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Research Article

# Differentiating under $q$–integral sign

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**Abstract**

The Leibniz integral rule enables us to interchange the order of differentiation and integration under some differentiability conditions on the functions. It can be very useful in the computing the exact value of certain integrals. In this paper, we will present analogs of such rule for $q$-integrals with functional borders and their properties.

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## 1. Introduction

The method of differentiation, under the integral sign, concerns integrals depending on a parameter. It was introduced by G. Leibniz in 1697, and now it is known as the Leibniz integral rule. It represents a formula for differentiation of a definite integral with the functional borders and the integrand which depends on a parameter more.

**Theorem 1.1.** (Leibniz integral rule) *Suppose that $f(x,y)$ and its partial derivative $\partial_x f(x,y)$ are continuous in the rectangle $\Delta = [a,b]\times[c,d]$. If the functions $\varphi,\psi :\ [a,b]\to[c,d]$ are continuously differentiable on $[a,b]$, then*

$$\begin{aligned}\frac{d}{dx}\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \\ = \int_{\varphi(x)}^{\psi(x)} \partial_x f(x,y)\,dy + f\left(x,\psi(x)\right)\psi'(x) - f\left(x,\varphi(x)\right)\varphi'(x).\end{aligned} \tag{1.1}$$

This rule can be used to evaluate unusual definite integrals, as it was done in [1,2,9].
An extension to the fractional calculus can be found in [5].
Our purpose is to state its analogy in the $q$-calculus.

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## 2. Preliminaries

In the theory of $q$-calculus (see [3, 4, 6–8]), for a real parameter $q \in (0,1)$, we introduce a $q$-real number $[a]_q$ by

$$[a]_q := \frac{1-q^a}{1-q} \qquad (a \in \mathbb{R}).$$

Notice that

$$[a]_q [b]_{q^a} = [ab]_q \qquad (a, b \in \mathbb{R}).$$

The $q$-analog of the Pochhammer symbol ($q$–shifted factorial) is defined by:

$$(a;q)_0 = 1\ , \qquad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \qquad k \in \mathbb{N}\cup\{\infty\}\,.$$

Also, $q$-analog of the power $(a-b)^k$ is

$$(a-b;q)^{(0)} = 1\ , \qquad (a-b;q)^{(k)} = \prod_{i=0}^{k-1}(a-bq^i), \qquad k \in \mathbb{N}\cup\{\infty\} \quad (a,b\in\mathbb{R}).$$

The relationship between them is given by

$$(a-b;q)^{(n)} = a^n\ (b/a;q)_n \qquad (a \neq 0).$$

In that way, for $n \in \mathbb{N}_0$, it holds

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} = \frac{(1-q;q)^{(n)}}{(1-q)^n}.$$

Their natural expansions to the reals are

$$(a-b;q)^{(\alpha)} = a^{\alpha}\frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}\ , \qquad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha \in \mathbb{R}),$$

and

$$(a-b;q)^{(\alpha)} = a^\alpha\,(1-b/a;q)^{(\alpha)} = a^\alpha(b/a;q)_\alpha\,.$$

It is true that

$$(qx-qt;q)^{(\beta)} = q^\beta(x-t;q)^{(\beta)}.$$

We can define $q$-binomial coefficients with

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q = \frac{(q^{\beta+1};q)_\infty\ (q^{\alpha-\beta+1};q)_\infty}{(q;q)_\infty\ (q^{\alpha+1};q)_\infty}\,, \qquad \alpha,\beta,\alpha-\beta \in \mathbb{R}\setminus\{-1,-2,\ldots\}\,.$$

Especially,

$$\begin{bmatrix}\alpha\\ k\end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\ (-1)^k q^{\alpha k} q^{-\binom{k}{2}} \quad (k\in\mathbb{N})\ .$$

