INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 17 NO. 2 PAGE 551–558 (2024)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1405824
RESEARCH ARTICLE

Lorentzian G-manifolds of Constant Positive Curvature

Reza Mirzaie* and Jafar Ebrahimi

(Communicated by Kazım İlarslan)

ABSTRACT

We study the orbits arising from isometric actions of connected Lie groups on Lorentzian manifolds with constant positive curvature.

Keywords: *Lorentzian manifold, Lorentzian space, timelike plane.*

AMS Subject Classification (2020): *Primary: 53C50; Secondary: 57S25; 53C30; 53B30.*

1. Introduction

Let M be a pseudo-Riemannian manifold and G be a Lie group with the isometric action of G on M. If G is large enough, then the geometry and topology of M is closely related to G and the orbits of the action of G on M. If $m = max\{dimG(x) : x \in M\}$, then $dimM - m$ is called the cohomogeneity of the action of G on M and is denoted by $coh(M, G)$. If the cohomogeneity is zero, then the action of G on M is transitive and M is a homogeneous manifold. Study of the homogeneous Riemannian manifolds is a classic subject in differential geometry and they are almost characterized. Also, cohomogeneity one and cohomogeneity two Rimannian manifolds and their orbits have been studied by many authors (see [6], [9], [13], [15], [16], [19], [20], [21]). Isometric actions on pseudo-Riemannian manifolds and their orbits are not as well studied as in the Riemannian manifolds. Although, some interesting partial results are obtained. See [24] for some results about homogeneous flat pseudo-Riemannian manifolds, and see [25] for useful theorems about the isometry groups of homogeneous pseudo-Riemannian manifolds of constant curvature. Also, we refer to [4], [8] and [18], for some results on homogeneous and symmetric Lorentzian manifolds. If the actions on pseudo-Riemannian manifolds are not transitive, the characterization of the orbits and orbit spaces is more complicated and there are not much results in the literature. We take in the present paper the case where M is a Lorentzian manifold of constant sectional curvature. The authors of [2] and [3] take the cases where M is a cohomogeneity one Minkowski, De Siter and anti-De Siter space. Then, they studied the acting group G and the orbits. We refer to [10] to see many more valuable results about cohomogeneity one actions on anti-De Siter spacetimes. In the present article, among the other results, we show that the characterization (up to diffeomorphism) of the orbits and orbit spaces of Lorentzian G-manifolds with constant positive curvature reduces to the characterization of the orbits and orbit spaces of the isometric actions on Riemannian manifolds of constant positive curvature (Theorem 3.8, Corollary 3.10). There are many results about the acting group and the orbits of isometric actions on Riemannian manifolds of constant positive curvature, which by using of our theorems, they can be transferred to the study of isometric actions on the Lorentzian manifolds of constant positive curvature (see Theorem 4.2 and Remark 4.3 for instance).

2. Preliminaries

Let $L^{n+1}(= R_1^{n+1})$ be the usual Lorentzian space with the scalar product $< v, w >= -v_1w_1 + v_2w_2 + ... + v_{n+1}w_{n+1}$. Let $SO_0(n, 1)$ be the connected component of the identity in the Lie group of all linear isometries of

Received : *16-12-2023,* ***Accepted :*** *27-09-2024*
**** Corresponding author***

L^{n+1} and G be a Lie subgroup of $SO_0(n,1)$. The action of G is said irreducible if G does not leave invariant any proper subspace of L^{n+1} and weakly irreducible if any G-invariant subspace has a degenerate induced metric. We will use the following theorem proved in [22].

