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PSEUDO-ABSORBING COMULTIPLICATION MODULES OVER A PULLBACK RING

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Abstract. In this paper, we introduce the notion of pseudo-absorbing comultiplication modules. A full description of all indecomposable pseudo-absorbing comultiplication modules with finite dimensional top over certain kinds of pullback rings are given and establish a connection between the pseudo-absorbing comultiplication modules and the pure-injective modules over such rings.

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1. Introduction

The idea of representing a complex mathematical object by a simpler one is as old as mathematics itself. It is particularly useful in classification problems. For instance, a single linear transformation on a finite dimensional vector space is very adequately characterized by its reduction to its rational or its Jordan canonical form. One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. The reader is referred to [3] and [30, Chapter 1 and 14] for a detailed discussion of classification problems, their representation types (finite, tame, or wild), and useful computational reduction procedures. Here we should point out that the classification of all indecomposable modules over an arbitrary unitary ring (including finite-dimensional algebras over an algebraically closed field) is an impossible task. In particular one is interested in the classification of certain 'significant" modules rather than in arbitrary modules; the pure-injective modules seem to form such a class of modules which arise in practice and where there is hope of some kind of classification. Pure-injective modules play a central role in the model theory of modules: for example classification of the complete theories of R-modules reduce to classifying the (complete theories of) pure-injectives. Also, for some rings the "small" (finite-dimensional, finitely generated, ...) modules are classified and in many cases this classification can be extended to give a classification of the (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and families of finitely generated modules (see [26,27,28,29]). One of our main concerns in this paper is to introduce a subclass of pure-injective modules in terms of simple (especially indecomposable) components.

In this paper all rings are commutative with identity and all modules are unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two discrete rank 1 valuation domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [19]). A typical example of local Dedekind domain pullback is the infinite-dimensional k-algebra $k[x, y : xy = 0]_{(x,y)}$ where k is a field (it is the pullback $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ of two local Dedekind domains $k[x]_{(x)}, k[y]_{(y)}$ (see [2, Section 6]). Modules over pullback rings have been studied by several authors (see for instance, [2,7,8,9,10,11,12,13,14,15,17,19,20,21,24]). Notably, there is an important work of Levy [21], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings.

The classification of subclasses of pure-injective modules over the pullback of two DVRs over a common factor field is very interesting and important in the literature. One point of this paper is that to introduce a subclass of pure-injective modules over such rings. Indeed, this article includes the classification of those indecomposable pseudo-absorbing comultiplication modules over $k[x, y : xy = 0]_{(x,y)}$ where k is a field, which have finite-dimensional top.

The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced and studied by Badawi in [4]. Various generalizations of prime ideals are also studied in [5] and [6]. Recall that a proper ideal I of a ring R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Recently (see [25]), the concept of 2-absorbing ideal is extended to the context of 2-absorbing submodule which is a generalization of prime submodule. Recall from [25] that a proper R-submodule N of a module M is said to be a 2-absorbing submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. In [12], the concept of 2-absorbing submodule is extended to the context of pseudo-absorbing submodule which is a generalization of 2-absorbing submodule. A proper submodule N of an R-module M is called pseudo-absorbing if $(N :_R M)$ is a 2-absorbing ideal of R (by [25, Theorem 2.3], if N is a 2-absorbing submodule of an R-module M, then $(N :_R M)$ is a 2-absorbing ideal of R; so every 2-absorbing submodule is a pseudo-absorbing submodule).

