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ON UNRESTRICTED DUAL-GENERALIZED COMPLEX HORADAM NUMBERS

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Abstract. This research introduces a novel category of dual-generalized complex numbers, with components represented by unrestricted Horadam numbers. We present various recurrence relations, summation formulas, the Binet formula, and the generating function associated with these numbers. Additionally, a comprehensive bilinear index-reduction formula is derived, which encompasses Vajda's, Catalan's, Cassini's, D'Ocagne's, and Halton's identities as specific cases.

1. Introduction

Hypercomplex numbers have many applications such as in physics, geometry, robotics, and quantum mechanics. There are many studies related to different types of hypercomplex numbers. One among them is dual-generalized complex numbers. They are defined by Gurses et al. [11] as a generalization of dual-complex numbers, hyper-dual numbers, and dual-hyperbolic numbers. The set of dual-generalized complex numbers is defined by

$$\mathbb{DC}_{\mathfrak{p}} = \left\{a_0 + a_1 J + a_2 \varepsilon + a_3 J\varepsilon \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}, \tag{1}$$

where the dual unit ε and the generalized complex unit J adhere to the following rules:

$$J^2 = \mathfrak{p}, -\infty < \mathfrak{p} < \infty, \varepsilon^2 = 0, \varepsilon \neq 0, \varepsilon J = J\varepsilon. \tag{2}$$

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Table 1. Multiplication table of units J, ε, $J\varepsilon$.

	1	J	ε	$J\varepsilon$
1	1	J	ε	$J\varepsilon$
J	J	$\mathfrak{p}$	$J\varepsilon$	$\mathfrak{p}\varepsilon$
ε	ε	$J\varepsilon$	0	0
$J\varepsilon$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	0	0

The multiplication scheme for the basis elements of dual generalized complex numbers can also be given in the following table.

Clearly, when the parameter $\mathfrak{p}$ takes the value of -1, the newly introduced commutative number system corresponds into dual-complex numbers. Similarly, for $\mathfrak{p} = 0$, it aligns with hyper-dual numbers, and for $\mathfrak{p} = 1$, it corresponds to dual-hyperbolic numbers. Consequently, an examination of dual-generalized complex numbers allows for the simultaneous understanding of dual-complex numbers, hyper-dual numbers, and dual-hyperbolic numbers. For a more in-depth understanding of dual-generalized complex numbers, one may refer to the relevant literature [5, 6, 9, 11, 17, 18] and the cited references therein.

Extensive research has been conducted on quaternion sequences within specific quaternion algebras. Notably, Horadam [14] explored Fibonacci quaternions within the realm of real quaternion algebra, focusing on quaternion sequences comprising Fibonacci number components. Expanding on the concept of Fibonacci quaternions, Sentürk et al. [19] introduced unrestricted Horadam quaternions within a generalized quaternion algebra by

$$H_n^{(x,y,z)} = w_n + w_{n+x}i + w_{n+y}j + w_{n+z}k,$$

where $\{w_n\}$ is the Horadam sequence [15] defined by

$$w_n = pw_{n-1} + qw_{n-2}, \quad n \geq 2 \tag{3}$$

with the arbitrary initial values w_0, w_1 and nonzero integers p, q. Here the basis $\{1, i, j, k\}$ satisfies the following multiplication rules:

$$\begin{aligned} i^2 &= -\lambda,\ j^2 = -\mu, k^2 = -\lambda\mu, \\ ij &= -ji = k,\ jk = -kj = \mu i,\ ki = -ik = \lambda j, \end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. For $\lambda = \mu = 1$, it simplifies to the real quaternion algebra, and when $x = 1, y = 2$, and $z = 3$, the unrestricted Horadam quaternions reduce to the Horadam quaternions in [13]. Some matrix representations of Horadam quaternions can be found in [22], and for some recent papers related to special types of quaternions with unrestricted subscripts can be found in [2, 3, 7, 8]. For more on Horadam sequences, see [16, 20].

Several researchers have explored the realm of dual-generalized complex numbers incorporating components resembling Fibonacci sequences. Specifically, Cihan et al. [4] pioneered the study of dual-hyperbolic Fibonacci and Lucas numbers, while

Gungor and Azak [10] established the framework for dual-complex Fibonacci and Lucas numbers. In a similar context, Tan et al. [21] introduced the concept of hyper-dual Horadam quaternions. Furthermore, Gurses et al. [12] innovatively presented the dual-generalized complex Fibonacci quaternions, utilizing dual Fibonacci numbers as coefficients in lieu of real numbers. Recently, Tan and Ocal [23] introduced the dual generalized complex Horadam quaternions.

