPAlg (R).

Lemma 3.8. *A d-split$^+$ epimorphism, of pre-R-algebroids, with an R-algebroid codomain is a precrossed module over R-algebroids.*

Proof. If $(\mathrm{V}, \mathrm{A}, \alpha; e_\alpha)$ is a d-split$^+$ epimorphism, of pre-R-algebroids, with an R-algebroid codomain then V is a pre-R-algebroid, A is an R-algebroid with $\mathrm{A}_0 = \mathrm{V}_0$ and α is a pre-R-algebroid morphism from V to A. Moreover, e_α is equal to the identity on A_0 by Corollary 3.3 and so $se_\alpha a = sa$ and $te_\alpha a = ta$ for all $a \in \mathrm{A}$. Thus, for all $v \in \mathrm{V}$ and $a, a' \in \mathrm{A}$ with $tv = sa$ and $ta' = sv$ the compositions $v\,(e_\alpha a)$ and $(e_\alpha a')\,v$ are both well-defined and the assignments

$$(v, a) \mapsto v^a = v\,(e_\alpha a) \quad \text{and} \quad (a', v) \mapsto {}^{a'}v = (e_\alpha a')\,v \tag{3.1}$$

obviously determine an associative A-action on V. Furthermore,

$$\alpha\,(v^a) = \alpha\,(v\,(e_\alpha a)) = (\alpha v)\,(\alpha e_\alpha a) = (\alpha v)\,a\,,$$

since $\alpha e_\alpha = id_{\mathrm{A}}$, and similarly $\alpha({}^{a'}v) = a'\,(\alpha v)$ for all $v \in \mathrm{V}$ and $a, a' \in \mathrm{A}$ with $tv = sa$ and $ta' = sv$. That is, CM1 is satisfied and the proof is completed. □

4. Split (Pre)crossed Modules over R-Algebroids

The result that a d-split$^+$ epimorphism with an R-algebroid codomain is a precrossed module motivates us to introduce the following definition:

Definition 4.1. Let $\eta : \mathrm{V} \to \mathrm{A}$ be a (pre)crossed module over R-algebroids. If, additionally, η is a split epimorphism of pre-R-algebroids, then it is called a *split (pre)crossed module over R-algebroids*.

Proposition 4.2. *If* $\eta : \mathrm{V} \to \mathrm{A}$ *is a split (pre)crossed module over R-algebroids, then it is a split*$^+$ *epimorphism of pre-R-algebroids.*

Proof. $\mathrm{V}_0 = \mathrm{A}_0$ and η is equal to the identity on A_0, since it is a (pre)crossed module. □

For the same concerns stated just before Definition 3.4 we introduce the following customized definition of split (pre)crossed modules:

Definition 4.3. If $\eta : \mathrm{V} \to \mathrm{A}$ is both a (pre)crossed module over R-algebroids and a d-split epimorphism, then we shall call it a *definite split (pre)crossed module* or shortly a *d-split (pre)crossed module, over R-algebroids*.

Theorem 4.4. $\eta : \mathrm{V} \to \mathrm{A}$ *is a d-split precrossed module, over R-algebroids, if and only if it is a d-split*$^+$ *epimorphism with an R-algebroid codomain.*

Proof. This is a direct result of Lemma 3.8, Proposition 4.2 and Definition 4.3. □

Then, after now, the splitting of a d-split (pre)crossed module $\eta : \mathrm{V} \to \mathrm{A}$ over R-algebroids will be denoted by e_η and to denote such a d-split (pre)crossed module we shall use the ordered quadruple $(\mathrm{V}, \mathrm{A}, \eta; e_\eta)$, for convenience.

Definition 4.5. Given two d-split (pre)crossed modules $(\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ and $(\mathrm{V}', \mathrm{A}', \eta'; e_{\eta'})$, over R-algebroids, a (pre)crossed module morphism (f, g) from $\eta : \mathrm{V} \to \mathrm{A}$ to $\eta' : \mathrm{V}' \to \mathrm{A}'$ is called a *definite split (pre)crossed module morphism* or shortly a *d-split (pre)crossed module morphism, over R-algebroids*, if additionally the equality $f e_\eta = e_{\eta'} g$ holds.