In the present paper we introduce a new class of R-modules, called pseudo-absorbing comultiplication modules (the dual notion of pseudo-absorbing multiplication modules), and we study it in details from the classification problem point of view. We are going to study pullbacks of $DVRs$, discrete rank 1 valuation domains. By DVR we will indicate discrete rank 1 valuation domains, which are exactly the local Dedekind domains. First, we give a complete description of the pseudo-absorbing comultiplication modules over a DVR. Let R be a pullback of two $DVRs$ over a common factor field. The main purpose of this paper is to give a complete description of the indecomposable pseudo-absorbing comultiplication R-modules with finite-dimensional top over $R/\mathrm{rad}(R)$ (for any module M we define its top as $M/\mathrm{Rad}(R)M$). The classification is divided into two stages: the description of all indecomposable separated pseudo-absorbing comultiplication R-modules and then, using this list of separated pseudo-absorbing comultiplication modules we show that non-separated indecomposable pseudo-absorbing comultiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable pseudo-absorbing comultiplication R-modules. Then we use the classification of separated indecomposable pseudo-absorbing comultiplication modules from Section 3, together with results of Levy [20,21] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable pseudo-absorbing comultiplication modules M with finite-dimensional top (see Theorem 4.9). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable pseudo-absorbing comultiplication modules (where infinite length pseudo-absorbing comultiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated pseudo-absorbing comultiplication modules.

For the sake of completeness, we state some definitions and notations used throughout. Let R be the pullback ring as mentioned in the beginning of Introduction. An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2))$. Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [19, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [19, Lemma 2.9]. Let M be an R-module. A separated representation of M is a pair (S, φ) where

(i) S is a separated R-module;

(ii) φ is an R-homomorphism of S onto M;

(iii) for every pair (S', φ') satisfying (i) and (ii), and for every R-homomorphism α of S in S' such that $\varphi'\alpha = \varphi$, α is $1-1$. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [19, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [19, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [19, Theorem 2.8].

Definition 1.1. (a) If R is a ring and N is a submodule of an R-module M, then the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be prime submodule if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r \in (N : M)$, so $(N : M) = P$ is a prime ideal of R, and N is said to be P-prime submodule. The set of all prime submodules in an R-module M is denoted $\mathrm{Spec}(M)$ [22,23].

(b) An R-module M is a comultiplication module provided for each submodule N of M there exists an ideal I of R such that N is the set of elements m in M such that $Im = 0$. In this case we can take $N = (0 :_M \mathrm{ann}(N))$ [1].

(c) A proper submodule N of an R-module M is called pseudo-absorbing if $(N :_R M)$ is a 2-absorbing ideal of R. The set of all pseudo-absorbing submodules in an R-module M is denoted by $\mathrm{pabSpec}(M)$ [12].

(d) An R-module M is defined to be a pseudo-absorbing multiplication module if for every pseudo-absorbing submodule N of M, $N = IM$ for some ideal I of R [12].

(e) A submodule N of an R-module M is called pure submodule if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called relatively divisible (or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$ [26,32].

(f) A module M is pure-injective if it has the injective property relative to all pure exact sequences [26,32].

Remark 1.2. (a) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M [26,32].

(b) Let N be an R-submodule of M. It is clear that N is an RD-submodule of M if and only if for all $m \in M$ and $r \in R$, $rm \in N$ implies that $rm = rn$ for some $n \in N$. Furthermore, if M is torsion-free, then N is an RD-submodule if and only if for all $m \in M$ and for all non-zero $r \in R$, $rm \in N$ implies that $m \in N$. In this case, N is an RD-submodule if and only if N is a prime submodule.

2. Pseudo-absorbing comultiplication modules

In this section, we give a complete description of the pseudo-absorbing comultiplication modules over a DVR. We begin with the key definition of this paper.

Definition 2.1. Let R be a commutative ring. An R-module M is defined to be a pseudo-absorbing comultiplication module if for every pseudo-absorbing submodule N of M, $N = (0 :_M I)$ for some ideal I of R.

One can easily show that if M is a pseudo-absorbing comultiplication module, then $N = (0 :_M \operatorname{ann}(N))$ for every pseudo-absorbing submodule N of M.

