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STRONGLY GRADED MODULES AND POSITIVELY GRADED MODULES WHICH ARE UNIQUE FACTORIZATION MODULES

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Abstract. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a strongly graded module over strongly graded ring $D = \oplus_{n\in\mathbb{Z}} D_n$. In this paper, we prove that if M_0 is a unique factorization module (UFM for short) over D_0 and D is a unique factorization domain (UFD for short), then M is a UFM over D. Furthermore, if D_0 is a Noetherian domain, we give a necessary and sufficient condition for a positively graded module $L = \oplus_{n\in\mathbb{Z}_0} M_n$ to be a UFM over positively graded domain $R = \oplus_{n\in\mathbb{Z}_0} D_n$.

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1. Introduction

Throughout this paper $R = \oplus_{n\in\mathbb{Z}_0} D_n$ is a positively graded domain which is a sub-domain of the strongly graded domain $D = \oplus_{n\in\mathbb{Z}} D_n$ and $L = \oplus_{n\in\mathbb{Z}_0} M_n$ is a positively graded module over R which is a subset of the strongly graded module $M = \oplus_{n\in\mathbb{Z}} M_n$.

In [12], the authors defined a concept of a Unique Factorization Module (UFM for short) by submodule approach. They also proved that the concept of UFM by [12] is equivalent to Nicolas's UFM (see [8] for the definition of a UFM by Nicolas), which is defined in terms of irreducible elements of D and M. In [12], the authors proved that if M is a UFM, then the polynomial module $M[x]$ is also a UFM. There are several papers on UFMs, see for example [1], [5], [8] and [9].

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On the other hand, in [11], it is shown that $D = \oplus_{n\in\mathbb{Z}} D_n$, a strongly graded domain, is a G-Dedekind domain if and only if D_0 is a G-Dedekind domain. Moreover, in [2], it is shown that $R = \oplus_{n\in\mathbb{Z}_0} D_n$, a positively graded domain, is a Unique Factorization Domain (UFD for short) if and only if D_0 is a UFD and D_1 is a principal D_0-module.

The aim of this paper is to extend the main results of [11] and [12] to a strongly graded module $M = \oplus_{n\in\mathbb{Z}} M_n$. The paper is organized as follows. In Section 2, we give some characterizations of a unique factorization module that will be very useful in Section 3 and Section 4. We refer the reader to [2], [6], [7], [11] and [13], and for details regarding graded rings, and to [3] regarding graded modules, that are not mentioned in this paper.

In Section 3, we prove that if M_0 is a UFM over D_0, then $M = \oplus_{n\in\mathbb{Z}} M_n$ is a UFM over D as a generalization of the result in [12]. We give some examples of strongly graded modules which are UFMs.

In Section 4, we give the necessary and sufficient condition for a positively graded module $L = \oplus_{n\in\mathbb{Z}_0} M_n$ to be a UFM. We end Section 4 with some examples of positively graded modules which are UFMs as the application of the main result of this paper, that is, Theorem 4.8.

We refer the reader to [10], [14], [15] for the concept and properties of v-submodules and [4] to some properties of multiplicative ideal theory which are not mentioned in this paper.

2. Unique factorization modules

Let M be a torsion-free module over an integral domain D with the quotient fi eld K. In [15], the authors defined the following concepts.

Definition 2.1. (1) A non-zero D-submodule N of KM is called a fractional D-submodule in KM if there is a non-zero $r \in D$ such that $rN \subseteq M$ and $KN = KM$.

(2) A non-zero D-submodule $\mathfrak{a}$ of K is called a fractional M-ideal in K if there is a non-zero $m \in M$ such that $\mathfrak{a}m \subseteq M$.

We represent $F(M)$ as the collection of all fractional D-submodules in KM, while $F_M(D)$ refers to the set containing all fractional M-ideals in K. Suppose $N \in F(M)$ and $\mathfrak{a} \in F_M(D)$. We define $N^- = \{k \in K \mid kN \subseteq M\}$ and $\mathfrak{a}^+ = \{m \in KM \mid \mathbf{a}m \subseteq M\}$. Then, it easily follows that $N^- \in F_M(D)$ and $\mathfrak{a}^+ \in F(M)$.

For $N \in F(M)$ and $\mathfrak{a} \in F_M(D)$, we define $N_v = (N^-)^+$ and $\mathfrak{a}_{v1} = (\mathfrak{a}^+)^-$. Consequently, $N_v \in F(M)$, and it satisfies $N_v \supseteq N$. Similarly, $\mathfrak{a}_{v1} \in F_M(D)$, and it satisfies $\mathfrak{a}_{v1} \supseteq \mathfrak{a}$. When $N = N_v$, we classify N as a fractional v-submodule in KM. Moreover, if $\mathfrak{a} = \mathfrak{a}_{v1}$, we refer to $\mathfrak{a}$ as a v_1-ideal (with respect to M). We refer the reader to [10], [14], [15] for details regarding v-submodules and v_1-operation.

In [12], the authors defined a concept of a unique factorization module by a submodule approach. The authors gave the definition and characterization of unique factorization modules as follows.

Definition 2.2. [[12], Definition 2] A torsion-free module M over an integral domain D is called a unique factorization module (UFM for short) if

(1) M is completely integrally closed (CIC for short), that is, $O_K(N) = \{k \in K \mid kN \subseteq N\} = D$ for every non-zero submodule N of M, where K is the quotient field of D,
(2) every v-submodule of M is principal,
(3) M satisfies the ascending chain condition on v-submodules of M.