## 3. Differential $q$–calculus

We define $q$-derivative of a function $f(x)$ by

$$(D_{q,x}f)(x) = \frac{f(x)-f(qx)}{x-qx} \quad (x \neq 0).$$

For a pair of functions $u(x)$ and $v(x)$ and constants $\alpha,\beta \in \mathbb{R}$, we have linearity and product rules

$$D_{q,x}(\alpha\ u(x)+\beta\ v(x)) = \alpha(D_{q,x}u)(x)+\beta(D_{q,x}v)(x),$$
$$D_{q,x}(u(x)\,v(x)) = u(qx)(D_{q,x}v)(x)+v(x)(D_{q,x}u)(x)\ .$$

It will be useful to notice the following:

$$\begin{aligned} D_{q,x}\big((x-a;q)^{(\alpha)}\big) &= [\alpha]_q(x-a;q)^{(\alpha-1)} \\ D_{q,x}\big((a-x;q)^{(\alpha)}\big) &= -[\alpha]_q(a-qx;q)^{(\alpha-1)}. \end{aligned}$$

The composite rule is not valid in general. A known case can be provided by next lemmas.

**Lemma 3.1.** *Let $F(x)=f(u(x))$, where $u=ax^c$, $x>0$ $(a\neq 0, c\neq 0)$. Then*

$$D_{q,x}F(x)=D_{q^c,u}f(u)\ D_{q,x}u(x).$$

**Lemma 3.2.** *Let $F(x)=f\big(u_1(x),u_2(x),\dots,u_n(x)\big)$, where*

$$u_k(x)=a_kx^{c_k},\quad x>0\qquad (a_k,c_k\neq 0,\ k=1,2,\dots,n).$$

*Then the following holds:*

$$\begin{aligned} D_{q,x}F(x)=&D_{q^{c_1},u_1}f\big(u_1,u_2,\dots,u_n\big)\ D_{q,x}u_1(x) \\ &+\sum_{k=2}^{n}D_{q^{c_k},u_k}f\big(u_1(qx),\dots,u_{k-1}(qx),u_k,u_{k+1},\dots,u_n\big)\ D_{q,x}u_k(x). \end{aligned}$$

## 4. On $q$-integrals with functional borders

The $q$-integral is defined by

$$(I_{q,0}f)(x)=\int_0^x f(t)\,d_qt=x(1-q)\sum_{k=0}^{\infty}f(xq^k)\,q^k\qquad(0<|q|<1),\tag{4.1}$$

and

$$(I_{q,a}f)(x)=\int_a^x f(t)\,d_qt=\int_0^x f(t)\,d_qt-\int_0^a f(t)\,d_qt.\tag{4.2}$$

For operators defined in this manner, the following is valid:

$$(D_{q,x}I_{q,a}f)(x)=f(x),\qquad (I_{q,a}D_{q,x}f)(x)=f(x)-f(a).$$

The formula for $q$-integration by parts is

$$\int_a^b u(t)\big(D_{q,x}v\big)(t)\,d_qt=\big[u(t)v(t)\big]_{t=a}^{t=b}-\int_a^b v(qt)\big(D_{q,x}u\big)(t)\,d_qt.$$

**Lemma 4.1.** *The $Q$-derivative of $q$-integral with functional upper border satisfies*

$$D_{Q,x}\int_0^{\varphi(x)}f(t)\ d_qt=\frac{1}{x(1-Q)}\int_{\varphi(Qx)}^{\varphi(x)}f(t)\ d_qt\qquad(x\neq 0).\tag{4.3}$$

***Proof.*** Let us denote by $F(x)=\displaystyle\int_0^{\varphi(x)}f(t)\,d_qt$. According to the definition of $Q$-derivative, we can write

$$D_{Q,x}F(x)=\frac{F(x)-F(Qx)}{(1-Q)x}=\frac{1}{x(1-Q)}\left(\int_0^{\varphi(x)}f(t)\ d_qt-\int_0^{\varphi(Qx)}f(t)\ d_qt\right),$$

wherefrom (4.3) follows. □

**Corollary 4.2.** *If $c\cdot d\in\mathbb{N}$, $c,d>0$, then*

$$D_{q^d,x}\int_0^{ax^c}f(t)\ d_qt=\frac{ax^{c-1}}{[d]_q}\sum_{k=0}^{cd-1}f(ax^cq^k)q^k\quad(x\neq 0).\tag{4.4}$$