Theorem 2.1. *If G is a connected Lie subgroup of the isometries of L^{n+1} with irreducible action, then $G = SO_0(n,1)$.*

Recall that the pseudo-sphere and the hyperbolic space of radius $r > 0$ are defined by:

$$S_1^n(r) = \{x \in L^{n+1} :< x, x >= r^2\},$$

$$H^n(r) = \{x \in L^{n+1} :< x, x >= -r^2\}.$$

$S_1^n(r)$ is a Lorentzian manifold of constant positive curvature and $H^n(r)$ is a Riemannian manifold of constant negative curvature. If $r = 1$, we will denote $S_1^n(r)$ and $H^n(r)$ by S_1^n and H^n.
We will denote by Λ the null cone of L^{n+1}, $\Lambda = \{x \in L^{n+1} :< x, x >= 0\}$. The upper and lower null cones are defined by $\Lambda^+ = \{x \in \Lambda : x_1 > 0\}$ and $\Lambda^- = \{x \in \Lambda : x_1 < 0\}$. By definition, the infinity $H^n(\infty)$ of the hyperbolic space $H^n(r)$, $r > 0$, is the classes of equivalence of asymptotic geodesics (see [11]). We refer to [11] also, to see the definition of the horosphere centered at a point $z \in H^n(\infty)$. Definitions of the infinity and horospheres of $H^n(r)$ in [11] are based on the Poincare model of $H^n(r)$. In the Lorentzian model of $H^n(r)$ (the above model), each null line corresponds to a point of $H^n(\infty)$. Let l be a null line in L^{n+1} containing the origin. There is a null vector v such that $l = tv$, $t \in R$, and there is a unique hyperplane W of L^{n+1} containing l and tangent to Λ. We denote by $[W]$ the collection of all hyperplanes parallel to W. If $W' \in [W]$ and $S = W' \cap H^n(r) \neq \emptyset$, then S is a horosphere centered at l (in the Lorentzian model of $H^n(r)$). We will use the following theorem, proved in [22].
Theorem 2.2. *If G is a connected subgroup of the isometries of L^{n+1} with weakly irreducible action, then either G acts transitively on $H^n(r)$, $r > 0$, or there is a point $l \in H^n(\infty)$ such that G acts transitively on each horosphere of $H^n(r)$ centered at l.*
Remark 2.3. In Theorem 2.2, if the action of G on $H^n(r)$ is not transitive, then there is a null hyperplane W such that for all $W' \in [W]$ with $W' \cap H^n(r) \neq \emptyset$, we have $G(W' \cap H^n(r)) = W' \cap H^n(r)$ (note that the sets $W' \cap H^n(r)$ are the horospheres with common center at infinity mentioned in the above theorem). Now, It is easy to show that:
(1) For all $W' \in [W]$, $G(W') = W'$.
(2) Since the action of G on all W' with $W' \cap H^n(r) \neq \emptyset$ is of cohomogeneity one, then the action of G on L^{n+1} is of cohomogeneity two. Then, from the fact that the union of all principal orbits of the action of G on L^{n+1} is a dense subset of L^{n+1}, we get that the action of G on S_1^n is also of cohomogeneity one.

The following theorem about the decompositioin of closed and connected subgroups of $SO_0(n,1)$ will be a useful tool in the proofs.

Theorem 2.4 ([14]). *If G is a closed and connected subgroup of $SO_0(n,1)$ without null eigenvector, then either $G = SO_0(n,1)$ or there is a non-negative integer $m < n$ and a closed and connected subgroup K of the isometries of $\mathbb{R}^{n-m}$ such that $G = SO_0(m,1) \times K$.*

3. Results

By the following theorem, in isometric actions of connected groups on pseudo-spheres with big cohomogeneity, there is a useful global decomposition of the orbits. In what follows, for simplicity we consider $S_1^n(r)$ with $r = 1$ (similar results are true for all $r > 0$).

Theorem 3.1. *Let G be a closed and connected subgroup of* $SO_0(n, 1)$.
If $coh(S_1^n, G) \geq 2$, *then there is a non-negative integer* $m < n$ *and a closed and connected subgroup* K *of* $O(n - m)$ *such that*

$$G = SO_0(m, 1) \times K.$$

Proof. Three cases may occur.
(1) The action of G on L^{n+1} is irreducible. This case is not possible. Because, by Theorem 2.1, $G = SO_0(n, 1)$ and $coh(S_1^n, SO_0(n, 1)) = 0$.
(2) The action of G on L^{n+1} is reducible without null eigenvector. Then, by Theorem 2.4, either $G = SO_0(n, 1)$ which is impossible as the above, or G splits as $G = SO_0(m, 1) \times K$, $K \subset O(n - m)$, for some positive integer $m < n$.
(3) The action of G is reducible with null eigenvector (i.e, weakly irreducible). Then, by Theorem 2.2 and Remark 2.3, $coh(S_1^n, G) = 1$.