Theorem 2.3. [[12], Theorem 1] *Suppose $O_K(M) = D$. The following conditions are equivalent:*

(1) *M is a UFM.*
(2) *M is a v-multiplication module and D is a UFD.*
(3) (a) *D is a UFD.*
    (b) *For every prime element p of D, pM is a maximal v-submodule.*
    (c) *For every v-submodule N of M, $\mathfrak{n} = (N : M) \neq \{0\}$, where $(N : M) = \{r \in D \mid rM \subseteq N\}$.*
(4) *Every v-submodule of M is principal and D is a UFD.*

Note that if M is a finitely generated torsion-free D-module, then $O_K(M) = D$ by Lemma 2.1 of [3]. Throughout this paper, *M is a finitely generated torsion-free D-module and satisfies the ascending chain condition on v-submodules of M.*

Lemma 2.4. *Let P be a maximal v-submodule of M. Then P is a prime submodule.*

Proof. Let $rm \in P$ where $r \in D$ and $m \in M$. If $m \notin P$, then $P \subset Dm + P \subseteq (Dm + P)_v \subseteq M$ and so $(Dm + P)_v = M$. Then $P \supseteq (Drm + rP)_v = (r(Dm + P))_v = r(Dm + P)_v = rM$. Hence P is prime. □

The following theorem will be very useful for Section 3 and Section 4.

Theorem 2.5. *Suppose that D is a UFD and M is a CIC module that satisfies the ascending chain condition on v-submodules of M. Then M is a UFM if and only if every prime v-submodule of M is principal.*

Proof. If M is a UFM, then it is clear that every prime v-submodule of M is principal, by Theorem 2.3. Conversely, we assume on the contrary that M is not a UFM. Let N be a v-submodule of M which is not principal and we may assume that N is maximal with this property because M satisfies the ascending chain condition on v-submodules of M. Let P be a maximal v-submodule of M containing N. Then $P = pM$ for some non-zero $p \in D$ by Lemma 2.4. Since $N \subset P \subset M$, we have $N \subseteq p^{-1}N \subset M$ and so $(p^{-1}N)_v = p^{-1}N_v = p^{-1}N$. Then $N = p^{-1}N$ or $p^{-1}N$ is principal by the maximality of N. If $p^{-1}N$ is principal, then $p^{-1}N = tM$ for some $t \in D$ and so $N = ptM$, a contradiction. Hence $N = p^{-1}N$, which implies $p^{-1} \in O_K(N) = D$. Then $P = pM \supseteq p(p^{-1}M) = M$, a contradiction. Hence every v-submodule N of M is principal and so M is a UFM. □

In a UFD, the concept of a principal ideal, a v-ideal, and an invertible ideal are equivalent.

Remark 2.6. Let D be a UFD and A be a v-ideal of D. Then

(1) D is a UFM over D.
(2) A is a UFM over D.
(3) If M is a finitely generated projective module over D, then M is a UFM. In particular, every finite direct sum of D is a UFM.

Proof. (1) It is clear.
(2) Note that A is principal since A is a v-ideal of D. Then A is isomorphic to D as a D-module. Hence A is a UFM by (1).
(3) By Theorem 3.1 of [15], M is a v-multiplication module. Then by Theorem 2.3, M is a UFM since D is a UFD. □

3. Strongly graded modules which are UFMs

Throughout this section, $D = \oplus_{n\in\mathbb{Z}} D_n$ is a strongly graded domain. It is known that D is a G-Dedekind domain if and only if D_0 is a G-Dedekind domain by Theorem 2.1 of [11]. Assume that K_0 and K are the quotient fields of D_0 and D respectively. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a strongly graded module over D where M_0 is a finitely generated torsion-free D_0-module and we assume that M satisfies the ascending chain condition on v-submodules of M. In this section, we will prove that M is a UFM over D if M_0 is a UFM over D_0.

Note that the concept of a principal ideal, a v-ideal, and an invertible ideal are equivalent in a UFD. We begin this section with the following proposition.

Proposition 3.1. *If D_0 is a UFD then $D = \oplus_{n\in\mathbb{Z}} D_n$ is a UFD.*

Proof. Suppose that D_0 is a UFD, that is, D_0 is a maximal order and each prime v-ideal P_0 of D_0 is principal (see Proposition 1 of [3]). Then D is a maximal order by Theorem 1 of [6]. Let P be a non-zero prime v-ideal of D. If $P_0 = P \cap D_0 \neq \{0\}$, then $P = P_0 D$ and P_0 is a v-ideal of D_0 (see the proof of Theorem 2.1 and Lemma 1.2 of [11]). So $P_0 = p_0 D_0 = D_0 p_0$ for some $p_0 \in P_0$ and $P = p_0 D = D p_0$ follows. If $P_0 = P \cap D_0 = (0)$, then $P = w A_0^{-1} B_0 D$ for some invertible ideals A_0, B_0 of D_0, which implies P is principal since D_0 is a UFD. Thus P is principal and hence D is a UFD by Proposition 1 of [2]. $\square$

Recall that a module M over a CIC domain D is a UFM if and only if every prime v-submodule P of M is principal, that is, $P = pM$ for some element $p \in D$ by Theorem 2.5 (see [12] for the definition of a UFM).

Note that M is a finitely generated torsion-free D-module since M_0 is a finitely generated torsion-free D_0-module. Furthermore, M_0 is CIC if and only if M is CIC by Theorem 3.1 of [3].

In the rest of this section, we assume that M_0 *is a UFM*. Then D_0 *is a UFD* (see Theorem 2.3).

Now, we study the structure of v-submodule P of M with $P \cap M_0 \neq \{0\}$.

