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Extension of Synthetic Division and Its Applications

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Abstract

This study focused on "Extension of synthetic division and its applications". The study was designed to show synthetic division and its extension and to point out the applications of synthetic division and its extension. The study found out that the concepts of polynomial and rational expressions in single variables are basic concepts to deal extension of synthetic division and its applications. Using the preliminary concepts, the concept of synthetic division is extended in this study. Also, the study found out that an extension of synthetic division is used for finding the oblique asymptote of the graph of a rational function, evaluating the integration of some rational functions, representing polynomial expression by factorial function in numerical analysis, and so on.

1. Introduction

Division is one of the arithmetic operations [1]. The division of two real numbers, saying a and b, is denoted by $a \div b$ *or* $\frac{a}{b}$ provided that b is different from zero. From this expression, a and b are said to be dividend and divisor, respectively [2]. The task of division of real numbers$(a \div b;\ b \neq 0)$ is finding real numbers k *and* t that satisfy the equation $a = kb + t$ [3]. From this equation, k and t are said to be quotient and remainder of the division, respectively [4]. Since human beings use the concept of division of real numbers for their day $-$ to $-$day activities frequently, they upgraded this concept beyond the set of real numbers. Polynomial division is one of this upgrading. In this division, the quotient and the remainder are polynomial expressions [5]. There are different techniques to perform polynomial division. Among these techniques, long division and synthetic division are the most known techniques. Long division uses variable $-$ wise division whereas synthetic division uses coefficient $-$ wise division [6]. This study is mainly focused on extension of synthetic division as well as its applications.
The main objectives of this study are

- to show synthetic division.
- to show the extension of synthetic division.
- to point out the applications of synthetic division and its extension.

2. Preliminaries

Polynomial and rational expressions in single variable are the preliminary concepts to deal synthetic division and its extension [7].

Definition 2.1 ([8]). *A polynomial expression in a single variable x of degree n is an expression in the form of*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \tag{2.1}$$

where n is a whole number and $a_n \neq 0$.

Remark 2.2. *From equation* (2.1),

- $a_n,\ a_{n-1},\ \ldots,\ a_1$ *and* a_0 are said to be coefficients of a polynomial expression.

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- $a_n x^n$ is said to be the leading term of a polynomial expression.
- a_n is said to be the leading coefficient of a polynomial expression.
- a_0 is said to be the constant term of a polynomial expression.
- For any doing in this paper, we use the set of complex numbers as a domain of a polynomial expression.

Example 2.3. *x^3+1 and 3 are polynomial expressions in x of degree 3 and 0, respectively but 0 is a polynomial expression in x with no degree.*

Definition 2.4 ([9]). *A complex number r is said to be a zero of a polynomial expression P(x) if P(r) = 0.*

Theorem 2.5 (**Linear Factorization Theorem**, [10]). *Let $a_n \neq 0$ be a leading coefficient of a polynomial expression $P(x)$ and the zeros of $P(x)$ with their multiplicities are listed in the following table (k is a natural number). Then the factorization form of $P(x)$ is $P(x) = a_n\ (x-r_1)^{m_1}\ (x-r_2)^{m_2}\ (x-r_3)^{m_3}\ \ldots\ (x-r_k)^{m_k}$.*

zeros	Multiplicities
r_1	m_1
r_2	m_2
r_3	m_3
$\vdots$	$\vdots$
r_k	m_k

Table 1: The zeros of P(x) with their multiplicities.

Definition 2.6 ([11]). *A rational expression in a single variable x is an expression in the form of*

$$\frac{P(x)}{Q(x)}, \tag{2.2}$$

where P(x) and Q(x) are polynomial expressions in x and Q(x)≠0.

Remark 2.7. *From equation (2.2),*

- $P(x)$ is said to be numerator of rational expression.
- $Q(x)$ is said to be denominator of rational expression.
- If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational expression is said to be proper rational expression.
- If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then the rational expression is said to be improper rational expression.