□

If in Theorem 3.1, we replace the assumption $coh(S_1^n, G) \geq 2$ with $coh(S_1^n, G) = 1$, and we consider three possibilities in the proof of that theorem, then we get that either a similar decomposition as Theorem 3.1 is true for G, or the action of G on L^{n+1} is weakly irreducible. In the later case, by similar arguments in Remark 2.3, there is a null hyperplane W such that $G(W') = W'$ for all $W' \in [W]$, which implies $G(S_1^n \cap W') = S_1^n \cap W'$. Consequently, the following corollary is true:

Corollary 3.2.
If G is a connected subgroup of $SO_0(n, 1)$ *and* $coh(S_1^n, G) = 1$, *then one of the following is true:*
1) *There is a non-negative integer* $m < n$ *and a connected subgroup* K *of* $O(n - m)$ *such that*

$$G = SO_0(m, 1) \times K.$$

2) *There is a null hyperplane* W *in* R_1^{n+1} *such that each G-orbit of* S_1^n *is included in* $S_1^n \cap W'$ *for some* $W' \in [W]$.

Now, we show that when the cohomogeneity is big, characterization of the orbits of isometric actions on pseudo-spheres reduces to characterization of the orbits of isometric actions on the usual spheres.

Theorem 3.3. *If G is a closed and connected subgroup of the isometries of* S_1^n *such that* $coh(S_1^n, G) \geq 2$, *then there is a positive integer* $m < n$ *and a connected subgroup* K *of* $O(n - m)$ *such that each G-orbit is equal to one of the following sets:*
(a) $\{0\} \times E$, $\Lambda^+ \times E$, $\Lambda^- \times E$, *where* E *is a K-orbit in* S^{n-m-1} *and* $\Lambda^{\pm}$ *are the upper and lower null cones in* L^{m+1}.
(b) $S_1^m(r) \times E$, *E is a K-orbit in* $S^{n-m-1}(1 - r), 0 < r < 1$,
(c) $H^m(r) \times E$, *E is a K-orbit in* $S^{n-m-1}(1 - r), r < 0$.
(d) $S_1^m \times \{0\}$

Proof. By Theorem 3.1, there is a positive integer m such that G splits as $G = SO(m, 1) \times K$, $K \subset O(n - m)$. Consider R_1^{n+1} as $R_1^{n+1} = R_1^{m+1} \times R^{n-m}$. Then, for each $(x, y) \in R_1^{n+1} (= R_1^{m+1} \times R^{n-m})$, $G(x, y) = SO(m, 1)(x) \times K(y)$. Denote by $<, >_0$ the usual inner product of R^{n-m}. Then,

$$S_1^n = \{(x, y) \in R_1^{m+1} \times R^{n-m} :< x, x > + < y, y >_0 = 1\}.$$

Put $< x, x >= r$. If $r = 0$ then from the fact that the orbits are connected we get (a). If $0 < r < 1$, then $x \in S_1^m(r)$ and $y \in S^{n-m-1}(1 - r)$ and we get (b). If $r < 0$, in a similar way, (c) is true and $r = 1$ implies (d). □

If in the above theorem $coh(S_1^n, G) = 1$, then the principal orbits must be of dimension $n-1$. Then, the K-orbits E must be of dimension $n-m-1$, which implies that the K-orbits E are diffeomorphic to S^{n-m-1}. Now, by the similar way in proof of the above theorem and by using of Corollary 3.2, we get the following Remark.