Lemma 2.2. *Assume that R is a commutative ring and let $K \subseteq N$ be submodules of an R-module M. Then N is a pseudo-absorbing submodule of M if and only if N/K is a pseudo-absorbing submodule of M/K.*

Proof. This follows from the fact that $(N :_R M) = (N/K :_R M/K)$. □

Proposition 2.3. (i) *Let M be a pseudo-absorbing comultiplication module over a commutative ring R. Then every direct summand of M is a pseudo-absorbing comultiplication R-submodule.*

(ii) *Let M be an R-module, N an R-submodule of M and I an ideal of R such that $I \subseteq (0 :_R M)$. Then M is pseudo-absorbing comultiplication as an R-module if and only if M is pseudo-absorbing comultiplication as an R/I-module.*

Proof. (i) Let K be a direct summand of M. Then $M = K \oplus N$ for some submodule N of M. It suffices to show that M/N is a pseudo-absorbing comultiplication R-module. Let L/N be a pseudo-absorbing submodule of M/N. Then by Lemma 2.2, L is a pseudo-absorbing submodule of M, so $L = (0 :_M J)$ for some ideal J

of R. We show that $L/N = (0 :_{M/N} J)$. Let $y + N \in L/N$. Then $Jy = 0$ gives $J(y + N) = 0$; so $y + N \in (0 :_{M/N} J)$. For the reverse inclusion, assume that $z + N \in (0 :_{M/N} J)$. Then $Jz \subseteq N \cap JM = JN \subseteq JL = 0$; hence $z \in L$, and we have equality.

(ii) It is easy to see that N is a pseudo-absorbing R-submodule of M if and only if N is pseudo-absorbing submodule of M as an R/I-module. Now the assertion follows the fact that $(0 :_M J) = (0 :_M (I + J)/I)$ for every ideal J of R. □

Remark 2.4. (a) Let R be a DVR with unique maximal ideal $P = Rp$.

(i) Since $(P :_R R) = P$, P is a pseudo-absorbing submodule of the R-module R. Now $(0 :_R \text{ann}(P)) = (0 :_R 0) = R \neq P$ gives R is not a pseudo-absorbing comultiplication R-module.

(ii) Each R/P^n $(n \geq 1)$ is a pseudo-absorbing comultiplication R-module since it is a comultiplication R-module.

(iii) Every non-zero proper submodule L of $E = E(R/P) = Q(R)/P$, the injective hull of R/P, is of the form $L = A_n = (0 :_E P^n)$ $(n \geq 1)$, $L = A_n = Ra_n$ and $PA_{n+1} = A_n$. Since E is divisible, $(A_n : E) = 0$; hence each A_n is a pseudo-absorbing submodule $(n \geq 1)$. Thus E is a pseudo-absorbing comultiplication R-module.

(iv) Let $Q(R)$ be the field of fractions of R and consider $N = R$ as a non-zero proper R-submodule of $Q(R)$. Since $(N :_R Q(R)) = 0$, N is a pseudo-absorbing submodule of $Q(R)$. Now $(0 :_{Q(R)} (0 :_R N)) = (0 :_{Q(R)} 0) = Q(R) \neq N$ gives $Q(R)$ is not a pseudo-absorbing comultiplication R-module.

(v) For a ring R, it is known that $R \neq 0$ if and only if $\text{Spec}(R) \neq \emptyset$. By [11, Theorem 3.2], $\text{Spec}E(R/P) = \emptyset$. Thus for a module M it is not always true that if $M \neq 0$, then $\text{Spec}(M) \neq \emptyset$. Now we study $\text{pabSpec}(M)$, where M is a pseudo-absorbing comultiplication R-module. We claim that $\text{pabSpec}(M) \neq \emptyset$. Assume to the contrary, $\text{pabSpec}(M) = \emptyset$. Since $\text{Spec}(M) \subseteq \text{pabSpec}(M) = \emptyset$, it follows from [22, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. By an argument like that in [8, Proposition 2.7 Case 2], $M \cong E(R/P)$ which is a contradiction by (a) (iii).

(b) Let R and R' be any commutative rings, $g : R \to R'$ a surjective homomorphism and M an R'-module.

(i) It is easy to see that N is a pseudo-absorbing R-submodule of M if and only if it is a pseudo-absorbing R'-submodule of M.