Then, each d-split (pre)crossed module morphism is a d-split$^+$epi morphism and thus we get the following result:

Corollary 4.6. *All d-split precrossed modules over R-algebroids form with their morphisms and with the pointwise composition, which is defined as in Proposition 2.11, a subcategory of both* $\mathrm{PXAlg}(R)$ *and* $\mathrm{DSplit^+EpiPAlg}(R)$*. We shall denote this category by* $\mathrm{DSplitPXAlg}(R)$*. Similarly, all d-split crossed modules over R-algebroids form with their morphisms a subcategory of both* $\mathrm{XAlg}(R)$ *and* $\mathrm{DSplitPXAlg}(R)$*, which we shall denote by* $\mathrm{DSplitXAlg}(R)$*.*

In Lemma 3.8, we have seen that a d-split$^+$ epimorphism, of pre-R-algebroids, with an R-algebroid codomain is a precrossed module in which the associative A-action on V is defined through the splitting morphism. That is to mean, there are d-split (pre)crossed modules $(\mathrm{V}, \mathrm{A}, \eta; e_\eta)$, in which the associative A-action on V is defined through the splitting morphism e_η as in (3.1):

Definition 4.7. Let $(\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ be a d-split (pre)crossed module over R-algebroids. We shall say that A *acts on* V *through the splitting morphism* e_η if the associative A-action on V is defined through e_η as in (3.1), i.e., by

$$(v, a) \mapsto v^a = v(e_\eta a) \quad \text{and} \quad (a', v) \mapsto {}^{a'}v = (e_\eta a')v$$

for all $v \in \mathrm{V}$ and $a, a' \in \mathrm{A}$ with $ta' = sv$, $tv = sa$.

Definition 4.8. Let $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ be a d-split (pre)crossed module over R-algebroids. If A acts on V through the splitting morphism e_η, then $\mathcal{V}$ will be called a *d-split* (pre)crossed module*.

Then, defining a d-split* (pre)crossed module morphism as in Definition 4.5 gives us the following result:

Corollary 4.9. *All d-split* precrossed modules over R-algebroids form with their morphisms a full subcategory of* $\mathrm{DSplitPXAlg}(R)$*, which will be denoted by* $\mathrm{DSplit^*PXAlg}(R)$*.*

Proposition 4.10. *Any d-split crossed module, over R-algebroids, is d-split*.*

Proof. If $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ is a d-split crossed module over R-algebroids, then

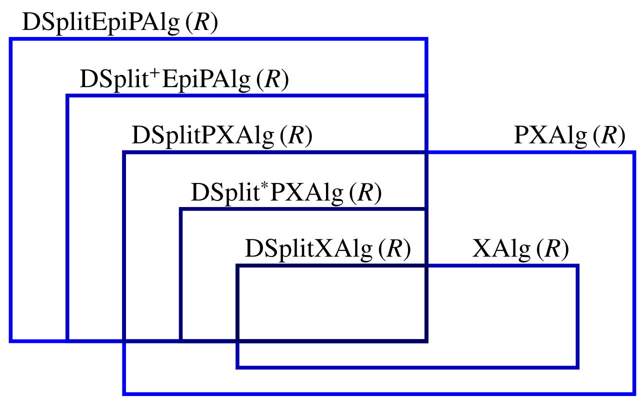
$$v^a = v^{\eta e_\eta a} = v(e_\eta a) \quad \text{and} \quad {}^{a'}v = {}^{\eta e_\eta a'}v = (e_\eta a')v\,,$$

for all $v \in \mathrm{V}$ and $a, a' \in \mathrm{A}$ with $ta' = sv$ and $tv = sa$, meaning that $\mathcal{V}$ is d-split*. □

Then, we get the following clear result:

Corollary 4.11. DSplitXAlg (R) *is a full subcategory of* DSplit*PXAlg (R).

Although they are not just sets, the resultant hierarchical structure above could be illustrated with the following schematic diagram:



5. More on Split (Pre)crossed Modules

Up to now, we have seen some basic features of split (pre)crossed modules, over R-algebroids. In what follows, we shall explore their further features:

Proposition 5.1. *Given two d-split precrossed modules* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *and* $\mathcal{V}' = (\mathrm{V}', \mathrm{A}', \eta'; e_{\eta'})$*, over R-algebroids, if the pair* (f, g) *is a d-split precrossed module morphism from* $\mathcal{V}$ *to* $\mathcal{V}'$*, then* $g = \eta' f e_\eta$.