Lemma 3.2. [Lemma 5.1 of [3]] *Let N_0 be a fractional D_0-submodule of M_0 with $N_0 \subseteq M_0$ and $N = DN_0$. Then*

(1) *$N^- = D(N_0)^-$, and*
(2) *$N_v = D(N_0)_v$.*

Lemma 3.3. *Let P be a prime D-submodule of M with $P_0 = P \cap M_0 \neq \{0\}$. Then*

(1) *P_0 is a prime submodule of M_0, and*
(2) *$P' = DP_0$ is a prime submodule of M.*
(3) *If P is a prime v-submodule, then P_0 is a prime v-submodule of M_0, and $P = DP_0$.*

Proof. (1) See Lemma 5.2 of [3].

(2) See Lemma 5.2 of [3].

(3) Let $P' = DP_0 \subseteq M$. Consider that $P = P_v \supseteq (P')_v = (DP_0)_v = D(P_0)_v$ by Lemma 3.2. Thus $P_0 = P \cap M_0 \supseteq D(P_0)_v \cap M_0 = (P_0)_v$. Hence $P_0 = (P_0)_v$ and so P_0 is a prime v-submodule by (1).

Note that $P' = DP_0 = Dp_0 M_0$ for some non-zero $p_0 \in D_0$ because M_0 is a UFM. Since Dp_0 is an invertible ideal, $(P')^- = (Dp_0)^{-1} = Dp_0^{-1} \supseteq P^-$, which implies $D \supseteq Dp_0P^-$ and $P^{'} = Dp_0M_0 = Dp_0M \supseteq Dp_0P^-P$. If $P \supset P'$ then $Dp_0P^-M \subseteq P' \subseteq Dp_0M$ since P' is a prime submodule by (2). Then $P^-M \subseteq M$ and so $P^- = D$ since M is CIC. Thus $P = P_v = (P^-)^+ = (D)^+ = M$, a contradiction. Hence $P = DP_0$. $\square$

In the rest of this section, we assume that M *satisfies the ascending chain condition on v-submodules of M.*

Proposition 3.4. *Let N be a v-submodule of M with $N_0 = N \cap M_0 \neq \{0\}$. Then*

(1) *N_0 is a v-submodule of M_0 and $N_0 = \mathfrak{n}_0M_0$ for some ideal $\mathfrak{n}_0$ of D_0.*
(2) *$N = D\mathfrak{n}_0M$ and $D\mathfrak{n}_0 = (N : M)$.*

Proof. (1) Similar to the previous lemma, we get that N_0 is a v-submodule of M_0. Furthermore, $N_0 = \mathfrak{n}_0M_0$ for some ideal $\mathfrak{n}_0$ of D_0 follows since M_0 is a UFM over D_0 and by Theorem 2.3.

(2) Suppose there is a v-submodule N such that $N \neq D\mathfrak{n}_0M$ where $\mathfrak{n}_0$ is an ideal of D_0. We may assume that N is maximal with this property because M satisfies the ascending chain condition on v-submodules of M. Then there is a maximal v-submodule P with $P \supseteq N$ and $P = D\mathfrak{p}_0M$, where $\mathfrak{p}_0$ is a maximal ideal of D_0. It follows that $M \supseteq (D\mathfrak{p}_0)^{-1}N \supseteq N$. If $(D\mathfrak{p}_0)^{-1}N = N$, then $(D\mathfrak{p}_0)^{-1} \subseteq D$, a contradiction because M is CIC. Thus $(D\mathfrak{p}_0)^{-1}N \supset N$ and it follows from Lemma 3.2 of [14] that $((D\mathfrak{p}_0)^{-1}N)_v = (D\mathfrak{p}_0)^{-1}N$. By the choice of N, $(D\mathfrak{p}_0)^{-1}N = D\mathfrak{t}_0M$ for some ideal $\mathfrak{t}_0$ of D_0. Hence $N = D\mathfrak{p}_0\mathfrak{t}_0M$, a contradiction. Hence $N = D\mathfrak{n}_0M$ for some ideal $\mathfrak{n}_0$ of D_0. The last statement easily follows since $D\mathfrak{n}_0$ is invertible. $\square$

Next we study the structure of a prime v-submodule P of M such that $P \cap M_0 = \{0\}$. Since $K^g = \oplus_{n\in\mathbb{Z}}K_0D_n = K_0D$ is a principal ideal domain by [6] and K_0M is a finitely generated torsion-free K^g-module, we have that a v-submodule P_1 of K_0M is prime if and only if $P_1 = \mathfrak{p}_1K_0M$, where $\mathfrak{p}_1$ is a maximal ideal of K^g such that $\mathfrak{p}_1 = (P_1 : K_0M)$ by Theorem 3.3 of [14].

Note that if D_0 is a UFD and $\mathfrak{p}$ is a prime v-ideal of D, then $\mathfrak{p} = \mathfrak{p}_0D$ for some prime v-ideal $\mathfrak{p}_0$ of D_0 or $\mathfrak{p} = \mathfrak{p}_1 \cap D$ for some prime ideal $\mathfrak{p}_1$ of K_0D by Lemma 2.6 of [11], and moreover $\mathfrak{p} = pD$ for some $p \in D$ by Proposition 3.1.

The following lemma is a graded version of Lemma 4.5 of [14].

Lemma 3.5. *Let N be a D-submodule of M. Then*

(1) $(K_0N : K_0M) = K_0\mathfrak{n}$, *where* $\mathfrak{n} = (N : M)$ *and* $K_0N^- = (K_0N)^-$.

(2) $(K_0N)_v = K_0N_v$.

Proof. See the proof of Lemma 5.4 of [3]. □

The following lemma is a graded version of Lemma 4.6 of [14]. We write the proof because we need v_1-operation to prove the last properties (see [10], [14], [15] for details regarding v-submodules and v_1-operation).