(ii) If M is a pseudo-absorbing comultiplication R'-module, then we will show that M is a pseudo-absorbing comultiplication R-module. Assume that M is a pseudo-absorbing comultiplication R'-module and let N be a pseudo-absorbing R-submodule of M. Then by (i), $N = (0 :_M J)$, where $J = (0 :_{R'} N)$; so $I = g^{-1}(J)$ is an ideal of R with $g(I) = J$. It is enough to show that $(0 :_M J) = (0 :_M I)$. Let $m \in (0 :_M J)$. If $r \in I$, then $g(r) \in J$, so $g(r)m = 0$. Thus $rm = 0$ for every $r \in I$; hence $m \in (0 :_M I)$. For the reverse inclusion, assume that $x \in (0 :_M I)$ and $s \in J$. Then $s = g(a)$ for some $a \in I$. It follows that $sx = g(a)x = ax = 0$ for every $s \in J$; hence $x \in (0 :_M J)$, and we have equality.

Theorem 2.5. *Let M be a pseudo-absorbing comultiplication module over a DVR with unique maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n ($n \geq 1$) and K is a direct sum of copies of $E(R/P)$. In particular, every pseudo-absorbing comultiplication R-module is pure-injective.*

Proof. By [16, Theorem 8], M possesses a unique largest divisible submodule N; $M = N \oplus K$, where K has no divisible submodule. As M is pseudo-absorbing comultiplication, N is too, by Proposition 2.3(i). By [16, Theorem 7] and Remark 2.4, N is a direct sum of copies of $E(R/P)$. We will prove that K is bounded. If $pRK = pK = K$, then it is easy to verify that K is divisible, a contradiction; hence $pK \neq K$. If $(0 :_R pK) = 0$, then $PK = pK = (0 :_K (0 :_R PK)) = (0 :_K 0) = K$ (because pK is a pseudo-absorbing submodule of K and K is pseudo-absorbing comultiplication), which is a contradiction. Therefore $(0 :_R pK) \neq 0$; and $p^t K = 0$ for some integer t. Thus K is bounded. Hence K is a direct sum of cyclic modules, by [18, Theorem 7.1]. As K is pseudo-absorbing comultiplication, by Remark 2.4, K is a direct sum of copies of R/P^n ($n \geq 1$). Then, in particular, statement follows from [7, Proposition 1.3]. □

Corollary 2.6. *Let M be a pseudo-absorbing comultiplication module over a DVR with unique maximal ideal $P = Rp$. Then M is torsion.*

Proof. By Theorem 2.5, every pseudo-absorbing comultiplication module over a DVR is a direct sum of torsion modules. It is easy to verify that a direct sum of torsion modules over a DVR is torsion. □

Theorem 2.7. *Let R be a DVR with unique maximal ideal $P = Rp$. Then the indecomposable pseudo-absorbing comultiplication modules over R, up to isomorphism, are the following:*

(i) *R/P^n, $n \geq 1$, the indecomposable torsion modules;*
(ii) *$E(R/P)$, the injective hull of R/P.*

Proof. By [7, Proposition 1.3], these modules are indecomposable. Being pseudo-absorbing comultiplication follows from Remark 2.4 (a). It remains to be shown that there are no more indecomposable pseudo-absorbing comultiplication modules. Let M be an indecomposable pseudo-absorbing comultiplication module. Then M is either R/P^n, $n \geq 1$, or $E(R/P)$, by Theorem 2.6. □

3. The separated pseudo-absorbing comultiplication modules

In this section we determine the indecomposable pseudo-absorbing comultiplication separated R-modules where

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two $DVRs$ R_1, R_2 with maximal ideals P_1, P_2 generated respectively by $p_1 \in P_1 \setminus P_1^2$, $p_2 \in P_2 \setminus P_2^2$, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field (we do not need the a priori assumption of finite-dimensional top for this classification). Then R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

Let $a = (r, s) \in R$ with $r \neq 0$ and $s \neq 0$. Then we can write $a = (p_1^n, p_2^m)$ for some positive integers m, n, so $\text{ann}(a) = 0$; hence $Ra \cong R$. If $a = (0, p_2^m)$ for some positive integer m, then $\text{ann}(a) = P_1 \oplus 0$, and so $R(0, p_2^m) \cong R/(P_1 \oplus 0) \cong R_2$. Similarly, $R(p_1^n, 0) \cong R/(0 \oplus P_2) \cong R_1$. The other ideals I of R are of the form $I = P_1^n \oplus P_2^m = (P_1^n, P_2^m) = (< p_1^n >, < p_2^m >)$ for some integers m, n since $I \subseteq P = P_1 \oplus P_2 = (P_1, P_2) = (< p_1 >, < p_2 >)$ and $p_1 p_2 = 0 = p_2 p_1$ (see [7, p. 4054]). We need the following lemma proved in [12, Proposition 3.1].