***Proof.*** According to the definition (4.1), we have

$$\int_0^{ax^c} f(t)\, d_qt = ax^c(1-q)\sum_{n=0}^{\infty} f\left(ax^cq^n\right)q^n,$$

and

$$\int_0^{ax^cq^{cd}} f(t)\, d_qt = ax^cq^{cd}(1-q)\sum_{m=0}^{\infty} f\left(ax^cq^{cd}q^m\right)q^m$$
$$= ax^c(1-q)\sum_{n=cd}^{\infty} f\left(ax^cq^n\right)q^n.$$

Applying the relation (4.3), the statement (4.4) follows. □

**Lemma 4.3.** *The following Leibniz type rule for $q$-integrals is valid:*

$$D_{q,x}\int_0^{bx} f(x,y)\, d_qy = \int_0^{bx} D_{q,x}f(x,y)\, d_qy + bf(qx,bx) \qquad (b,x>0).$$

***Proof.*** The left side can be written in the form

$$\begin{aligned}
\mathrm{LS} &= D_{q,x}\int_0^{bx} f(x,y)\, d_qy = D_{q,x}\left(bx(1-q)\sum_{k=0}^{\infty} f(x,bxq^k)q^k\right)\\
&= b(1-q)D_{q,x}\left(\sum_{k=0}^{\infty} xf(x,bq^kx)q^k\right) = b(1-q)\sum_{k=0}^{\infty} q^kD_{q,x}\left(x\cdot f(x,bq^kx)\right)\\
&= b(1-q)\sum_{k=0}^{\infty} q^k\frac{xf(x,bq^kx)-qxf(qx,bxq^{k+1})}{x(1-q)}\\
&= b\sum_{k=0}^{\infty}\left(f(x,bxq^k)-qf(qx,bxq^{k+1})\right)q^k.
\end{aligned}$$

Let us denote by

$$\mathrm{RS} = \int_0^{bx} D_{q,x}f(x,y)\, d_qy = \int_0^{bx}\frac{f(x,y)-f(qx,y)}{(1-q)x}\, d_qy.$$

According to definition of $q$-integral, we have

$$\mathrm{RS} = b\sum_{k=0}^{\infty}\left(f(x,bxq^k)-f(qx,bxq^k)\right)q^k.$$

Hence

$$\mathrm{LS}-\mathrm{RS} = -bq\sum_{k=0}^{\infty} f(qx,bxq^{k+1})q^k + b\sum_{k=0}^{\infty} f(qx,bxq^k)q^k = bf(qx,bx).$$

□

By the previous lemma, the next theorem follows.

**Theorem 4.4.** *The following Leibniz type rule for $q$-integrals on the interval $[ax,bx]$, where $0\le a<b$ and $x>0$, is valid:*

$$D_{q,x}\int_{ax}^{bx} f(x,y)\, d_qy = \int_{ax}^{bx} D_{q,x}f(x,y)\, d_qy + bqf(qx,bx) - aqf(qx,ax).$$

In the sequel we give a general result related to interchanging $q$-derivative and $q$-integral.

**Theorem 4.5.** *The following Leibniz type rule for q-integrals is valid:*

$$D_{q,x}\int_0^{\varphi(x)} f(x,y)\;d_qy=\int_0^{\varphi(x)} D_{q,x}f(x,y)\;d_qy+R(f,\varphi,x,q), \tag{4.5}$$

*where the remainder term is given by*

$$R(f,\varphi,x,q)=\frac{1}{(1-q)x}\int_{\varphi(qx)}^{\varphi(x)} f(qx,y)\;d_qy \qquad (x>0). \tag{4.6}$$

***Proof.*** Let us introduce

$$F(x)=\int_0^{\varphi(x)} f(x,y)d_qy, \qquad G(x)=\int_0^{\varphi(x)} D_{q,x}f(x,y)\;d_qy.$$