Lemma 3.6. *Let M_0 be a UFM over D_0 and $P_1 = \mathfrak{p}_1K_0M$ be a prime v-submodule of K_0M, where $\mathfrak{p}_1$ is a maximal ideal of K_0D, $P = P_1 \cap M$ and $\mathfrak{p} = \mathfrak{p}_1 \cap D$. Then*

(1) *P is a prime submodule of M and $\mathfrak{p} = (P : M)$.*

(2) *$K_0P = P_1$ and $P \cap M_0 = \{0\}$.*

(3) *$P = \mathfrak{p}M$ and P is a maximal v-submodule of M*

Proof. (1) See Lemma 5.5 (1) of [3].

(2) See Lemma 5.5 (2) of [3].

(3) By Lemma 3.5 and (2), we have $P_1 = (P_1)_v = (K_0P)_v = K_0P_v$, so P is a v-submodule of M. Since M is a v-Noetherian D-module, there are finite elements $m_i \in P$ such that $P = (Dm_1 + \ldots + Dm_k)_v$. Note that $K_0P = K_0(Dm_1 + \ldots + Dm_k)_v = (K_0Dm_1 + \ldots + K_0Dm_k)_v$ by Lemma 3.5. Further since $K_0P = P_1 = K_0\mathfrak{p}K_0M = \mathfrak{p}K_0M$, for m_i there are finite $p_{ij} \in \mathfrak{p}$ and $l_{ij} \in K_0M$ such that $m_i = \sum_j p_{ij}l_{ij}$. Then there is a non-zero $c \in D_0$ with $cl_{ij} \in M$ for all l_{ij} so that $cm_i \in \mathfrak{p}M$. Put $\mathfrak{a} = \{r_0 \in D_0 \mid r_0P \subseteq \mathfrak{p}M\}$, an ideal of D_0 with $\mathfrak{a}P \subseteq \mathfrak{p}M$. If $\mathfrak{a} = D_0$, then $P = \mathfrak{p}M$ and we are done. If $\mathfrak{a} \subset D_0$, by Lemma 3.2 of [10], $\mathfrak{a}_{v_1}P \subseteq (\mathfrak{a}_{v_1}P)_v = (\mathfrak{a}P)_v \subseteq (\mathfrak{p}M)_v = \mathfrak{p}M_v = \mathfrak{p}M$ because $\mathfrak{p}$ is an invertible ideal. By the definition of $\mathfrak{a}$, we have that $\mathfrak{a}_{v_1} \subseteq \mathfrak{a}$, which implies $\mathfrak{a}_{v_1} = \mathfrak{a}$, that is, $\mathfrak{a}$ is a v_1-ideal of D_0. Since $\mathfrak{a}$ is a v_1-ideal of D_0, $\mathfrak{a}^+$ is a v-submodule of M_0 by Lemma 2.3 of [15], which implies $\mathfrak{a}^+ = r_0M_0$ because M_0 is a UFM. Then $\mathfrak{a} = \mathfrak{a}_{v_1} = (\mathfrak{a}^+)^- = (r_0M_0)^- = r_0^{-1}D_0$ and so $\mathfrak{a}$ is an invertible ideal. Note that $\mathfrak{p}^{-1}\mathfrak{a}P \subseteq M$ and $K_0\mathfrak{p}^{-1}\mathfrak{a}P = K_0\mathfrak{p}^{-1}\mathfrak{p}_1K_0M = K_0M$, since $K_0D\mathfrak{p} = \mathfrak{p}_1$. It follows that $\mathfrak{p}^{-1}\mathfrak{a}P \cap M \neq \{0\}$ and $\left(\mathfrak{p}^{-1}\mathfrak{a}P\right)_v = \mathfrak{p}^{-1}\mathfrak{a}P_v = \mathfrak{p}^{-1}\mathfrak{a}P$ by Lemma 3.2 of [14] since $\mathfrak{p}^{-1}\mathfrak{a}$ is an invertible D-ideal in K^g. Then by Proposition 3.4, $\mathfrak{p}^{-1}\mathfrak{a}P = \mathfrak{n}DM$ for some ideal $\mathfrak{n}$ of D_0 and $P = \mathfrak{p}\mathfrak{a}^{-1}\mathfrak{n}DM$. It follows that $\mathfrak{p} = (P : M) = \mathfrak{p}\mathfrak{a}^{-1}\mathfrak{n}D$ and that $D = \mathfrak{a}^{-1}\mathfrak{n}D$. Hence $P = \mathfrak{p}M$.

To prove that P is a maximal v-submodule of M, let N be a maximal

v-submodule of M containing P. Then K_0N is a v-submodule of K_0M containing $K_0P = P_1$ by Lemma 3.5, so $K_0N = P_1$ by the assumption. Thus $P = P_1 \cap M \supseteq N$ and $N = P$ follows. Hence P is a maximal v-submodule of M. □

Lemma 3.7. *Let M_0 be a UFM over D_0 and P be a prime v-submodule of M such that $P \cap M_0 = \{0\}$. Then there is a maximal v-submodule P_1 of K_0M such that $P = P_1 \cap M$.*