Remark 3.4. If G is a closed and connected subgroup of the isometries of S_1^n such that $coh(S_1^n, G) = 1$, then one of the followings is true:
(1) There is a positive integer $m < n$ such that each orbit is equal to one of the following sets:
(a) $\{0\} \times S^{n-m-1}$, $\Lambda^+ \times S^{n-m-1}$, $\Lambda^- \times S^{n-m-1}$, $\Lambda^{\pm}$ are the upper and lower null cones in L^{m+1}.
(b) $S_1^m(r) \times S^{n-m-1}(1-r)$, $0 < r < 1$.
(c) $H^m(r) \times S^{n-m-1}(1-r), r < 0$.
(d) $S_1^m \times \{0\}$.
(2) There is a null hyperplane W such that each orbit D is included in $W' \cap S_1^n$ for some $W' \in [W]$ depended to D . If D is a principal orbit, by dimensional reasons, $D = W' \cap S_1^n$.

Remark 3.5. (a consequence of the arguments in [5], pages 62-64). If M is a connected semi-Riemannian manifold and G is a connected subgroup of $Iso(M)$, and if $\widetilde{M}$ is the universal semi-Riemannian covering manifold of M with the covering map $\kappa : \widetilde{M} \to M$, then there is a connected covering $\widetilde{G}$ of G with the covering map $\pi : \widetilde{G} \to G$, such that $\widetilde{G}$ acts isometrically on $\widetilde{M}$ and
(1) Each deck transformation δ of the covering $\kappa : \widetilde{M} \to M$ maps $\widetilde{G}$-orbits on to $\widetilde{G}$-orbits.
(2) If $x \in M$ and $\widetilde{x} \in \widetilde{M}$ such that $\kappa(\widetilde{x}) = x$, then $\kappa(\widetilde{G}(\widetilde{x})) = G(x)$.
(3) The desk transformation group, which we denote it by Γ, centralizes $\widetilde{G}$ (i.e., for each $\delta \in \Gamma$ and $\widetilde{g} \in \widetilde{G}, \delta\widetilde{g} = \widetilde{g}\delta$).

The following theorem proved in [12] will play an important role in the proof of Theorem 3.7.

Theorem 3.6. *A complete, connected and homogeneous Riemannian manifold of negative curvature is simply connected.*

By the following theorem, characterization of the orbits and orbit spaces of isometric actions on Lorentzian manifolds of constant positive curvature is actually a problem in Riemannian manifolds of constant positive curvature.

Remark 3.7. If ϕ and ψ are isometries of a connected pseudo-Riemannian manifold M and there is a $p \in M$ such that $\phi(p) = \psi(p)$ and $d\phi_p = d\psi_q$, then $\phi = \psi$ (see [17], Section 3, Proposition 62). Consequently, if two isometries are equal on a open subset of M then they are equal.

Theorem 3.8. *Let M_1^n be a Lorentzian manifold of constant positive curvature and G be a closed and connected subgroup of the isometries.*
If $coh(M, G) \geq 2$, then there is a positive integer m and a connected subgroup G' of $O(n-m)$ such that:
(a) M is diffeomorphic to $S_1^m \times M'$, where $M' = \frac{S^{n-m-1}}{\Gamma}, \Gamma = \pi_1(M)$.
(b) G is covered by $SO(m, 1) \times G'$ and there is an isometric action of G' on M' such that:
(b1) $\pi_1(M) = \pi_1(M')$, $coh(M, G) = coh(M', G')$.
(b2) Each G-orbit is diffeomorphic to one of the following spaces:

$$S_1^m, \ E, \ \Lambda^+ \times E, \ \Lambda^- \times E, \ S_1^m \times E, \ H^m \times E.$$

Where, Λ^+ and Λ^- are the upper and lower null cones in L^{m+1} and E is a G'-orbit in M'.

Proof. Without lose of generality, let M be of constant curvature $c = 1$. The universal covering manifold of M is $S_1^n (= S_1^n(1))$. Keeping the symbols used in Remark 3.5, $\widetilde{G}$ acts isometrically on S_1^n with $Coh(S_1^n, \widetilde{G}) = Coh(M, G)$. By Theorem 3.1, there is a positive integer $m < n$ such that $\widetilde{G}$ decomposes as $\widetilde{G} = SO(m, 1) \times G'$, $G' \subset O(n-m)$. Then, the $\widetilde{G}$-orbits of S_1^n are equal to one of the cases mentioned in Theorem 3.3. For each $r < 0$, the set $H^m(r) \times S^{n-m-1}(1-r)$ is the union of all orbits in the form (c) of Theorem 3.3. Let Γ be the decktransformation group (which is isomorphic to $\pi_1(M)$). By Remark 3.5, Γ maps orbits to orbits and for each