Example 2.8. *$\frac{x^3+1}{x-1}$ and x^2+3 are rational expressions in x but $\frac{x^{0.5}}{x-1}$ is not rational expression because $x^{0.5}$ is not a polynomial expression.*

Theorem 2.9 ((**Polynomial Division Theorem**), [12]). *Let f(x) and d(x) are polynomial expressions such that d(x)≠0, and $\frac{f(x)}{d(x)}$ is improper rational expression. Then there exist unique polynomial expressions q(x) and r(x) such that*

$$f(x) = q(x)d(x) + r(x), \tag{2.3}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.

Remark 2.10. *From equation (2.3), $q(x)$ and $r(x)$ are the quotient and remainder of $f(x) \div d(x)$, respectively.*

3. Synthetic Division

Synthetic division is a way to divide a polynomial expression in x by the binomial $x-c$, where c is a constant [13]. In synthetic division, we follow the following steps

Step 1: Set up the synthetic division.
Step 2: Bring down the leading coefficient to the 2^{nd} cell of the bottom row.
Step 3: Multiply c by the value just written on the 2^{nd} cell of the bottom row, and then write this result on the 3^{rd} cell of the 3^{rd} row.
Step 4: Add the column created in Step 3, and then write this result on the 3^{rd} cell of the bottom row.
Step 5: Repeat Steps 3 and 4 until done.
Step 6: Write out the answer [14].

The numbers in the bottom row make up our coefficients of the quotient and remainder. The final value on the last cell of the bottom row is the remainder and the rests are the coefficients of the quotient [15].

Example 3.1. *Consider a rational expression $\frac{x^3+1}{x-2}$.*

x^3+1 means x^3+0x^2+0x+1 and the zero of $x-2$ is 2.

- Write the coefficients of x^3+0x^2+0x+1 and the zero of $x-2$ in the following manner.

Zero of the divisor	
2	1 0 0 1

Table 2: Step-1 of synthetic division for Example 3.1.

- Bring down the leading coefficient, 1 to the 2^{nd} cell of the bottom row. That is

Zero of the divisor				
2	1	0	0	1
	1			

Table 3: Step-2 of synthetic division for Example 3.1.

- Multiply 1 by the zero of the divisor, 2. This is equal to 2. Then write 2 on the 3^{rd} cell of the 3^{rd} row. That is

Zero of the divisor				
2	1	0	0	1
		2		
	1			

Table 4: Step-3 of synthetic division for Example 3.1.

- Add 0 and 2. This is equal to 2. Then write 2 on the 3^{rd} cell of the bottom row. That is

Zero of the divisor				
2	1	0	0	1
		2		
	1	2		

Table 5: Step-4 of synthetic division for Example 3.1.

- Multiply 2 by the zero of the divisor, 2. This is equal to 4. Then write 4 on the 4^{th} cell of the 3^{rd} row. That is

Zero of the divisor				
2	1	0	0	1
		2	4	
	1	2		

Table 6: Step-5 of synthetic division for Example 3.1.

- Add 0 and 4. This is equal to 4. Then write 4 on the 4^{th} cell of the bottom row. That is

Zero of the divisor				
2	1	0	0	1
		2	4	
	1	2	4	

Table 7: Step-6 of synthetic division for Example 3.1.

- Multiply 4 by the zero of the divisor, 2. This is equal to 8. Then write 8 on the 5^{th} cell of the 3^{rd} row. That is

Zero of the divisor				
2	1	0	0	1
		2	4	8
	1	2	4	

Table 8: Step-7 of synthetic division for Example 3.1.

- Add 1 and 8. This is equal to 9. Then write 9 on the 5^{th} cell of the bottom row. That is

Zero of the divisor				
2	1	0	0	1
		2	4	8
	1	2	4	**9=r**

Table 9: Step-8 of synthetic division for Example 3.1.

Therefore, 9 is the remainder and 1, 2, and 4 are the coefficients of the quotient of the given polynomial division. That is, $r(x) = 9$ and $q(x) = x^2 + 2x + 4$.