Proof. Let $\mathfrak{p} = (P : M)$. Then $\mathfrak{p}$ is a prime v-ideal of D, so $\mathfrak{p}$ is a non-zero minimal prime ideal. Thus $\mathfrak{p}$ is in one of the following form: $\mathfrak{p} = \mathfrak{p}_0D$ for some prime ideal $\mathfrak{p}_0$ of D_0 or $\mathfrak{p} = \mathfrak{p}_1 \cap D$ for some prime ideal $\mathfrak{p}_1$ of K_0D by Theorem 2.1 and Lemma 2.6 of [11]. In the first case, $P \supseteq \mathfrak{p}_0DM \supseteq \mathfrak{p}_0M_0 \neq \{0\}$, a contradiction. Hence $\mathfrak{p} = \mathfrak{p}_1 \cap D$ with $K_0\mathfrak{p} = \mathfrak{p}_1$. Since $P \cap M_0 = \{0\}$, $K_0M \supset K_0P = (K_0P)_v$ by Lemma 3.5. Thus there is a maximal v-submodule P_1 of K_0M such that $P_1 \supseteq K_0P$. By Lemma 3.5, $(P_1 : K_0M) \supseteq (K_0P : K_0M) = K_0(P : M) = K_0\mathfrak{p} = \mathfrak{p}_1$. Since $(P_1 : K_0M)$ is a prime ideal of K_0D, $\mathfrak{p}_1 = (P_1 : K_0M)$. Hence $P_1 = \mathfrak{p}_1K_0M$ and $P_1 \cap M \supseteq P$. By Lemma 3.6, $P_1 \cap M = \mathfrak{p}M \subseteq P$ and hence $P = P_1 \cap M$ and $P = \mathfrak{p}M$. □

Proposition 3.8. *Let P be a prime v-submodule of M with $P \cap M_0 = \{0\}$. Then $P = \mathfrak{p}M$ for some prime v-ideal D where $\mathfrak{p} \cap D_0 = \{0\}$.*

From Lemma 3.3 and Proposition 3.8 we get the following theorem.

Theorem 3.9. *Let $D = \oplus_{n\in\mathbb{Z}}D_n$ be a strongly graded domain and $M = \oplus_{n\in\mathbb{Z}}M_n$ be a strongly graded module over D where M satisfies the ascending chain condition on v-submodules of M. If M_0 is a UFM over D_0 then M is a UFM over D.*

Proof. Since D_0 is a UFD, D is a UFD by Proposition 3.1, and so every prime v-ideal of D is principal. It is clear that D is a maximal order by Proposition 1 of [2], which implies D_0 is a maximal order by Theorem 1 of [6] and so M is CIC by Theorem 3.1 of [3]. To prove M is a UFM, let P be a prime v-submodule of M and $P_0 = P \cap M_0$.

(1) Case $P_0 \neq \{0\}$. Then $P = DP_0$ by Lemma 3.3. Note that P_0 is a prime v-submodule of D_0 by Lemma 3.3. Since M_0 is a UFM, $P_0 = p_0M_0$ for some $p_0 \in D_0$ and so $P = DP_0 = Dp_0M_0 = p_0DM_0 = p_0M$.

(2) Case $P_0 = \{0\}$. Then $P = \mathfrak{p}M$ for some v-ideal $\mathfrak{p}$ of D where $\mathfrak{p} \cap D_0 = \{0\}$ by Proposition 3.8. Since D is a UFD, $\mathfrak{p} = pD$ for some element $p \in D$ and so $P = \mathfrak{p}M = pDM = pM$, for some element $p \in D$.

Hence every prime v-submodule of M is principal and so M is a UFM by Theorem 2.5. □

As an application of Theorem 3.9, we have the following examples.

Example 3.10. If M is a UFM over an integral domain D then the Laurent polynomial module $M[x, x^{-1}]$ is a UFM over $D[x, x^{-1}]$.

Example 3.11. Let T be an arbitrary UFD and A, B be two non-zero v-ideals of T. Let K be the quotient field of T. Then $M = \oplus_{n\in\mathbb{Z}} AB^n x^n = \ldots + AB^{-2}x^{-2} + AB^{-1}x^{-1} + A + ABx + AB^2x^2 + \ldots$ is a UFM over $D = \oplus_{n\in\mathbb{Z}} B^n x^n = \ldots + B^{-2}x^{-2}B^{-1}x^{-1} + T + Bx + B^2x^2 + \ldots \subseteq K[x, x^{-1}]$, a Laurent polynomial ring over K.

4. Positively graded modules which are UFMs

Let $R = \oplus_{n\in\mathbb{Z}_0} D_n$ be a positively graded domain which is a sub-domain of the strongly graded domain $D = \oplus_{n\in\mathbb{Z}} D_n$. It is known that R is Noetherian if and only if D_0 is Noetherian by Proposition 2.1 of [13]. In this section, we will prove that $L = \oplus_{n\in\mathbb{Z}_0} M_n$, a positively graded module over R, is a UFM if and only if M_0 is a UFM over D_0 when D_0 is a Noetherian domain.

In the rest of this section, $R = \oplus_{n\in\mathbb{Z}_0} D_n$ and $L = \oplus_{n\in\mathbb{Z}_0} M_n$, where D_0 is a Noetherian domain and M_0 is a finitely generated torsion-free D_0-module.

In [2], it is shown that R is a UFR if and only if D_0 is a UFR and D_1 is a principal D_0-module. We begin this section with the following proposition that is a commutative case of Theorem 1 of [2].

Proposition 4.1. *A positively graded domain $R = \oplus_{n\in\mathbb{Z}_0} D_n$ is a UFD if and only if*

(1) *D_0 is a UFD, and*
(2) *D_1 is a principal D_0-module, that is, there is $p_1 \in D_1$ such that $D_1 = D_0 p_1$.*

Note that L is a finitely generated torsion-free R-module since M_0 is a fi nitely generated torsion-free D_0-module (see [3], Lemma 4.4.). Furthermore, M_0 is CIC if and only if L is CIC by Theorem 4.1 of [3].