Proof. This is a direct result of Proposition 3.6. □

The converse of Proposition 5.1 does not have to be correct always. That is, given two d-split precrossed modules $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ and $\mathcal{V}' = (\mathrm{V}', \mathrm{A}', \eta'; e_{\eta'})$, over R-algebroids, a pre-R-algebroid morphism $f : \mathrm{V} \to \mathrm{V}'$ and an R-algebroid morphism $g : \mathrm{A} \to \mathrm{A}'$, if $g = \eta' f e_\eta$, then the pair (f, g) may fail to be a d-split precrossed module morphism from $\mathcal{V}$ to $\mathcal{V}'$. Instead, we have the following:

Proposition 5.2. *Given two d-split* precrossed modules* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *and* $\mathcal{V}' = (\mathrm{V}', \mathrm{A}', \eta'; e_{\eta'})$*, over R-algebroids, a pre-R-algebroid morphism* $f : \mathrm{V} \to \mathrm{V}'$ *and an R-algebroid morphism* $g : \mathrm{A} \to \mathrm{A}'$*, if* $f = e_{\eta'} g \eta$*, then the pair* (f, g) *is a d-split* precrossed module morphism from* $\mathcal{V}$ *to* $\mathcal{V}'$.

Proof. Using the equalities $\eta' e_{\eta'} = id_{\mathrm{A}'}$ and $\eta e_\eta = id_{\mathrm{A}}$ we get

$$\eta' f = \eta' e_{\eta'} g \eta = id_{\mathrm{A}'} g \eta = g \eta \quad , \qquad f e_\eta = e_{\eta'} g \eta e_\eta = e_{\eta'} g id_{\mathrm{A}} = e_{\eta'} g$$

and thus,

$$f(v^a) = f(v(e_\eta a)) = f(v) f(e_\eta a) = f(v) f e_\eta(a) = f(v) e_{\eta'} g(a) = f(v)^{g(a)}$$

for all $v \in \mathrm{V}$ and $a \in \mathrm{A}$ with $tv = sa$. It can similarly be shown that $f({}^{a'}v) = {}^{g(a')}f(v)$ for all $v \in \mathrm{V}$ and $a' \in \mathrm{A}$ with $ta' = sv$, and this completes the proof. □

Lemma 5.3. *If* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *is a d-split* (pre)crossed module over R-algebroids, then* V *is an R-algebroid.*

Proof. For each $x \in \mathrm{A}_0$ and for all $v, v' \in \mathrm{V}$ with $tv = sv' = x$

$$v = v^{1_{\mathrm{A}(x)}} = v(e_\eta(1_{\mathrm{A}(x)})) \quad \text{and} \quad v' = {}^{1_{\mathrm{A}(x)}}v' = (e_\eta(1_{\mathrm{A}(x)}))v'$$

meaning that $e_\eta(1_{\mathrm{A}(x)})$ is the identity morphism of V on $x \in \mathrm{A}_0$, i.e.

$$1_{\mathrm{V}(x)} = e_\eta(1_{\mathrm{A}(x)})\,, \tag{5.1}$$

and this completes the proof. □

Lemma 5.4. *If* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *is a d-split** *(pre)crossed module over R-algebroids, then both* η *and* e_η *are R-algebroid morphisms.*

Proof. Both η and e_η are pre-R-algebroid morphisms. In addition, $e_\eta(1_{\mathrm{A}(x)}) = 1_{\mathrm{V}(x)}$ and $\eta(1_{\mathrm{V}(x)}) = (\eta e_\eta)(1_{\mathrm{A}(x)}) = 1_{\mathrm{A}(x)}$ for all $x \in \mathrm{A}_0$, by (5.1) and by the equality $\eta e_\eta = id_{\mathrm{A}}$. Moreover, each of η and e_η is equal to the identity on A_0 because $\mathcal{V}$ is d-split$^+$ by Proposition 4.2. Thus, $e_\eta(1_{\mathrm{A}(x)}) = 1_{\mathrm{V}(x)} = 1_{\mathrm{V}(e_\eta x)}$ and $\eta(1_{\mathrm{V}(x)}) = 1_{\mathrm{A}(x)} = 1_{\mathrm{A}(\eta x)}$, and this completes the proof. □

Theorem 5.5. *A d-split** *precrossed module* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *is a d-split crossed module if and only if* $e_\eta\eta = id_{\mathrm{V}}$.

Proof. First of all, V is an R-algebroid by Lemma 5.3, since $\mathcal{V}$ is d-split*, and thus V has an identity morphism $1_{\mathrm{V}(x)}$ on each $x \in \mathrm{V}_0 = \mathrm{A}_0$. Moreover, both η and e_η are R-algebroid morphisms by Lemma 5.4.