Inspired by the studies mentioned earlier, we now present the unrestricted dual generalized complex Horadam numbers. We obtain some recurrence relations, the generating function, and the Binet formula of these numbers. We also obtain the general bilinear index-reduction formula of these numbers which reduces to the Vajda's, Halton's, Catalan's, Cassini's, and D'Ocagne's identities as a special case. Moreover, we give summation formulas and a matrix representation of them.

We conclude this section with some preliminaries related to the Horadam sequence.

The Horadam sequence $\{w_n\}$ transforms into the (p,q)-Fibonacci sequence $\{u_n\}$ when $w_0 = 0, w_1 = 1$, and into the (p,q)-Lucas sequence $\{v_n\}$ when $w_0 = 2, w_1 = p$. When $p = q = 1$, these sequences simplify to the traditional Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$, respectively.

The Binet formula of Horadam sequence $\{w_n\}$ is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{4}$$

where $A := w_1 - w_0\beta, B := w_1 - w_0\alpha$, and α, β are the roots of the characteristic polynomial $x^2 - px - q$, that is; $\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2}$. Also we have $\alpha\beta = -q, \alpha + \beta = p, \Delta := \alpha - \beta = \sqrt{p^2 + 4q}$ with $p^2 + 4q > 0$.

2. Main Results

In this section, we initially establish the concept of unrestricted dual-generalized complex Horadam numbers, followed by an exploration of some fundamental properties associated with these numbers. Throughout this section, we simply denote the unrestricted dual-generalized complex Horadam numbers as unrestricted DGC Horadam numbers. Let also x, y and z be arbitrary positive integers.

Definition 1. *The nth unrestricted DGC Horadam number is defined as*

$$\widetilde{w}_n^{(x,y,z)} = w_n + w_{n+x}J + w_{n+y}\varepsilon + w_{n+z}J\varepsilon,$$

where w_n is the nth Horadam number, ε is dual unit, and J is generalized complex unit adhering to the multiplication rules in (2).

In the following table, we give some special cases of the unrestricted dual-generalized complex DGC Horadam numbers $\widetilde{w}_n^{(1,2,3)}$. We should note that when

TABLE 2. Special cases of the unrestricted DGC Horadam numbers.

$\mathfrak{p}$	p	q	w_0	w_1	
$\mathfrak{p}$	1	1	0	1	DGC Fibonacci numbers [12]
$\mathfrak{p}$	1	1	2	1	DGC Lucas numbers [12]
-1	1	1	0	1	Dual-complex Fibonacci numbers [10]
-1	1	1	2	1	Dual-complex Lucas numbers [10]
-1	k	1	0	1	Dual-complex k-Fibonacci numbers [1]
1	1	1	0	1	Dual-hyperbolic Fibonacci numbers [4]
1	1	1	2	1	Dual-hyperbolic Lucas numbers [4]
0	1	1	w_0	w_1	Hyper-dual Fibonacci numbers [22]

$x = 1, y = 2$, and $z = 3$, the unrestricted dual-generalized complex Horadam numbers $\widetilde{w}_n^{(x,y,z)}$ reduce to the conventional dual generalized complex Horadam numbers in [23].

The addition, subtraction, and multiplication of two unrestricted DGC Horadam numbers $\widetilde{w}_n^{(x,y,z)}$ and $\widetilde{w}_m^{(x,y,z)}$ are defined as

$$\begin{aligned}\widetilde{w}_n^{(x,y,z)} \pm \widetilde{w}_m^{(x,y,z)} &= (w_n \pm w_m) + (w_{n+x} \pm w_{m+x}) J \\ &\quad + (w_{n+y} \pm w_{m+y}) \varepsilon + (w_{n+z} \pm w_{m+z}) J\varepsilon\end{aligned}$$

and

$$\begin{aligned}\widetilde{w}_n^{(x,y,z)} \widetilde{w}_m^{(x,y,z)} &= (w_n w_m + \mathfrak{p} w_{n+x} w_{m+x}) + (w_n w_{m+x} + w_{n+x} w_m) J \\ &\quad + (w_n w_{m+y} + w_{n+y} w_m + \mathfrak{p} w_{n+x} w_{m+z} + \mathfrak{p} w_{n+z} w_{m+x}) \varepsilon \\ &\quad + (w_n w_{m+z} + w_{n+x} w_{m+y} + w_{n+y} w_{m+x} + w_{n+z} w_m) J\varepsilon,\end{aligned}$$

respectively.