The following lemma is a module version of Lemma 2.5 (2) of [13] and can be proved in a similar way as in Lemma 5.1 of [3].

Lemma 4.2. *Let N_0 be a fractional D_0-submodule of M_0 with $N_0 \subseteq M_0$ and $N = RN_0$. Then*

(1) $N^- = R(N_0)^-$,
(2) $N_v = R(N_0)_v$.

The following lemma is a graded version of Lemma 4.2 and Lemma 4.3 of [14].

Lemma 4.3. *Let M_0 be a UFM over D_0 and P be a prime R-submodule of L with $P_0 = P \cap M_0 \neq \{0\}$. Then*

(1) *P_0 is a prime submodule of M_0, and*
(2) *$P' = RP_0$ is a prime submodule of L.*
(3) *If P is a prime v-submodule, then P_0 is a prime v-submodule of M_0, and $P = RP_0$.*

Proof. (1) The proof is similar to the proof of Lemma 4.2 (1) of [14].

(2) The proof is similar to the proof of Lemma 4.2 (2) of [14].

(3) Let $P' = RP_0 \subseteq L$. Consider that $P = P_v \supseteq (P')_v = (RP_0)_v = R(P_0)_v$ by Lemma 4.2. Thus $P_0 = P \cap M_0 \supseteq R(P_0)_v \cap M_0 = (P_0)_v$. Hence $P_0 = (P_0)_v$ and so P_0 is a prime v-submodule by (1).
Note that $P' = RP_0 = Rp_0M_0$ for some non-zero $p_0 \in D_0$ because M_0 is a UFM. Since Rp_0 is an invertible ideal, $(P')^- = (Rp_0)^{-1} = Rp_0^{-1} \supseteq P^-$, which implies $R \supseteq Rp_0P^-$ and $P^{'} = Rp_0M_0 = Rp_0L \supseteq Rp_0P^-P$. If $P \supset P'$ then $Rp_0P^-L \subseteq P' = Rp_0L$ since P' is a prime submodule by Lemma 4.3 (2). Then $P^-L \subseteq L$ and so $P^- = D$. Thus $P = P_v = (P^-)^+ = (D)^+ = L$, a contradiction. Hence $P = RP_0$. □

The following proposition is a graded version of Proposition 4.4 of [14].

Proposition 4.4. *Let M_0 be a UFM over D_0 and N be a v-submodule of L with $N_0 = N \cap M_0 \neq \{0\}$. Then*

(1) *N_0 is a v-submodule of M_0 and $N_0 = \mathfrak{n}_0M_0$ for some ideal $\mathfrak{n}_0$ of D_0.*
(2) *$N = R\mathfrak{n}_0L$ and $R\mathfrak{n}_0 = (N : L)$.*

Proof. (1) Similar to the previous lemma, we get that N_0 is a v-submodule of M_0. Furthermore, $N_0 = \mathfrak{n}_0M_0$ for some ideal $\mathfrak{n}_0$ of D_0 follows since M_0 is a UFM over D_0 and by Theorem 2.3.

(2) Suppose there is a v-submodule N such that $N \neq R\mathfrak{n}_0L$ where $\mathfrak{n}_0$ is an ideal of D_0. We may assume that N is maximal with this property because

M is Noetherian. Then there is a maximal v-submodule P with $P \supseteq N$ and $P = R\mathfrak{p}_0 L$, where $\mathfrak{p}_0$ is a maximal ideal of D_0. It follows that $L \supseteq (R\mathfrak{p}_0)^{-1}N \supseteq N$. If $(R\mathfrak{p}_0)^{-1}N = N$, then $(R\mathfrak{p}_0)^{-1} \subseteq R$, a contradiction because L is CIC. Thus $(R\mathfrak{p}_0)^{-1}N \supset N$ and it follows from Lemma 3.2 of [14] that $((R\mathfrak{p}_0)^{-1}N)_v = (R\mathfrak{p}_0)^{-1}N$. By the choice of N, $(R\mathfrak{p}_0)^{-1}N = R\mathfrak{t}_0 L$ for some ideal $\mathfrak{t}_0$ of D_0. Hence $N = R\mathfrak{p}_0\mathfrak{t}_0 L$, a contradiction. Hence $N = R\mathfrak{n}_0 L$ for some ideal $\mathfrak{n}_0$ of D_0. The last statement easily follows since $R\mathfrak{n}_0$ is invertible. □

Next we study the structure of a prime v-submodule P of L such that $P \cap M_0 = \{0\}$. Since $Q^g = \oplus_{n\in\mathbb{Z}_0} K_0 D_n = K_0 R$ is a principal ideal domain by Lemma 2.1 of [13] and $K_0 L$ is a finitely generated torsion-free Q^g-module, we have that a v-submodule P_1 of $K_0 L$ is prime if and only if $P_1 = \mathfrak{p}_1 K_0 L$, where $\mathfrak{p}_1$ is a maximal ideal of Q^g such that $\mathfrak{p}_1 = (P_1 : K_0 L)$ by Theorem 3.3 of [14].

The following lemma is a graded version of Lemma 4.5 of [14].

Lemma 4.5. *Let N be an R-submodule of L. Then*

(1) $(K_0 N : K_0 L) = K_0\mathfrak{n}$, *where* $\mathfrak{n} = (N : L)$ *and* $K_0 N^- = (K_0 N)^-$.

(2) $(K_0 N)_v = K_0 N_v$.

Proof. (1) The proof is similar to the proof of Lemma 4.5 (1) of [14].