According to the definition of $q$-integral, we have

$$F(x)=(1-q)\varphi(x)\sum_{n=0}^{\infty} f\left(x,\varphi(x)q^n\right)q^n=(1-q)\sum_{n=0}^{\infty}\varphi(x)f(x,\varphi(x)q^n)q^n,$$

wherefrom

$$D_{q,x}F(x)=(1-q)\sum_{n=0}^{\infty}D_{q,x}\Big(\varphi(x)f(x,\varphi(x)q^n)\Big)q^n.$$

Having in mind that

$$D_{q,x}\left(\varphi(x)f\left(x,\varphi(x)q^n\right)\right)=\frac{\varphi(x)f\left(x,\varphi(x)q^n\right)-\varphi(qx)f\left(qx,\varphi(qx)q^n\right)}{x(1-q)}\;,$$

we get

$$D_{q,x}F(x)=\frac{1}{x}\sum_{n=0}^{\infty}\Big(\varphi(x)f(x,\varphi(x)q^n)-\varphi(qx)f(qx,\varphi(qx)q^n)\Big)q^n.$$

From the other side,

$$\begin{aligned}G(x)&=\int_0^{\varphi(x)}\frac{f(x,y)-f(qx,y)}{x(1-q)}\;d_qy\\&=\frac{1}{x}\varphi(x)\sum_{n=0}^{\infty}\Big(f(x,\varphi(x)q^n)-f(qx,\varphi(x)q^n)\Big)q^n.\end{aligned}$$

Now,

$$\begin{aligned}D_{q,x}F(x)-G(x)&=\frac{1}{x}\sum_{n=0}^{\infty}\Big(\varphi(x)f\left(qx,\varphi(x)q^n\right)-\varphi(qx)f(qx,\varphi(qx)q^n)\Big)q^n\\&=\frac{1}{x(1-q)}\left(\int_0^{\varphi(x)} f(qx,y)\;d_qy-\int_0^{\varphi(qx)} f(qx,y)\;d_qy\right)\\&=\frac{1}{x(1-q)}\int_{\varphi(qx)}^{\varphi(x)} f(qx,y)\;d_qy.\end{aligned}$$

□

Now, we are able to formulate the rule very close to (1.1).

**Corollary 4.6.** *The following Leibniz type rule for q-integrals is valid*

$$D_{q,x}\int_{\varphi(x)}^{\psi(x)} f(x,y)\;d_qy=\int_{\varphi(x)}^{\psi(x)} D_{q,x}f(x,y)\;d_qy+\mathcal{R}(f,\varphi,\psi,x,q),$$

*where the remainder term is*

$$\mathcal{R}(f,\varphi,\psi,x,q)=\frac{1}{x(1-q)}\left(\int_{\psi(qx)}^{\psi(x)} f(qx,y)\;d_qy-\int_{\varphi(qx)}^{\varphi(x)} f(qx,y)\;d_qy\right) \qquad (x>0).$$

In some special cases, we can get the expression for $R(f,\varphi,x,q)$ similar to (1.1) including some mean value relations [8].

**Theorem 4.7.** *Let $\varphi(x)=bx^m$, where $b>0$, $m\in\mathbb{N}_0$ and $x>0$. Then the rule (4.5) for q-integrals is valid with*

$$R(f,\varphi,x,q)=\frac{1}{[m]_q}\sum_{k=0}^{m-1}q^k f\Big(qx,\varphi(x)q^k\Big)\; D_{q,x}\varphi(x).$$

***Proof.*** Using the notation from the proof of the previous theorem, we can write

$$\begin{aligned}&D_{q,x}F(x)-G(x)\\&=\frac{1}{x}\left(-\sum_{k=0}^{\infty}q^k\varphi(qx)\,f\big(qx,\varphi(qx)q^k\big)+\varphi(x)\sum_{k=0}^{\infty}q^k f\big(qx,\varphi(x)q^k\big)\right).\end{aligned}\tag{4.7}$$