Theorem 1. *The unrestricted DGC Horadam numbers satisfy the following relation:*

$$\widetilde{w}_n^{(x,y,z)} = p\widetilde{w}_{n-1}^{(x,y,z)} + q\widetilde{w}_{n-2}^{(x,y,z)}, \; n \geq 2.$$

Proof. Using the definition of unrestricted DGC Horadam numbers and the definition of classical Horadam numbers, we get

$$\begin{aligned}p\widetilde{w}_{n-1}^{(x,y,z)} + q\widetilde{w}_{n-2}^{(x,y,z)} &= p\,(w_{n-1} + w_{n-1+x}J + w_{n-1+y}\varepsilon + w_{n-1+z}J\varepsilon) \\ &\quad + q\,(w_{n-2} + w_{n-2+x}J + w_{n-2+y}\varepsilon + w_{n-2+z}J\varepsilon) \\ &= (pw_{n-1} + qw_{n-2}) + (pw_{n-1+x} + qw_{n-2+x}) J \\ &\quad + (pw_{n-1+y} + qw_{n-2+y}) \varepsilon + (pw_{n-1+z} + qw_{n-2+z}) J\varepsilon \\ &= w_n + w_{n+x}J + w_{n+y}\varepsilon + w_{n+z}J\varepsilon = \widetilde{w}_n^{(x,y,z)}.\end{aligned}$$

□

In the following Theorem, we give a relation between (p, q)-Fibonacci numbers and the unrestricted DGC Horadam numbers.

Theorem 2. *For $n \geq 1$, we have*

$$\widetilde{w}_n^{(x,y,z)} = u_n \widetilde{w}_1^{(x,y,z)} + q u_{n-1} \widetilde{w}_0^{(x,y,z)}.$$

Proof. From the definition of (p,q)-Fibonacci numbers and the definition of the unrestricted DGC Horadam numbers, we get

$$\begin{aligned} & u_n \left(w_1 + w_{x+1} J + w_{y+1}\varepsilon + w_{z+1} J\varepsilon\right) + q u_{n-1} \left(w_0 + w_x J + w_y \varepsilon + w_z J\varepsilon\right) \\ = \;& u_n w_1 + q u_{n-1} w_0 \\ & + \left(u_n w_{x+1} + q u_{n-1} w_x\right) J + \left(u_n w_{y+1} + q u_{n-1} w_y\right)\varepsilon + \left(u_n w_{z+1} + q u_{n-1} w_z\right) J\varepsilon \\ = \;& w_n + w_{n+x} J + w_{n+y}\varepsilon + w_{n+z} J\varepsilon \\ = \;& \widetilde{w}_n^{(x,y,z)}. \end{aligned}$$

□

Theorem 3. *The generating function for unrestricted DGC Horadam numbers is*

$$G(t) = \frac{\widetilde{w}_0^{(x,y,z)} + \left(\widetilde{w}_1^{(x,y,z)} - p\widetilde{w}_0^{(x,y,z)}\right) t}{1 - pt - qt^2}.$$

Proof. Let

$$G(t) := \sum_{n=0}^{\infty} \widetilde{w}_n^{(x,y,z)} t^n = \widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)} t + \sum_{n=2}^{\infty} \widetilde{w}_n^{(x,y,z)} t^n.$$

From Theorem 1, we have

$$\begin{aligned} & \left(1 - pt - qt^2\right) G(t) \\ = \;& \widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)} t + \sum_{n=2}^{\infty} \widetilde{w}_n^{(x,y,z)} t^n - p\widetilde{w}_0^{(x,y,z)} t - p\sum_{n=2}^{\infty} \widetilde{w}_{n-1}^{(x,y,z)} t^n - q\sum_{n=2}^{\infty} \widetilde{w}_{n-2}^{(x,y,z)} t^n \\ = \;& \widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)} t - p\widetilde{w}_0^{(x,y,z)} t + \sum_{n=2}^{\infty} \left(\widetilde{w}_n^{(x,y,z)} - p\widetilde{w}_{n-1}^{(x,y,z)} - q\widetilde{w}_{n-2}^{(x,y,z)}\right) t^n \\ = \;& \widetilde{w}_0^{(x,y,z)} + \left(\widetilde{w}_1^{(x,y,z)} - p\widetilde{w}_0^{(x,y,z)}\right) t. \end{aligned}$$