$r < 0$ the submanifold of S_1^n in the form $H^m(r) \times S^{n-m-1}(1-r)$ is unique. Then,

$$\Gamma(H^m(r) \times S^{n-m-1}(1-r)) = H^m(r) \times S^{n-m-1}(1-r), r < 0.$$

A differential geometric argument involving sectional curvatures shows that each $\gamma \in \Gamma$, has the following decomposition when we consider its action on $H^m(r) \times S^{n-m-1}(1-r)$:

$$\gamma = \gamma_1 \times \gamma_2 \in Iso(H^m(r)) \times Iso(S^{n-m-1}(1-r)).$$

Since $\bigcup_{r<0}(H^m(r) \times S^{n-m-1}(1-r))$ is open in S_1^n, then by Remark 3.7, each $\gamma \in \Gamma$ decomposes as $\gamma = \gamma_1 \times \gamma_2 \in O(m,1) \times O(n-m)$ all over S_1^n. Put

$$\Gamma_1 = \{\gamma_1 : \ \gamma_1 \times \gamma_2 \in \Gamma \ \ for \ \ some \ \ \gamma_2\}$$

and define Γ_2 similarly. Thus, we have

$$\frac{H^m(r) \times S^{n-m-1}(1-r)}{\Gamma} = \frac{H^m(r)}{\Gamma_1} \times \frac{S^{n-m-1}(1-r)}{\Gamma_2}.$$

$\frac{H^m(r)}{\Gamma_1}$ is homogeneous (since $\frac{H^m(r)}{\Gamma_1} \times \frac{S^{n-m-1}(1-r)}{\Gamma_2}$ is a G-orbit and the action of G on this orbit decomposes by the above argument). Thus, by Theorem 3.6, $\gamma_1 = \{I\}$. Therefore,

$$\Gamma = \{I\} \times \Gamma_2 \quad (*)$$

For each $c = 1 - r > 0$, Put $M'(c) = \frac{S^{n-m-1}(c)}{\Gamma_2}$ and let $M' = M'(1)$. By $(*)$ part (a) of the theorem is true.
Now, Consider the universal covering map $\kappa : S^{n-m-1}(c) \to M'(c)$. G' acts on $M'(c)$ as follows:

$$g \in G', x \in M'(c), g(x) = \kappa(gy), \ \ for \ \ some \ \ y \in \kappa^{-1}(x).$$

This action of G' on $M'(c)$ is well defined(Because, if $y_1, y_2 \in \kappa^{-1}(x)$, then for some $\delta \in \Gamma_2$, $\delta(y_1) = y_2$. Since by Remark 3.5(3), $g\delta = \delta g$, then $\kappa(gy_2) = \kappa(g\delta(y_1)) = \kappa(\delta g(y_1)) = \kappa(g(y_1))$).
Now, we get the result easily from Theorem 3.3. □

Remark 3.9. If M_1^n is a Lorentzian manifold of constant curvature then it is isometric to $\frac{S_1^n}{\Gamma}$ such that Γ is a finite subgroup of $O(1) \times O(n)$.

Proof. The proof is a simple consequence of Proposition 12, Proposition 15, and Lemma 15 in Section 9 of [17]. □

The following Corollary characterizes the orbits, when the cohomogeneity is one.