Let $\frac{f(x)}{d(x)}$ be improper rational expression such that $d(x) = t(x-c); t \neq 0$. Assume that the quotient and remainder of this rational expression are $q(x)$ and $r(x)$, respectively.

$$\frac{f(x)}{d(x)} = \frac{f(x)}{t(x-c)} = q(x) + \frac{r(x)}{t(x-c)} \Longrightarrow \frac{1}{t}\ \frac{f(x)}{x-c} = \frac{1}{t}\left[tq(x) + \frac{r(x)}{x-c}\right] \Longrightarrow \frac{f(x)}{x-c} = t\ q(x) + \frac{r(x)}{x-c}$$

From the preceding equation, we can conclude that

- The quotient of $\frac{f(x)}{x-c}$ is t times of the quotient of $\frac{f(x)}{t(x-c)}$.
- The remainder of $\frac{f(x)}{x-c}$ is the same as the remainder of $\frac{f(x)}{t(x-c)}$.

Remark 3.2. *Two proportional improper rational expressions have the same quotient but not the same remainder.*

4. Extension of Synthetic Division

Let $\frac{P(x)}{Q(x)}$ be improper rational expression and $Q(x) = (x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)$; each $a_i\,'s$ is a constant.

- $\frac{P(x)}{x-a_1} = q_1(x) + \frac{r_1}{x-a_1}$ where $q_1(x)$ and r_1 are the quotient and remainder of $\frac{P(x)}{x-a_1}$, respectively.
- Multiply the preceding equation by $\frac{1}{x-a_2}$. Then we get

$$\frac{P(x)}{(x-a_1)(x-a_2)} = \frac{q_1(x)}{x-a_2} + \frac{r_1}{(x-a_1)(x-a_2)} \Longrightarrow \frac{P(x)}{(x-a_1)(x-a_2)} = q_2(x) + \frac{r_2}{x-a_2} + \frac{r_1}{(x-a_1)(x-a_2)},$$

  where $q_2(x)$ and r_2 are the quotient and remainder of $\frac{q_1(x)}{x-a_2}$, respectively.

$$\Longrightarrow \frac{P(x)}{(x-a_1)(x-a_2)} = q_2(x) + \frac{r_2(x-a_1)+r_1}{(x-a_1)(x-a_2)},$$

- Multiply the preceding equation by $\frac{1}{x-a_3}$. Then we get

$$\frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)} = \frac{q_2(x)}{x-a_3} + \frac{r_2(x-a_1)+r_1}{(x-a_1)(x-a_2)(x-a_3)}$$
$$\Longrightarrow \frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)} = q_3(x) + \frac{r_3}{x-a_3} + \frac{r_2(x-a_1) + r_1}{(x-a_1)(x-a_2)(x-a_3)},$$

  where $q_3(x)$ and r_3 are the quotient and remainder of $\frac{q_2(x)}{x-a_3}$, respectively.

$$\Longrightarrow \frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)} = q_3(x) + \frac{r_3(x-a_1)(x-a_2) + r_2(x-a_1) + r_1}{(x-a_1)(x-a_2)(x-a_3)}$$

- Using similar pattern

$$\frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)} = q_n(x) + \frac{r_1 + r_2(x-a_1) + r_3(x-a_1)(x-a_2) + \ldots + r_n(x-a_1)(x-a_2)\ \ldots\ (x-a_{n-1})}{(x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)}$$

Hence, $q(x) = q_n(x)$ is the quotient of $\frac{P(x)}{Q(x)}$ and $r(x) = r_1 + r_2(x-a_1) + r_3(x-a_1)(x-a_2) + \ldots + r_n(x-a_1)(x-a_2)\ \ldots\ (x-a_{n-1})$ is the remainder of $\frac{P(x)}{Q(x)}$. To find $q_1(x),\ q_2(x),\ \ldots,\ q_n(x),\ r_1,\ r_2,\ \ldots,$ *and* r_n, we use synthetic division successively in the following manner.