Thus, we get the desired result. □

Theorem 4. *The Binet formula of unrestricted DGC Horadam numbers is*

$$\widetilde{w}_n^{(x,y,z)} = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta},$$

where $\underline{\alpha} = 1 + \alpha^x J + \alpha^y \varepsilon + \alpha^z J\varepsilon$ *and* $\underline{\beta} = 1 + \beta^x J + \beta^y \varepsilon + \beta^z J\varepsilon$.

Proof. Using the Binet formula of Horadam numbers in (4), we have

$$\widetilde{w}_n^{(x,y,z)} = w_n + w_{n+x} J + w_{n+y}\varepsilon + w_{n+z} J\varepsilon$$

$$\begin{aligned} &= \left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right) + \left(\frac{A\alpha^{n+x} - B\beta^{n+x}}{\alpha-\beta}\right) J \\ &\quad + \left(\frac{A\alpha^{n+y} - B\beta^{n+y}}{\alpha-\beta}\right)\varepsilon + \left(\frac{A\alpha^{n+z} - B\beta^{n+z}}{\alpha-\beta}\right) J\varepsilon \\ &= \frac{A\alpha^n}{\alpha-\beta}\left(1+\alpha^x J+\alpha^y\varepsilon+\alpha^z J\varepsilon\right) - \frac{B\beta^n}{\alpha-\beta}\left(1+\beta^x J+\beta^y\varepsilon+\beta^z J\varepsilon\right) \\ &= \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha-\beta}. \end{aligned}$$

□

From Theorem 4, we derive the Binet formulas of unrestricted DGC (p,q)-Fibonacci and Lucas cases:

$$\widetilde{u}_n^{(x,y,z)} = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha-\beta} \quad \text{and} \quad \widetilde{v}_n^{(x,y,z)} = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n, \tag{5}$$

respectively. By considering (5), the following relation can be easily derived:

$$\widetilde{v}_n^{(x,y,z)} = \widetilde{u}_{n+1}^{(x,y,z)} + q\widetilde{u}_{n-1}^{(x,y,z)}.$$

To establish various properties of unrestricted DGC Horadam numbers, we require the following lemma.

Lemma 1. *Let x, y, z be positive integers with $z > y > x$. Then we have*

$$\underline{\alpha}\underline{\beta} = \widetilde{v}_0^{(x,y,z)} - 1 + (-q)^x\left(\left(1+v_{z-x}\varepsilon\right)\mathfrak{p} + v_{y-x}J\varepsilon\right).$$

Proof.

$$\begin{aligned} \underline{\alpha}\underline{\beta} &= \left(1+\alpha^x J+\alpha^y\varepsilon+\alpha^z J\varepsilon\right)\left(1+\beta^x J+\beta^y\varepsilon+\beta^z J\varepsilon\right) \\ &= 1+\mathfrak{p}\left(\alpha\beta\right)^x \\ &\quad + \left(\alpha^x+\beta^x\right) J \\ &\quad + \left(\alpha^y+\beta^y+\mathfrak{p}\left(\alpha^x\beta^z+\alpha^z\beta^x\right)\right)\varepsilon \\ &\quad + \left(\alpha^z+\beta^z+\alpha^x\beta^y+\alpha^y\beta^x\right) J\varepsilon \end{aligned}$$