(2) Let $m' \in (K_0 N)_v = ((K_0 N)^-)^+ = (K_0 N^-)^+$, that is, $K_0 L \supseteq K_0 N^- m' \supseteq N^- m'$. Then there is $r \in D_0$ such that $N^- r m' = r N^- m' \subseteq L$. Thus $r m' \in (N^-)^+ = N_v$ and so $m' \in r^{-1} N_v \subseteq K_0 N_v$.

Conversely, let $m' \in K_0 N_v$. We write $m' = \sum_{i=1}^{t} k_{0_i} m_i$ where $k_{0_i} \in K_0$ and $m_i \in N_v$ for all $i = 1, 2, \ldots, t$. Then for all $i = 1, 2, \ldots, t$, we have $N^- m_i \subseteq L$ and so $K_0 N^- m' = K_0 N^- \left(\sum_{i=1}^{t} k_{0_i} m_i\right) \subseteq N^-(K_0 m_1 + \ldots + K_0 m_t) \subseteq K_0 L$. Then

$$m' \in (K_0 N^-)^+ = ((K_0 N)^-)^+ = (K_0 N)_v.$$

Hence $(K_0 N)_v = K_0 N_v$. □

The following lemma is a graded version of lemma 4.6 of [14]. We write the proof because we need v_1-operation to prove the last properties (see [10], [14], [15] for details regarding v-submodules and v_1-operation).

Lemma 4.6. *Let M_0 be a UFM over D_0 and $P_1 = \mathfrak{p}_1 K_0 L$ be a prime v-submodule of K_0L, where $\mathfrak{p}_1$ is a maximal ideal of K_0R, $P = P_1 \cap L$ and $\mathfrak{p} = \mathfrak{p}_1 \cap R$. Then*

(1) *P is a prime submodule of L and $\mathfrak{p} = (P : L)$.*
(2) *$K_0P = P_1$ and $P \cap M_0 = \{0\}$.*
(3) *$P = \mathfrak{p}L$ and P is a maximal v-submodule of L.*

Proof. (1) Let $r \in R$ and $m \in L$ such that $rm \in P$ and $m \notin P$. Since $m \notin P_1$ and P_1 is prime, we have $rL \subseteq rK_0L \subseteq P_1$ and so $rL \subseteq P$. Hence P is a prime submodule of L.
Since $\mathfrak{p}L \subseteq \mathfrak{p}K_0L = P_1$, we have $\mathfrak{p}L \subseteq P$, so $\mathfrak{p} \subseteq (P : L)$. Conversely let $r \in (P : L)$, that is, $r \in R$ and $rL \subseteq P$. Then $rK_0L \subseteq K_0P \subseteq P_1$, so $r \in (P_1 : K_0L) = \mathfrak{p}_1$. Thus $r \in \mathfrak{p}_1 \cap R = \mathfrak{p}$. Hence $\mathfrak{p} = (P : L)$.

(2) Let $m' \in P_1$ and we write $m' = \sum_{i=1}^{n} t_i m_i$ where $t_i \in \mathfrak{p}_1$ and $m'_i \in K_0L$. Then there are $\alpha, \beta \in D_0$ such that $\alpha t_i \in \mathfrak{p}$ and $\beta m'_i \in L$ and so $\alpha\beta m' \in \mathfrak{p}L \subseteq P$. Thus $m' \in (\alpha\beta)^{-1}P \subseteq K_0P$. Hence $K_0P = P_1$.
Note that $\mathfrak{p}_1 = \langle t \rangle = tK_0R$ for some prime element $t \in K_0R$ with $\deg(t) \geq 1$. If $P \cap M_0 \neq \{0\}$ and let $0 \neq m \in P \cap M_0$. Then $m = tm'$ for some $m' \in K_0L$, since $K_0P = P_1 = tK_0L$. Write $t = t_n + t_{n-1} + \ldots + t_0$ ($t_i \in K_0D_i$, with $t_n \neq 0$) and $m' = m_l + \ldots + m_0$ ($m_j \in K_0M_j$). Then we get $t_n m_l = 0$, so $m_l = 0$ and so on. Then we have $m = 0$, a contradiction. Hence $P \cap M_0 = \{0\}$.

(3) The proof is similar to Lemma 3.6 (3). □

Lemma 4.7. *Let M_0 be a UFM over D_0 and P be a prime v-submodule of L such that $P \cap M_0 = \{0\}$. Then $P = \oplus_{n \geq 1} M_n = D_1L$ or there is a maximal v-submodule P_1 of K_0L such that $P = P_1 \cap L$.*

Proof. Let $\mathfrak{p} = (P : L)$. Then $\mathfrak{p}$ is a prime v-ideal of R, so $\mathfrak{p}$ is a non-zero minimal prime ideal. Thus $\mathfrak{p}$ is in one of the following form: $\mathfrak{p} = \mathfrak{p}_0R$ for some prime ideal $\mathfrak{p}_0$ of D_0, $\mathfrak{p} = \oplus_{n \geq 1} D_n$ or $\mathfrak{p} = \mathfrak{p}_1 \cap R$ for some prime ideal $\mathfrak{p}_1$ of K_0R by Proposition 3.1 of [13]. In the first case, $P \supseteq \mathfrak{p}_0RL \supseteq \mathfrak{p}_0M_0 \neq \{0\}$, a contradiction. In the second case, if $P \supseteq (\oplus_{n \geq 1} D_n)L = RD_1L = D_1L = \oplus_{n \geq 1} M_n$. If $P \supset \oplus_{n \geq 1} M_n$, there is a non-zero submodule T_0 of M_0 such that $P = T_0 + \oplus_{n \geq 1} M_n$. Then $P \cap M_0 \supseteq T_0 \neq \{0\}$, a contradiction. Hence $P = \oplus_{n \geq 1} M_n$. In the last case, $\mathfrak{p} = \mathfrak{p}_1 \cap R$ with $K_0\mathfrak{p} = \mathfrak{p}_1$. Since $P \cap M_0 = \{0\}$, $K_0L \supset K_0P = (K_0P)_v$ by Lemma 4.5. Thus there is a maximal v-submodule P_1 of K_0L such that $P_1 \supseteq K_0P$. By Lemma 4.5, $(P_1 : K_0L) \supseteq (K_0P : K_0L) = K_0(P : L) = K_0\mathfrak{p} = \mathfrak{p}_1$. Since $(P_1 : K_0L)$