Zeros of the divisor	
$\boxed{a_1}$	**Coefficients of** $P(x)$ *This line consists numbers which are obtained using steps of synthetic division*
	Coefficients of $q_1(x)$ *and* r_1
$\boxed{a_2}$	**Coefficients of** $q_1(x)$ *This line consists numbers which are obtained using steps of synthetic division*
	Coefficients of $q_2(x)$ *and* r_2
$\vdots$	$\vdots$
	Coefficients of $q_{n-1}(x)$ & r_{n-1}
$\boxed{a_n}$	**Coefficients of** $q_{n-1}(x)$ *This line consists numbers which are obtained using steps of synthetic division*
	Coefficients of $q_n(x)$ *and* r_n

Table 10: Steps of extension of synthetic division.

Hence, such type of successive use of synthetic division is called **extension of synthetic division**.

Remark 4.1. *The order of* $a_1,\ a_2, \ldots$ & a_n *in the preceding table does not affect the desired results.*

Example 4.2. *Let us find the quotient and remainder of rational expressions* $\frac{x^5+x+1}{(x^2-1)(x+2)}$, $\frac{x^4+x}{x^2+1}$ *and* $\frac{x^4+x}{x^2+x}$ *using extension of synthetic division.*

(a) $\frac{x^5+x+1}{(x^2-1)(x+2)} = \frac{x^5+0x^4+0x^3+0x^2+x+1}{(x+2)(x+1)(x-1)}$. Hence, the coefficients of a polynomial expression in the numerator of this rational expression are 1, 0, 0, 0, 1 & 1, and the zeros of a polynomial expression in the denominator of this rational expression are -2, -1 & 1. Now let us construct a table.

Zeros of the divisor						
-2	1	0 − 2	0 4	0 − 8	1 16	1 − 34
	1	− 2	4	− 8	17	$\mathbf{-33} = r_1$
-1	1	− 2 − 1	4 3	− 8 − 7	17 15	
	1	− 3	7	− 15	$\mathbf{32} = r_2$	
1	1	− 3 1	7 − 2	− 15 5		
	1	− 2	5	$\mathbf{-10} = r_3$		

Table 11: Steps of synthetic division for Example 4.2 (a).

Therefore,

- the quotient is $q(x) = x^2 - 2x + 5$.
- the remainder is

$$r(x) = r_1 + r_2(x+2) + r_3(x+2)(x+1) = -33 + 32(x+2) - 10(x+2)(x+1)$$
$$\Longrightarrow r(x) = -10x^2 + 2x + 11$$

(b) $\frac{x^4+x}{x^2+1} = \frac{x^4+0x^3+0x^2+x+0}{(x+i)(x-i)}$. Hence, the coefficients of a polynomial expression in the numerator of this rational expression are 1, 0, 0, 1 & 0, and the zeros of a polynomial expression in the denominator of this rational expression are $-i$ & i. Now let us construct a table.

Zeros of the divisor					
− i	1	0 −i	0 −1	1 i	0 1−i
	1	−i	−1	1 + i	$\mathbf{1-i} = r_1$
i	1	−i i	−1 0	1 + i −i	
	1	0	−1	$\mathbf{1} = r_2$	

Table 12: Steps of synthetic division for Example 4.2 (b).

Therefore,

- the quotient is $q(x) = x^2 - 1$.
- the remainder is $r(x) = r_1 + r_2(x+i) = 1\text{-}i + 1(x+i) \Longrightarrow r(x) = x + 1$

(c) $\frac{x^4+x}{x^2+x} = \frac{x^4+0x^3+0x^2+x+0}{x(x+1)}$. Hence, the coefficients of a polynomial expression in the numerator of this rational expression are 1, 0, 0, 1 & 0, and the zeros of a polynomial expression in the denominator of this rational expression are 0 & − 1. Now let us construct a table.

Zeros of the divisor					
0	1	0 0	0 0	1 0	0 0
	1	0	0	1	$\mathbf{0} = r_1$
− 1	1	0 − 1	0 1	1 − 1	
	1	−1	1	$\mathbf{0} = r_2$	

Table 13: Steps of synthetic division for Example 4.2 (c).

Therefore,

- the quotient is $q(x) = x^2 - x + 1$.
- the remainder is $r(x) = r_1 + r_2 x = 0 + 0x \Longrightarrow r(x) = 0$. Hence, a polynomial expression in the denominator of the given rational expression is a factor of a polynomial expression in the numerator of the given rational expression.