$$\begin{aligned} \underline{\alpha}\underline{\beta} &= 1+\mathfrak{p}\left(-q\right)^x + v_x J + v_y\varepsilon + \mathfrak{p}\left(\alpha^x\beta^z+\alpha^z\beta^x\right)\varepsilon + v_z J\varepsilon + \left(\alpha^x\beta^y+\alpha^y\beta^x\right) J\varepsilon \\ &= 1+v_x J+v_y\varepsilon+v_z J\varepsilon+\mathfrak{p}\left(-q\right)^x+\mathfrak{p}\left(\alpha^x\beta^z+\alpha^z\beta^x\right)\varepsilon+\left(\alpha^x\beta^y+\alpha^y\beta^x\right) J\varepsilon \\ &= \widetilde{v}_0^{(x,y,z)} - 1 + \mathfrak{p}\left(-q\right)^x + \mathfrak{p}\left(\alpha^x\beta^z+\alpha^z\beta^x\right)\varepsilon + \left(\alpha^x\beta^y+\alpha^y\beta^x\right) J\varepsilon \\ &= \widetilde{v}_0^{(x,y,z)} - 1 + \mathfrak{p}\left(-q\right)^x + \mathfrak{p}\left(\left(\alpha\beta\right)^x\left(\alpha^{z-x}+\beta^{z-x}\right)\right)\varepsilon + \left(\left(\alpha\beta\right)^x\left(\alpha^{y-x}+\beta^{y-x}\right)\right) J\varepsilon \\ &= \widetilde{v}_0^{(x,y,z)} - 1 + \left(-q\right)^x\left(\mathfrak{p}+\mathfrak{p}\left(\alpha^{z-x}+\beta^{z-x}\right)\varepsilon+\left(\alpha^{y-x}+\beta^{y-x}\right) J\varepsilon\right) \\ &= \widetilde{v}_0^{(x,y,z)} - 1 + \left(-q\right)^x\left(\left(1+v_{z-x}\varepsilon\right)\mathfrak{p}+v_{y-x}J\varepsilon\right). \end{aligned}$$

□

Utilizing the Binet formula for unrestricted DGC Horadam numbers and applying Lemma 1, we derive the following identity.

Theorem 5. *(General bilinear index-reduction formula) For nonnegative integers* a, b, c, d *such that* $a+b=c+d$, $b>a, d>c$, *we have*

$$\widetilde{w}_a^{(x,y,z)}\widetilde{w}_b^{(x,y,z)}-\widetilde{w}_c^{(x,y,z)}\widetilde{w}_d^{(x,y,z)}=-\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^a v_{b-a}-(-q)^c v_{d-c}\right).$$

Proof. Let $\Delta=\alpha-\beta$. Using the Binet formula of unrestricted DGC Horadam numbers, we have

$$\begin{aligned}
&(\alpha-\beta)^2\left(\widetilde{w}_a^{(x,y,z)}\widetilde{w}_b^{(x,y,z)}-\widetilde{w}_c^{(x,y,z)}\widetilde{w}_d^{(x,y,z)}\right)\\
&=\left(A\underline{\alpha}\alpha^a-B\underline{\beta}\beta^a\right)\left(A\underline{\alpha}\alpha^b-B\underline{\beta}\beta^b\right)-\left(A\underline{\alpha}\alpha^c-B\underline{\beta}\beta^c\right)\left(A\underline{\alpha}\alpha^d-B\underline{\beta}\beta^d\right)\\
&=A^2\underline{\alpha}^2\alpha^{a+b}-AB\underline{\alpha}\underline{\beta}\alpha^a\beta^b-AB\underline{\beta}\underline{\alpha}\alpha^b\beta^a+B^2\underline{\beta}^2\beta^{a+b}\\
&\quad-A^2\underline{\alpha}^2\alpha^{c+d}+AB\underline{\alpha}\underline{\beta}\alpha^c\beta^d+AB\underline{\beta}\underline{\alpha}\beta^c\alpha^d-B^2\underline{\beta}^2\beta^{c+d}\\
&=A^2\underline{\alpha}^2\left(\alpha^{a+b}-\alpha^{c+d}\right)-AB\underline{\alpha}\underline{\beta}\left(\alpha^a\beta^b-\alpha^c\beta^d+\alpha^b\beta^a-\alpha^d\beta^c\right)+B^2\underline{\beta}^2\left(\beta^{a+b}-\beta^{c+d}\right)
\end{aligned}$$

$$\begin{aligned}
&= -AB\underline{\alpha}\underline{\beta}\left(\alpha^a\beta^b+\alpha^b\beta^a-\alpha^c\beta^d-\alpha^d\beta^c\right)\\
&= -AB\underline{\alpha}\underline{\beta}\left[\left((\alpha\beta)^a\left(\alpha^{b-a}+\beta^{b-a}\right)\right)-\left((\alpha\beta)^c\left(\alpha^{d-c}+\beta^{d-c}\right)\right)\right]\\
&= -AB\underline{\alpha}\underline{\beta}\left((-q)^a v_{b-a}-(-q)^c v_{d-c}\right).
\end{aligned}$$

Thus we get the desired result. □

From Theorem 5, we have the following corollaries.