Theorem 3.10. *Let M_1^n be a Lorentzian manifold of constant positive curvature and G be a closed and connected subgroup of the isometries. If $coh(M, G) = 1$, then one of the followings is true:*
(1) *There is a positive integer m such that M is diffeomorphic to $S_1^m \times M'$, where $M' = \frac{S^{n-m-1}}{\Gamma}$, $\Gamma = \pi_1(M)$. G is covered by $SO(m,1) \times G'$ and there is an isometric action of G' on M' such that $\pi_1(M) = \pi_1(M')$, $coh(M,G) = coh(M',G')$. Each G-orbit is diffeomorphic to one of the following spaces:*
S_1^m, M', $\Lambda^+ \times M'$, $\Lambda^- \times M'$, $S_1^m \times M'$, $H^m \times M'$.
Where, Λ^+ and Λ^- are the upper and lower null cones in L^{m+1}.
(2) $M = \frac{S_1^n}{Z_2}$ *and all orbits are diffeomorphic to R^{n-1}.*

Proof. Keep the symbols used in the proof of Theorem 3.8 and consider Corollary 3.2 and Remark 3.4 for the action of $\widetilde{G}$ on S_1^n. If the case (1) in Corollary 3.2 is true for the action of $\widetilde{G}$ on S_1^n, then we can repeat the proof of Theorem 3.8 to get part (1) of the theorem (note that by dimensional reasons E is diffeomorphic to M'). If the case (2) in Corollary 3.2 is true, we get part (2) of the theorem by the following argument.
Consider the null hyperplane W with the property that all $\tilde{G}$-orbits are included in $W' \cap S_1^n$ for some $W' \in [W]$. By Remark 3.9, each $\delta \in \Gamma$ belongs to $O(1) \times O(n-1)$. Let $\delta = (A, B) \in O(1) \times O(n-1)$. Since δ maps orbits to orbits, it is easy to see that δ preserves the foliation $[W]$. Since $\delta(o) = o$ and W is the unique member of $[W]$

which contains o (only one hyperplane in $[W]$ is a vector subspace of R_1^{n+1}), then $\delta(W) = W$. In other way, we have

$$S^{n-1} = S_1^n \cap (\{0\} \times R^n) \subset R_1^1 \times R^n (= R_1^{n+1}).$$

If $b = (0, x) \in S^{n-1} \subset R_1^1 \times R^n$, then $\delta(b) = (A, B)(0, x) = (0, Bx) \in S^{n-1}$. Thus, $\delta(S^{n-1}) = S^{n-1}$. Since W is null, then $W \cap S^{n-1}$ has two points, say $\{b, -b\}$. We have $\delta(W) = W$ and $\delta(S^{n-1}) = S^{n-1}$. Then, $\delta(b) = \pm$ b. If $\delta(b) = b$, then $\delta = I$(a decktransformation with a fixed point will be identity).
If $\delta(b) = -b$, then $\delta^2(b) = b$ and $\delta^2 = I$. Thus, $B^2 = I$. Since $B \in O(n-1)$, we get from $B^2 = I$ that B is symmetric. Then, there is an orthonormal basis for R^n consisting of eigenvectors of B (with eigenvalues ± 1). If one eigenvalue of B is 1, then B has fixed point and as the above argument, we get that $\delta = I$. If not, then all eigenvalues of B are -1. Thus, $B = -I$ and δ is equal to one of the following:

$$(1, I), \ (-1, I), \ (1, -I), \ (-1, -I).$$

If $\delta = (-1, I)$ then $\delta(0, x) = (0, x)$ which is contradiction (since a decktransformation with fixed point must be identity). In similar way since $(1, -I)(a, o) = (a, o)$ we have contradiction. Therefore,
$\Gamma = \{(1, I), (-1, -I)\}$ which is isomorphic to Z_2. Consider the universal covering map $\kappa : S_1^n \to M$ and let et $G(x)$ be a principal orbit in M and let $\kappa(\tilde{x}) = x$, $\tilde{x} \in S_1^n$. $\widetilde{G}(\tilde{x})$ is a connected component of $W' \cap S_1^n$, for some $W' \in [W]$ ($S_1^n \cap W'$ has two connected components and each component is diffeomorphic to R^{n-1}). But, if $\delta \in \Gamma$ is nonidentity, then by the above argument, $\delta = (-1, -I)$. Then, δ maps a connected component of $S_1^n \cap W'$ to the other one. Thus, $\kappa : \widetilde{G}(\tilde{x}) \to G(x)$ is isometry and $G(x)$ must be diffeomorphic to R^{n-1}.