Remark 4.3. *Let $\frac{f(x)}{d(x)}$ be improper rational expression such that $d(x) = (a_1x - c_1)(a_2x - c_2)(a_3x - c_3)\ \ldots\ (a_nx - c_n)$; each c_i 's is a constant & each a_i 's is a non − zero constant, then*

- *the quotient of $\frac{f(x)}{d(x)}$ is $\frac{1}{a_1\ a_2\ a_3\ \ldots\ a_n}$ of the quotient of $\frac{f(x)}{(x-\frac{c_1}{a_1})(x-\frac{c_2}{a_2})(x-\frac{c_3}{a_3})\ldots(x-\frac{c_n}{a_n})}$.*
- *the remainder of $\frac{f(x)}{d(x)}$ is the same as the remainder of $\frac{f(x)}{(x-\frac{c_1}{a_1})(x-\frac{c_2}{a_2})(x-\frac{c_3}{a_3})\ldots(x-\frac{c_n}{a_n})}$.*

5. Applications of Extension of Synthetic Division

Extension of synthetic division has applications for

- finding the oblique asymptote of the graph of a rational function
- evaluating the integration of some rational functions
- representing polynomial expression by factorial function in numerical analysis
- finding an equation of a line tangent to a graph of polynomial function at a point
- finding the greatest common factor of two polynomial expressions.

5.1. Oblique asymptote of the graph of a rational function

Extension of synthetic division is applicable to find oblique asymptote of the graph of a rational function. The graph of a rational function has oblique asymptote if the degree of a polynomial expression in the numerator exceeds the degree of a polynomial expression in the denominator by one.
Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function. If the degree of P(x) exceeds the degree of $Q(x)$ by one, then the oblique asymptote of the graph of $f(x)$ is a line$y = q(x)$ where $q(x)$ is a quotient of $P(x) \div Q(x)$.

Example 5.1. *Let us find the oblique asymptote of the graph of f(x) =* $\frac{x^3}{x^2-25}$ *using extension of synthetic division.*

Since the degree of x^3 exceeds the degree of x^2-25 by 1, then the graph of $f(x)$ has oblique asymptote.
$x^3 = x^3 + 0x^2 + 0x + 0$ and $x^2-25 = (x+5)(x-5)$.

Zeros of the divisor				
-5	1	0	0	0
		-5	25	-125
	1	-5	25	$-125 = r_1$
5	1	-5	25	
		5	0	
	1	0	**25** = r_2	

Table 14: Steps of synthetic division for Example 5.1.

From the preceding table, $q(x) = x$. Hence, the oblique asymptote of the graph of the given function is a line $y = x$.

5.2. Integration of some rational functions

If the integrand of a given integration is improper rational expression, then extension of synthetic division has application in a time of evaluation of the given integration using a method of integration by partial fractions.

Example 5.2. *Consider* $\int \frac{x^4}{x^2-2x-3}\,dx$.

To evaluate this integral using a method of integration by partial fractions, the integrand must be expressed as the sum of polynomial expression and proper rational expression. To do this, we apply extension of synthetic division.
$x^4 = x^4 + 0x^3 + 0x^2 + 0x + 0$ and $x^2-2x-3 = (x+1)(x-3)$.

Zeros of the divisor					
-1	1	0	0	0	0
		-1	1	-1	1
	1	-1	1	-1	**1** = r_1
3	1	-1	1	-1	
		3	6	21	
	1	2	7	**20** = r_2	

Table 15: Steps of synthetic division for Example 5.2.

From the preceding table,

- the quotient of $\frac{x^4}{x^2-2x-3}$ is $q(x) = x^2 + 2x + 7$.
- the remainder of $\frac{x^4}{x^2-2x-3}$ is $r(x) = 1+20(x+1) = 20x+21$.