Corollary 1. *(Vajda's identity) For* $a=m+k, b=n-k, c=m$, *and* $d=n$, *we have*

$$\begin{aligned}
&\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_{n-k}^{(x,y,z)}-\widetilde{w}_m^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}\\
&= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^{m+k}v_{n-m-2k}-(-q)^m v_{n-m}\right)\\
&= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}(-q)^m\left((-q)^k v_{n-m-2k}-v_{n-m}\right).
\end{aligned}$$

Since $v_{n-m}-(-q)^k v_{n-m-2k}=\Delta^2 u_k u_{n-m-k}$, we also have

$$\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_{n-k}^{(x,y,z)}-\widetilde{w}_m^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}=AB\underline{\alpha}\underline{\beta}(-q)^m u_k u_{n-m-k}.$$

Corollary 2. *(Catalan's identity) For* $a=n-m, b=n+m$ *and* $c=d=n$, *we have*

$$\begin{aligned}
&\widetilde{w}_{n-m}^{(x,y,z)}\widetilde{w}_{n+m}^{(x,y,z)}-\widetilde{w}_n^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}\\
&= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^{n-m}v_{2m}-2(-q)^n\right)
\end{aligned}$$

$$= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left(-q\right)^{n-m}\left(v_{2m}-2\left(-q\right)^m\right).$$

Since $v_{2m}-2\left(-q\right)^m=\Delta^2 u_m^2$, we also have

$$\widetilde{w}_{n-m}^{(x,y,z)}\widetilde{w}_{n+m}^{(x,y,z)}-\widetilde{w}_{n}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}=-AB\underline{\alpha}\underline{\beta}(-q)^{n-m}u_m^2.$$

Corollary 3. *(Cassini's identity) For $a=n-1, b=n+1$ and $c=d=n$, we have* $\widetilde{w}_{n-1}^{(x,y,z)}\widetilde{w}_{n+1}^{(x,y,z)}-\widetilde{w}_{n}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}=-AB\underline{\alpha}\underline{\beta}(-q)^{n-1}$

By using Lemma 1, we get

$$\widetilde{w}_{n-1}^{(x,y,z)}\widetilde{w}_{n+1}^{(x,y,z)}-\widetilde{w}_{n}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}$$
$$=-AB(-q)^{n-1}\left(\widetilde{v}_0^{(x,y,z)}-1+(-q)^x\left(\left(1+v_{z-x}\varepsilon\right)\mathfrak{p}+v_{y-x}J\varepsilon\right)\right).$$

Corollary 4. *(d'Ocagne's identity) For $a=n, b=m+1, c=n+1,$ and $d=m$, we have*

$$\widetilde{w}_{n}^{(x,y,z)}\widetilde{w}_{m+1}^{(x,y,z)}-\widetilde{w}_{n+1}^{(x,y,z)}\widetilde{w}_{m}^{(x,y,z)}=-\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left(-q\right)^n\left(v_{m-n+1}+qv_{m-n-1}\right).$$

Since $v_{m-n+1}+qv_{m-n-1}=-\Delta^2 u_{m-n}$, we also have

$$\widetilde{w}_{n}^{(x,y,z)}\widetilde{w}_{m+1}^{(x,y,z)}-\widetilde{w}_{n+1}^{(x,y,z)}\widetilde{w}_{m}^{(x,y,z)}=AB\underline{\alpha}\underline{\beta}(-q)^n u_{m-n}.$$

Corollary 5. *(Halton's identity) For $a=m+k, b=n, c=k,$ and $d=m+n$, we have*

$$\begin{aligned}\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}-\widetilde{w}_{k}^{(x,y,z)}\widetilde{w}_{m+n}^{(x,y,z)} &= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left(\left(-q\right)^{m+k}v_{n-m-k}-\left(-q\right)^k v_{m+n-k}\right)\\ &= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left(-q\right)^k\left(\left(-q\right)^m v_{n-k-m}-v_{n-k+m}\right).\end{aligned}$$

Since $v_{n-k+m}-\left(-q\right)^m v_{n-k-m}=\Delta^2 u_m u_{n-k}$, we have

$$\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}-\widetilde{w}_{k}^{(x,y,z)}\widetilde{w}_{m+n}^{(x,y,z)}=AB\underline{\alpha}\underline{\beta}(-q)^k u_m u_{n-k}.$$

Next, we give a relation between the unrestricted DGC (p,q)-Fibonacci numbers and the unrestricted DGC (p,q)-Lucas numbers.