Now, let $\mathcal{V}$ be a d-split crossed module. Then, for each $v \in \mathrm{V}$

$$e_\eta\eta v = (1_{\mathrm{V}(sv)})(e_\eta\eta v) = (1_{\mathrm{V}(sv)})^{\eta v} = 1_{\mathrm{V}(sv)}v = v\,,$$

where the first equality holds because each of η and e_η is equal to the identity on A_0, the second equality holds because $\mathcal{V}$ is d-split* and the third one holds because $\mathcal{V}$ is a crossed module. So we get $e_\eta\eta = id_{\mathrm{V}}$, as required.

Let, on the other hand, $e_\eta\eta = id_{\mathrm{V}}$. Then, for each $v, v' \in \mathrm{V}$ with $tv = sv'$

$$v^{\eta v'} = v(e_\eta\eta v') = v(id_{\mathrm{V}}v') = vv' = (id_{\mathrm{V}}v)v' = (e_\eta\eta v)v' = {}^{\eta v}v'\,,$$

where the first and last equalities hold because $\mathcal{V}$ is d-split*. So, $\mathcal{V}$ is a d-split crossed module, and this completes the proof. □

Corollary 5.6. *If* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *is a d-split crossed module over R-algebroids, then* η *is an isomorphism of R-algebroids.*

Proof. $\eta e_\eta = id_{\mathrm{A}}$, since $\mathcal{V}$ is d-split, and $e_\eta\eta = id_{\mathrm{V}}$ by Theorem 5.5, proving that η is an isomorphism of R-algebroids. □

Then, the following result is obvious:

Corollary 5.7. *If* $\mathcal{V} = (\mathrm{V}, \mathrm{A}, \eta; e_\eta)$ *is a d-split crossed module over R-algebroids, then* $\mathrm{V} \cong \mathrm{A}$.

6. Notes on Split Epimorphisms of and Split (Pre)crossed Modules over Associative R-Algebras

Categorically, pre-R-algebroids could be considered as many-object versions of associative R-algebras or associative R-algebras as pre-R-algebroids with one object. Therefore, all the findings above can be directly adjusted to split epimorphisms and split (pre)crossed modules over associative R-algebras. What is more, to make things easier, all object sets and all source and target functions could be completely ignored simply by assuming all associative R-algebras have the same unique object. Then the only thing to do is the restatement of all above-mentioned findings without considering source and target functions. However, it is worth to note the following further results:

Proposition 6.1. *Every (d-)split epimorphism* $\alpha : \mathrm{V} \to \mathrm{A}$ *of associative R-algebras is (d-)split*$^+$.

Proof. This is a direct result of Definition 3.2, since we assume that both N and A have the same unique object. □

Then, any d-split-epi morphism is a d-split$^+$epi morphism and thus, we get the following immediate result:

Corollary 6.2. *The categories of d-split and d-split*$^+$ *epimorphisms of associative R-algebras are the same.*

It is stated in Lemma 3.8 that a d-split$^+$ epimorphism with an R-algebroid codomain is a precrossed module. Then, by Proposition 6.1 we get the following:

Corollary 6.3. *A d-split epimorphism, of associative R-algebras, with a unitary R-algebra codomain is a precrossed module.*

Theorem 6.4. $\eta : \mathrm{V} \to \mathrm{A}$ *is a d-split precrossed module, over associative R-algebras, if and only if it is a d-split epimorphism with a unitary R-algebra codomain.*

Proof. This is a direct result of Theorem 4.4 and Proposition 6.1. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

References

[1] Akça, İ.İ., Avcıoğlu, O., *Equivalence between (pre)cat^1-R-algebroids and (pre)crossed modules of R-algebroids*, Bull. Math. Soc. Sci. Math. Roumanie, Tome **65**(113)(2022), no. 3, 267–288.

[2] Alp, M., *Pullback crossed modules of algebroids*, Iran. J. Sci. Technol., **32**(A1)(2008), 1–5.

[3] Alp, M., *Pushout crossed modules of algebroids*, Iran. J. Sci. Technol., **32**(A3)(2008), 175–181.

[4] Amgott, S.M., *Separable categories*, J. Pure Appl. Algebra, **40**(1986), 1–14.

[5] Arvasi, Z., Porter, T., *Simplicial and crossed resolutions of commutative algebras*, Journal of Algebra, **181**(2)(1996), 426–448.

[6] Arvasi, Z., *Crossed modules of algebras*, Math. Comput. Appl., **9**(2)(2004), 173–182.