□

4. Application, cohomogeneity one and cohomogeneity two Lorentzian manifolds of constant positive curvature

Cohomogeneity one Riemannian manifolds have been investigated by several authors. See [6] and [19] for some interesting results about cohomogeneity one Riemannian manifolds of non-positive curvature. When the curvature is positive the classification of the manifold and the orbits is a more complicated problem and it is not solved completely. Among other achievements in the positive curvature case, we refer to the work by C. Searle in [21], which provided a complete classification, up to diffeomorphism, when the manifold is compact of positive curvature with dimension less than seven. The seven dimensional case has been studied by F. Podesta and L. Verdiani in [20]. A. Kollross obtained a classification of cohomogeneity one hyperpolar actions on irreducible Riemannian symmetric spaces of compact type ([13]). The authors of [1] studied cohomogeneity one actions on non-simply connected Riemannian manifolds of constant positive curvature with arbitrary dimension.
In direction of the papers (mentioned in Introduction) about cohomogeneity one actions on Lorentzian manifolds, we use our theorem 3.10 to characterize the orbits of cohomogeneity one isometric actions on Lorentzian manifolds of constant positive curvature. First, we mention the following theorem which comes from Corollary 2.7.2 in chapter 2 of [25]. For definition of the groups D_k^*, T^*, O^*, I^* see [25] chapter 2.

Theorem 4.1. *A connected homogeneous n-dimensional Riemannian manifold of constant positive curvature (say 1) is isometric to one of the following spaces: RP^n, $\frac{S^n}{Z_k}$ or $\frac{S^n}{D_k^*}$ for some $k > 2$, $\frac{S^n}{T^*}$, $\frac{S^n}{O^*}$, $\frac{S^n}{I^*}$.*

We call each space mentioned in the above theorem a Wolf space and we denote it briefly by WS.
In Theorem 3.10, M' is a homogeneous Riemannian manifold of constant positive curvature (because, by the proof of Theorem 3.10, M' is a G'-orbit). Then, M' is a Wolf space and we get the following theorem from Theorem 3.10 and Theorem 4.1.

Theorem 4.2. *If M_1^n is a cohomogeneity one Lorentzian manifold of constant positive curvature, then one of the following is true:*
(1) *There is a positive integer $m < n$ and a Wolf space WS such that M_1^n is diffeomorphic to $S_1^m \times WS$. Each orbit is diffeomorphic to one of the spaces: S_1^m, $\Lambda^+ \times WS$, $\Lambda^- \times WS$, $S_1^m \times WS$, $H^m \times WS$. Where, Λ^+, Λ^- are the upper and lower null cones in R_1^{m+1}.*
(2) *$M = \frac{S_1^n}{Z_2}$ and all orbits are diffeomorphic to R^{n-1}.*

Remark 4.3. In Theorem 3.3, if the action of G on S_1^n is of cohomogeneity two, then the action of K on the spheres in parts (a),(b),(c) must be of cohomogeneity one. It is proved that if a connected group acts by chomogeneity one on S^n, then its action on R^{n+1} is hyperpolar (of cohomogeneity two). All groups with hyperpolar actions on R^{n+1} have been characterized (see [9] for definition of hyperpolar actions and the classification theorems). Thus, by our theorem 3.3, all groups which can act by cohomogeneity two on S_1^n are characterized. Similarly, in Theorem 3.8, M' is a cohomogeneity one Riemannian manifolds of constant positive curvature and this kind of manifolds and the acting group have been studied (see [1]). Consequently, our Theorem 3.8 leads to many useful results about cohomogeneity two actions on Lorentzian manifolds of constant positive curvature.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Affiliations

REZA MIRZAIE
ADDRESS: Imam Khomeini International University (IKIU), Dept. of Pure Mathematics, Qazvin-Iran.
E-MAIL: r.mirzaei@sci.ikiu.ac.ir
ORCID ID:0000-0002-9480-0222

JAFAR EBRAHIMI
ADDRESS: Imam Khomeini International University (IKIU), Dept. of Pure Mathematics, Qazvin-Iran.
E-MAIL: ebrahimijafar38@gmail.com
ORCID ID: 0009-0003-4760-8785