Theorem 6. *For nonnegative integers n and m such that $m \geq n$, we have*

$$\begin{aligned}\widetilde{v}_{n}^{(x,y,z)}\widetilde{u}_{m}^{(x,y,z)}-\widetilde{v}_{m}^{(x,y,z)}\widetilde{u}_{n}^{(x,y,z)} &= 2\left(-q\right)^n u_{m-n}\Bigg(\widetilde{v}_0^{(x,y,z)}-1\\ &+ \left(-q\right)^x\left(\left(1+v_{z-x}\varepsilon\right)\mathfrak{p}+v_{y-x}J\varepsilon\right)\Bigg).\end{aligned}$$

Proof. Using the Binet formula of unrestricted DGC Horadam numbers, we have

$$\begin{aligned}&\Delta\left(\widetilde{v}_{n}^{(x,y,z)}\widetilde{u}_{m}^{(x,y,z)}-\widetilde{v}_{m}^{(x,y,z)}\widetilde{u}_{n}^{(x,y,z)}\right)\\ =&\left(\underline{\alpha}\alpha^n+\underline{\beta}\beta^n\right)\left(\underline{\alpha}\alpha^m-\underline{\beta}\beta^m\right)-\left(\underline{\alpha}\alpha^m+\underline{\beta}\beta^m\right)\left(\underline{\alpha}\alpha^n-\underline{\beta}\beta^n\right)\end{aligned}$$

$$\begin{aligned}
&= \underline{\alpha}^2\alpha^{n+m} - \underline{\alpha}\underline{\beta}\alpha^n\beta^m + \underline{\alpha}\underline{\beta}\alpha^m\beta^n - \underline{\beta}^2\beta^{n+m} \\
&\quad -\underline{\alpha}^2\alpha^{n+m} + \underline{\alpha}\underline{\beta}\alpha^m\beta^n - \underline{\alpha}\underline{\beta}\alpha^n\beta^m + \underline{\beta}^2\beta^{n+m} \\
&= 2\left(\alpha\beta\right)^n \underline{\alpha}\underline{\beta}\left(\alpha^{m-n} - \beta^{m-n}\right) \\
&= 2\left(-q\right)^n \underline{\alpha}\underline{\beta}\Delta u_{m-n}.
\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}
\widetilde{v}_n^{(x,y,z)}\widetilde{u}_m^{(x,y,z)} - \widetilde{v}_m^{(x,y,z)}\widetilde{u}_n^{(x,y,z)} &= 2\left(-q\right)^n u_{m-n}\Big(\widetilde{v}_0^{(x,y,z)} - 1 \\
&+ \left(-q\right)^x\left(\left(1 + v_{z-x}\varepsilon\right)\mathfrak{p} + v_{y-x}J\varepsilon\right)\Big).
\end{aligned}$$

□

Presently, we provide a sum formula for unrestricted DGC Horadam numbers.

Theorem 7. *For $n \geq 2$, we have*

$$\sum_{r=1}^{n-1}\widetilde{w}_r^{(x,y,z)} = \frac{\widetilde{w}_n^{(x,y,z)} - \widetilde{w}_1^{(x,y,z)} + q\left(\widetilde{w}_{n-1}^{(x,y,z)} - \widetilde{w}_0^{(x,y,z)}\right)}{p+q-1}.$$