Hence,

$$\frac{x^4}{x^2-2x-3} = x^2 + 2x + 7 + \frac{20x + 21}{x^2-2x-3} \Longrightarrow \frac{x^4}{x^2-2x-3} = x^2 + 2x + 7 + \frac{81/4}{x-3} + \frac{-1/4}{x+1}$$

since $\frac{20x+21}{x^2-2x-3}$ is decomposed as $\frac{81/4}{x-3} + \frac{-1/4}{x+1}$ using the concept of partial fraction decomposition.

$$\int \frac{x^4}{x^2-2x-3}dx = \int (x^2 + 2x + 7)dx + \int \frac{81/4}{x-3}\,dx + \int \frac{-1/4}{x+1}dx.$$

After this, the evaluations of these integrals are too simple.

5.3. Representation of polynomial expression by factorial function

Extension of synthetic division has application in a time of expression of a polynomial expression by factorial function.
A product of the form $x(x-h)(x-2h)(x-3h)\dots(x-(n-1)h)$ is called factorial function and denoted by $x^{(n)}$. The name of h is step size of $\{x, x-h, x-2h, x-3h, \dots, x-(n-1)h\}$. Thus $x^{(n)} = x(x-h)(x-2h)(x-3h)\dots(x-(n-1)h)$. If $h=1$, then $x^{(n)} = x(x-1)(x-2)(x-3)\dots(x-(n-1))$.

Example 5.3. *Let us represent a polynomial expression $x^4-12x^3+42x^2-30x+9$ by factorial function using the help of extension of synthetic division(assuming $h=1$).*

Suppose

$$x^4-12x^3+42x^2-30x+9 = Ax^{(4)}+Bx^{(3)}+Cx^{(2)}+Dx^{(1)}+E$$
$$\Longrightarrow x^4-12x^3+42x^2-30x+9 = Ax(x-1)(x-2)(x-3)+Bx(x-1)(x-2)+Cx(x-1)+Dx+E$$

Let us divide both sides of the preceding equation by x$(x-1)(x-2)(x-3)(x-4)$. Then we get

$$\frac{x^4-12x^3+42x^2-30x+9}{x(x-1)(x-2)(x-3)(x-4)} = \frac{Ax(x-1)(x-2)(x-3)+Bx(x-1)(x-2)+Cx(x-1)+Dx+E}{x(x-1)(x-2)(x-3)(x-4)}.$$

Thus, 0 and $Ax(x-1)(x-2)(x-3)+Bx(x-1)(x-2)+Cx(x-1)+Dx+E$ are the quotient and the remainder of $\frac{x^4-12x^3+42x^2-30x+9}{x(x-1)(x-2)(x-3)(x-4)}$, respectively. To find all the unknown coefficients, let us apply extension of synthetic division.

$\Longrightarrow A=r_5,\ B=r_4,\ C=r_3,\ D=r_2$ *and* $E=r_1$.

Zeros of the divisor					
0	1	−12 0	42 0	−30 0	9 0
	1	−12	42	−30	**9 =** r_1 **= E**
1	1	−12 1	42 −11	−30 31	
	1	−11	31	**1 =**r_2 **= D**	
2	1	−11 2	31 −18		
	1	−9	**13 =** r_3 **= C**		
3	1	−9 3			
	1	**−6 =** r_4 **= B**			
4	1				
	1 = r_5 **= A**				

Table 16: Steps of synthetic division for Example 5.3.

From the preceding table, the factorial representation of the given polynomial expression is $x^{(4)}-6x^{(3)}+13x^{(2)}+x^{(1)}+9$.

5.4. Equation of a line tangent to a graph of polynomial function at a point

Extension of synthetic division has application to find the equation of a line which is tangent to the graph of a polynomial function $f(x)$ at a point $(r, f(r))$.
The equation of a line which is tangent to the graph of a polynomial function $f(x)$ at a point $(r, f(r))$ is $y=r(x)$ where $r(x)$ is the remainder of $f(x) \div (x-r)^2$.

Example 5.4. *Let us find the equation of a line which is tangent to the graph of a polynomial function $f(x)=x^5-1$ at a point $(1, f(1))$ using the concept of extension of synthetic division.*

The equation of a line which is tangent to the graph of a given polynomial function $f(x)$ at a point $(1, f(1))$ is the remainder of $f(x) \div (x-1)^2$.