is a prime ideal of K_0R, $\mathfrak{p}_1 = (P_1 : K_0L)$. Hence $P_1 = \mathfrak{p}_1 K_0 L$ and $P_1 \cap L \supseteq P$. By Lemma 4.6, $P_1 \cap L = \mathfrak{p}L \subseteq P$ and hence $P = P_1 \cap L$ and $P = \mathfrak{p}L$. So by the last two cases, $P = \oplus_{n\geq 1} M_n$ or there is a maximal v-submodule P_1 of K_0L such that $P = P_1 \cap L$. □

Note that if $R = \oplus_{n\in\mathbb{Z}_0} D_n$ is a Noetherian UFD, then $R = D_0[p_1]$ for some element $p_1 \in D_1$ by Theorem 1 of [2] and so $M = M_0[p_1]$, a polynomial module. Then the necessary condition of Theorem 4.8 is already proved in [12], but we give another proof by using v_1-operator.

Theorem 4.8. *Let $R = \oplus_{n\in\mathbb{Z}_0} D_n$ be a Noetherian UFD and $L = \oplus_{n\in\mathbb{Z}_0} M_n$ be a positively graded module over R. Then L is a UFM if and only if M_0 is a UFM.*

Proof. ($\Rightarrow$) Suppose that $L = \oplus_{n\in\mathbb{Z}_0} M_n$ is a UFM over R. Then L is CIC and so M_0 is CIC by Theorem 4.1 of [3]. Let P_0 be a non-zero prime v-submodule of M_0. Then $P = RP_0$ is a v-submodule of L by Lemma 4.2. Furthermore, as in Lemma 4.3 (2), RP_0 is a prime submodule of L. Hence RP_0 is a prime v-submodule of L and so it is principal since L is a UFM. Then $RP_0 = rL$ for some $r \in R$. Since $\{0\} \neq P_0 \subset RP_0 = rL$, $r \in D_0$, which implies $P_0 = rM_0$. Hence P_0 is principal and so M_0 is a UFM by Theorem 2.5.

($\Leftarrow$) Suppose that M_0 is a UFM over D_0. Since R is a UFD, it is clear that R is a maximal order by Proposition 1 of [2], which implies D_0 is a maximal order by Theorem 2.1 of [13] and so L is CIC by Theorem 4.1 of [3]. Note that D_0 is a UFD and D_1 is a principal D_0-module since R is a UFD. To prove L is a UFM, let P be a prime v-submodule of L and $P_0 = P \cap M_0$. By Lemma 4.3 (3), P_0 is a prime v-submodule.

(1) Case $P_0 \neq \{0\}$. Then $P = RP_0$ by Lemma 4.3 (3). Since M_0 is a UFM, $P_0 = p_0 M_0$ for some $p_0 \in D_0$ and so $P = RP_0 = Rp_0M_0 = p_0RM_0 = p_0L$.
(2) Case $P_0 = \{0\}$. Then $P = \oplus_{n\geq 1} M_n = D_1L$ or $P = \mathfrak{p}L$ for some v-ideal $\mathfrak{p}$ of R by Lemma 4.7. If $P = \oplus_{n\geq 1} M_n = D_1L$, then $P = d_1D_0L = d_1L$ for some $d_1 \in D_1$ since D_1 is a principal D_0-module. If $P = \mathfrak{p}L$, then $P = \mathfrak{p}L = pRL = pL$ for some $p \in R$ since R is a UFD.

Hence every prime v-submodule of L is principal and so L is a UFM by Theorem 2.5. □

We end this section with examples of a positively graded module which is a UFM.

Example 4.9. Let $R = \oplus_{n\in\mathbb{Z}_0} D_n$ be a positively graded domain where D_0 is a Noetherian UFD and D_1 is a principal D_0-module. Let $M = R \oplus R \oplus \ldots \oplus R$ be a positively graded module over R and P be a graded submodule of M such that $M = P \oplus T$. Then P is a UFM.

Proof. Note that P is a projective module. Then P is a G-Dedekind module and it is a v-multiplication module by Theorem 3.1 of [15]. Furthermore, since P is a v-multiplication module and R is a UFD, P is a UFM by Theorem 2.3. □

Lemma 4.10. *Let D be a domain, B be an invertible ideal of D and A be a non-zero ideal of D. Let $R = D + Bx + B^2x^2 + \ldots \subseteq D[x]$, where $D[x]$ is a polynomial ring over D and $L = A + ABx + AB^2x^2 + \ldots = AR$. Then L is a positively graded module over positively graded domain R.*

From Remark 2.6 and Lemma 4.10 we have the following example.

Example 4.11. Let D be an arbitrary Noetherian UFD and A, B be two non-zero v-ideals of D. Then $L = A + ABx + AB^2x^2 + \ldots$ is a UFM over $R = D + Bx + B^2x^2 + \ldots$.

Proof. Note that R is a UFD since D is a UFD and Bx is a principal D-module. Since A is a non-zero v-ideal of D, A is a UFM by Remark 2.6. Then by Theorem 4.8, L is a UFM over R. □

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