[7] Avcıoğlu, O., *Homotopies of crossed modules of R-algebroids*, Applied Categorical Structures, **29**(2021), 827–847.

[8] Avcıoğlu, O., Akça, İ.İ., *On generators of Peiffer ideal of a pre-R-algebroid in a precrossed module and applications*, New Trends in Mathematical Sciences, **5**(4)(2017), 148–155.

[9] Avcıoğlu, O., Akça, İ.İ., *Free modules and crossed modules of R-algebroids*, Turk. J. Math., **42**(6)(2018), 2863–2875.

[10] Brown, R., Higgins, P.J., Sivera, R., Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids, EMS Tracts in Mathematics, 15, European Mathematical Society (EMS), Zürich, 2011.

[11] Brown, R., İçen, İ., *Homotopies and automorphisms of crossed modules of groupoids*, Applied Categorical Structures, **11**(2003), 185–206.

[12] Brown, R., Spencer, C.B., *$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group*, Indagationes Mathematicae (Proceedings), **79**(4)(1976), 296–302.

[13] Böhm, G., *Crossed modules of monoids II: Relative crossed modules*, Applied Categorical Structures, **28**(2020), 601–653.

[14] Carrasco, P., Porter, T., *Coproduct of 2-crossed modules: applications to a definition of a tensor product for 2-crossed complexes*, Collect. Math., **67**(2016), 485–517.

[15] Dedecker, P., Lue, A.S.-T., *A nonabelian two-dimensional cohomology for associative algebras*, Bull. Amer. Math. Soc. **72**(1966), 1044–1050.

[16] Ege Arslan, U., Akça, İ.İ., Onarlı Irmak, G., Avcıoğlu, O., *Fibrations of 2-crossed modules*, Math. Meth. Appl. Sci., **42**(2019), 5293–5304.

[17] Gerstenhaber, M., *On the deformation of rings and algebras: II*, Annals of Mathematics, **84**(1)(1966), 1–19.

[18] Janalidze, G., *Internal Crossed Modules*, Georgian Mathematical Journal, **10**(1)(2003), 99–114.

[19] Lavendhomme, R., Roisin, J.R., *Cohomologie non abélienne de structures algébriques*, Journal of Algebra, **67**(2)(1980), 385–414.

[20] Lichtenbaum, S., Schlessinger, M., *The cotangent complex of a morphism*, Trans. Amer. Math. Soc., **128**(1967), 41–70.

[21] Lue, A.S.-T., *Non-abelian cohomology of associative algebras*, The Quarterly Journal of Mathematics, **19**(1)(1968), 159–180.

[22] Mitchell, B., *Rings with several objects*, Advances in Mathematics, **8**(1)(1972), 1–161.

[23] Mitchell, B., *Some applications of module theory to functor categories*, Bull. Amer. Math. Soc., **84**(1978), 867–885.

[24] Mitchell, B., Separable algebroids, Mem. Amer. Math. Soc., **57**, no. 333, iv+96pp., 1985.

[25] Mosa, G.H., Higher dimensional algebroids and crossed complexes, Ph.D. Thesis, University of Wales, Bangor, 1986.

[26] Peiffer, R., *Über Identitäten zwischen Relationen*, Math. Ann., **121**(1949), 67–99.

[27] Reidemeister, K., *Über Identitäten von Relationen*, Abh. Math. Semin. Univ. Hambg., **16**(1949), 114–118.

[28] Porter, T., *Homology of commutative algebras and an invariant of simis and vasconcelos*, Journal of Algebra, **99**(2)(1986), 458–465.

[29] Porter, T., *Some categorical results in the theory of crossed modules in commutative algebras*, Journal of Algebra, **109**(2)(1987), 415–429.

[30] Porter, T., The Crossed Manegarie: an introduction to crossed gadgetry and cohomology in algebra and topology. 17 April 2018. https://ncatlab.org/nlab/files/menagerie12a.pdf

[31] Shammu, N.M., Algebraic and categorical structure of categories of crossed modules of algebras, Ph.D. Thesis, University of Wales, Bangor, 1992.

[32] Temel, S., Şahan, T., Mucuk, O., *Crossed modules, double group-groupoids and crossed squares*, Filomat, **34**(6)(2020), 1755–1769.

[33] Whitehead, J.H.C., *On adding relations to homotopy groups*, Annals of Mathematics, **42**(2)(1941), 409–428.

[34] Whitehead, J.H.C., *Note on a previous paper entitled "On adding relations to homotopy groups."*, Annals of Mathematics, **47**(4)(1946), 806–810.