Zeros of the divisor						
1	1	0 1	0 1	0 1	0 1	−1 1
	1	1	1	1	1	**0 =** r_1
1	1	1 1	1 2	1 3	1 4	
	1	2	3	4	**5 =** r_2	

Table 17: Steps of synthetic division for Example 5.4.

From the preceding table, $r(x) = 0+5(x\text{-}1) = 5x-5$. Hence, the equation of a line which is tangent to the graph of a polynomial function $f(x)=x^5-1$ at a point $(1, f(1))$ is $y(x)=5x-5$.

5.5. Greatest common factor (GCF) of two polynomial expressions

Using the concept of Euclidean algorithm theorem and extension of synthetic division, we can find GCF of two polynomial expressions. **Euclidean algorithm theorem**: Let $f(x)$ and $g(x)$ are polynomial expressions such that $g(x) \neq 0$ and degree of $f(x) \geq$ degree of $g(x)$. Apply polynomial division theorem successively as follows

$$f(x) = q_0(x)g(x) + r_0(x) \quad \text{with degree of } g(x) > \text{degree of } r_0(x)$$
$$g(x) = q_1(x)r_0(x) + r_1(x) \quad \text{with degree of } r_0(x) > \text{degree of } r_1(x)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$r_{n-2}(x) = q_n(x)r_{n-1}(x) + r_n(x) \quad \text{with degree of } r_{n-1}(x) > \text{degree of } r_n(x)$$
$$r_{n-1}(x) = q_{n+1}(x)r_n(x) + 0.$$

Therefore, $GCF[f(x), g(x)] = \frac{1}{c}r_n(x)$ where c is the leading coefficient of $r_n(x)$ [6].
We use extension of synthetic division to find $q_0(x), q_1(x), q_2(x), \dots, q_{n+1}(x), r_0(x), r_1(x), r_2(x), \dots$ and $r_n(x)$.

Example 5.5. *Consider polynomial expressions $x^4 - x^3$ and $x^3 - x$.*

Firstly Let us find the quotient and the remainder of $x^4 - x^3 \div x^3 - x$ using extension of synthetic division.

Zeros of the divisor					
0	1	−1	0	0	0
		0	0	0	0
	1	−1	0	0	**0 =** r_1
−1	1	−1	0	0	
		−1	2	−2	
	1	−2	2	**-2 =** r_2	
1	1	−2	2		
		1	−1		
	1	−1	**1 =** r_3		

Table 18: Step-1 of synthetic division for Example 5.5.

From the preceding table, $x-1$ and $x(x-1)$ are the quotient and the remainder of $x^4 - x^3 \div x^3 - x$, respectively. Hence, $x^4 - x^3 = (x-1)(x^3 - x) + x(x-1)$.
Secondly Let us find the quotient and remainder of $x^3 - x \div$ [x(x−1)] using extension of synthetic division.

Zeros of the divisor				
0	1	0	−1	0
		0	0	0
	1	0	−1	**0 =** r_1
1	1	0	−1	
		1	1	
	1	1	**0 =** r_2	

Table 19: Step-2 of synthetic division for Example 5.5.

From the preceding table, $x+1$ and 0 are the quotient and remainder of $x^3 - x \div [x(x-1)]$, respectively. Hence, $x^3 - x = (x+1)[x(x-1)] + 0$. Therefore, the GCF of the given two polynomial expressions is $\frac{1}{1}x(x-1) = x^2 - x$ since the leading coefficient of $x(x-1)$ is 1.

6. Conclusion

Using variable − wise division(long division), we can able to find the quotient and remainder of improper rational expression $\frac{P(x)}{Q(x)}$. Sometimes, this division is more complex and tedious. To find the quotient and remainder of $\frac{P(x)}{Q(x)}$ in easiest and shortest way, we need to have an easiest and simplest way. Extension of synthetic division is among of them. To deal the extension of synthetic division, we need to have the concepts of polynomial and rational expressions as preliminary concepts. This division has great applications in different ares of Mathematics as stated earlier. We would like to recommend that mathematics teachers should use the extension of synthetic division for polynomial divisions in their teaching-learning tasks.

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