Including the definition of $\varphi(x)$ from the assumptions in this theorem, we have

$$\begin{aligned}D_{q,x}&F(x)-G(x)\\&=bx^{m-1}\left(-\sum_{k=0}^{\infty}q^{m+k}\,f\big(qx,bx^mq^{m+k}\big)+\sum_{k=0}^{\infty}q^k f\big(qx,bx^mq^k\big)\right)\\&=bx^{m-1}\sum_{k=0}^{m-1}q^k f\big(qx,bx^mq^k\big),\end{aligned}$$

wherefrom we finish the proof. □

**Theorem 4.8.** *Let $f(x,y)=f_1(x)f_2(y)$, with $f_2(x)=y^a$, and $\varphi(x)=bx^c$, where $a,b,c\in\mathbb{R}^+$ and $x>0$. Then the rule (4.5) for q-integrals is valid with*

$$R(f,\varphi,x,q)=f\big(qx,\varphi(x)\big)D_{q^{a+1},x}\varphi(x).$$

***Proof.*** The formula (4.7) with supposed forms of functions $f(x,y)$ and $\varphi(x)$, becomes

$$\begin{aligned}D_{q,x}&F(x)-G(x)\\&=\frac{1}{x}\left(-\sum_{k=0}^{\infty}q^kbq^cx^c\,f_1(qx)f_2\big(bq^cx^cq^k\big)+bx^c\sum_{k=0}^{\infty}q^kf_1(qx)f_2\big(bx^cq^k\big)\right)\\&=f_1(qx)bx^{c-1}\left(-\sum_{k=0}^{\infty}q^{c+k}\big(bq^{c+k}x^c\big)^a+\sum_{k=0}^{\infty}q^k\big(bx^cq^k\big)^a\right)\\&=f_1(qx)b^{a+1}x^{ac+c-1}\left(-q^{(a+1)c}\sum_{k=0}^{\infty}\big(q^{a+1}\big)^k+\sum_{k=0}^{\infty}\big(q^{a+1}\big)^k\right)\\&=f_1(qx)b^{a+1}x^{ac+c-1}\frac{1-q^{(a+1)c}}{1-q^{a+1}}\\&=f_1(qx)\big(bx^c\big)^abx^{c-1}[c]_{q^{a+1}}=f\big(qx,\varphi(x)\big)D_{q^{a+1},x}\varphi(x).\end{aligned}$$

□

## 5. Limit cases

In this section, we show that the previous considerations lead us to the remainder in the classical Leibniz integral rule when $q$ tends to 1.

**Theorem 5.1.** *Let $\varphi(x)=bx^m$, where $b>0$, $m\in\mathbb{N}_0$ and $x>0$. Then the remainder term (4.6) satisfies the following property:*

$$\lim_{q\to1}R(f,\varphi,x,q)=f\left(x,\varphi(x)\right)\varphi'(x).$$

***Proof.*** According to Theorem 4.7 we have

$$R(f,\varphi,x,q) = \frac{1}{[m]_q}\sum_{k=0}^{m-1} q^k f\Big(qx,\varphi(x)q^k\Big)\ D_{q,x}\varphi(x)$$
$$= bx^{m-1}\sum_{k=0}^{m-1} f(xq,bx^m q^k)q^k.$$

Hence

$$\lim_{q\to 1} R(f,\varphi,x,q) = bx^{m-1}\sum_{k=0}^{m-1} f(x,bx^m) = bmx^{m-1}f(x,bx^m),$$

what we wanted to prove. □

**Theorem 5.2.** *Let $f(x,y)=f_1(x)f_2(y)$, with $f_2(x)=y^a$, and $\varphi(x)=bx^c$, where $a,b,c\in\mathbb{R}^+$ and $x>0$. For the remainder term (4.6), the following property is valid*

$$\lim_{q\to 1} R(f,\varphi,x,q) = f\left(x,\varphi(x)\right)\varphi'(x).$$

*Proof.* Under conditions of Theorem 4.8, we have

$$R(f,\varphi,x,q) = f\big(qx,\varphi(x)\big)D_{q^{a+1},x}\varphi(x),$$

and consequently

$$\lim_{q\to 1} R(f,\varphi,x,q) = \lim_{q\to 1} f\big(qx,\varphi(x)\big)D_{q^{a+1},x}\varphi(x) = f\left(x,\varphi(x)\right)\varphi'(x).\ \Box$$

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