Proof. Using the Binet formula for unrestricted DGC Horadam numbers, we have

$$\begin{aligned}
\sum_{r=1}^{n-1}\widetilde{w}_r^{(x,y,z)} &= \sum_{r=1}^{n-1}\frac{A\underline{\alpha}\alpha^r - B\underline{\beta}\beta^r}{\alpha-\beta} \\
&= \frac{A\underline{\alpha}}{\alpha-\beta}\sum_{r=1}^{n-1}\alpha^r - \frac{B\underline{\beta}}{\alpha-\beta}\sum_{r=1}^{n-1}\beta^r \\
&= \frac{A\underline{\alpha}}{\alpha-\beta}\left(\frac{\alpha^n-\alpha}{\alpha-1}\right) - \frac{B\underline{\beta}}{\alpha-\beta}\left(\frac{\beta^n-\beta}{\beta-1}\right) \\
&= \frac{1}{\left(\alpha-\beta\right)\left(1-p-q\right)}\left(-\left(A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n\right) - q\left(A\underline{\alpha}\alpha^{n-1} - B\underline{\beta}\beta^{n-1}\right)\right. \\
&\quad \left. +q\left(A\underline{\alpha} - B\underline{\beta}\right) + \left(A\underline{\alpha}\alpha - B\underline{\beta}\beta\right)\right) \\
&= \frac{-\widetilde{w}_n^{(x,y,z)} - q\widetilde{w}_{n-1}^{(x,y,z)} + q\widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)}}{1-p-q}.
\end{aligned}$$

□

Theorem 8. *For nonnegative integers n and r, we have*

$$\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^m\widetilde{w}_{m+r}^{(x,y,z)} = \widetilde{w}_{2n+r}^{(x,y,z)}.$$

Proof. Using the Binet formula for unrestricted DGC Horadam numbers, we obtain

$$
\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\widetilde{w}_{m+r}^{(x,y,z)}\\
&=\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\left(\frac{A\underline{\alpha}\alpha^{m+r}-B\underline{\beta}\beta^{m+r}}{\alpha-\beta}\right)\\
&=\frac{A\underline{\alpha}\alpha^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}q^{n-m}\left(p\alpha\right)^{m}-\frac{B\underline{\beta}\beta^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}q^{n-m}\left(p\beta\right)^{m}\\
&=\frac{A\underline{\alpha}\alpha^{r}}{\alpha-\beta}\left(q+p\alpha\right)^{n}-\frac{B\underline{\beta}\beta^{r}}{\alpha-\beta}\left(q+p\beta\right)^{n}\\
&=\frac{A\underline{\alpha}\alpha^{2n+r}-B\underline{\beta}\beta^{2n+r}}{\alpha-\beta}=\widetilde{w}_{2n+r}^{(x,y,z)}.
\end{aligned}
$$

□

Ultimately, we present a matrix representation for unrestricted DGC Horadam numbers.

Theorem 9. *For $n \geq 0$, we have*

$$
\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^{n}\begin{bmatrix} \widetilde{w}_{2}^{(x,y,z)} & \widetilde{w}_{1}^{(x,y,z)} \\ \widetilde{w}_{1}^{(x,y,z)} & \widetilde{w}_{0}^{(x,y,z)} \end{bmatrix}=\begin{bmatrix} \widetilde{w}_{n+2}^{(x,y,z)} & \widetilde{w}_{n+1}^{(x,y,z)} \\ \widetilde{w}_{n+1}^{(x,y,z)} & \widetilde{w}_{n}^{(x,y,z)} \end{bmatrix}.
$$

Proof. By using Theorem 2, it can be easily demonstrated through mathematical induction on n. □

By computing the determinant on both sides of the matrix equality mentioned earlier, we derive Cassini's identity for the sequence $\{\widetilde{w}_n\}$ in a straightforward manner as:

$$
\widetilde{w}_{n+2}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}-\widetilde{w}_{n+1}^{(x,y,z)}\widetilde{w}_{n+1}^{(x,y,z)}=(-q)^{n}\left(\widetilde{w}_{2}^{(x,y,z)}\widetilde{w}_{0}^{(x,y,z)}-\widetilde{w}_{1}^{(x,y,z)}\widetilde{w}_{1}^{(x,y,z)}\right).
$$

3. Conclusion

In this paper we define a novel category of dual-generalized complex numbers, with components represented by unrestricted Horadam numbers. The main advantage to introducing unrestricted dual-generalized complex Horadam numbers is that many unrestricted dual-generalized complex numbers with the well-known numbers such as Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell, Pell-Lucas can be deduced as particular cases of these unrestricted DGC numbers. We state recurrence relations, summation formulas, Binet formula, and generating function associated with these numbers. In addition, a comprehensive bilinear index-reduction formula is derived, which encompasses Vajda's, Catalan's, Cassini's, D'Ocagne's, and Halton's identities as specific cases. For interested readers, the results of this paper could be applied for any other type of hypercomplex numbers.

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