Lemma 3.1. *Let R be a pullback ring as in (1). Then the following hold:*

(i) *The ideals $0, 0 \oplus P_2, 0 \oplus P_2^2, P_1 \oplus P_2, P_1 \oplus P_2^2, P_1 \oplus 0, P_1^2 \oplus 0, P_1^2 \oplus P_2^2$ and $P_1^2 \oplus P_2$ of R are 2-absorbing.*

(ii) *If T is a pseudo-absorbing submodule of a non-zero separated R-module $S = (S_1 \to \bar{S} \leftarrow S_2)$, then T_1 is a pseudo-absorbing submodule of S_1 and T_2 is a pseudo-absorbing submodule of S_2.*

(iii) *If T is a pseudo-absorbing submodule of a non-zero separated R-module S, then $(T :_R S) = 0$ or $0 \oplus P_2$ or $0 \oplus P_2^2$ or $P_1 \oplus P_2$ or $P_1 \oplus P_2^2$ or $P_1 \oplus 0$ or $P_1^2 \oplus 0$ or $P_1^2 \oplus P_2^2$ or $P_1^2 \oplus P_2$.*

Remark 3.2. Let R be a pullback ring as in (1). Let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \twoheadrightarrow S_i$. Set $T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\}$ and $T_2 = \{t_2 \in S_2 : (t_1, t_2) \in$

T for some $t_1 \in S_1\}$. Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi_1' = \pi_1|T : T \twoheadrightarrow T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus Ker(f_2) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\mathrm{Ker}(f_1) = P_1S_1$ and $\mathrm{Ker}(f_2) = P_2S_2$).

Proposition 3.3. *Let R be a pullback ring as in (1).*

(i) *The class of 2-absorbing ideals of R consists of the following: $0, 0 \oplus P_2, 0 \oplus P_2^2, P_1 \oplus P_2, P_1 \oplus P_2^2, P_1 \oplus 0, P_1^2 \oplus 0, P_1^2 \oplus P_2^2$ and $P_1^2 \oplus P_2$.*

(ii) *Let $S = (S_1 \rightarrow \bar{S} \leftarrow S_2)$ be a non-zero separated pseudo-absorbing comultiplication R-module. Then $\mathrm{pabSpec}(S) \neq \emptyset$.*

Proof. (i) By Lemma 3.1 (i), it remains to be shown that there are no more 2-absorbing ideals. If $I = P_1^n \oplus P_2^m$ is a non-zero 2-absorbing ideal with $m \geq 3$ or $n \geq 3$, say $I = P_1 \oplus P_2^3$ (resp. $I = 0 \oplus P_2^3$), then $(p_1^3, p_2^3) \in I \subseteq P$ $((0, p_2^3) \in I \subseteq P)$ but $(p_1^2, p_2^2) \notin I$ (resp. $(0, p_2^2) \notin I$). Thus $P_1^n \oplus P_2^m$ is not a 2-absorbing ideal for all either $m \geq 3$ or $n \geq 3$.

(ii) Let π_1 be the projection map of R onto R_1. By Remark 2.4 (a) (v), $\mathrm{pabSpec}(S_1) \neq \emptyset$, so there is a pseudo-absorbing submodule T_1 of S_1. Then by Remark 2.4 (b) and Remark 3.2, there exists a submodule $T = (T_1 \rightarrow \bar{T} \leftarrow T_2)$ of S such that $T_1 \cong (T + (0 \oplus P_2)S)/(0 \oplus P_2)S$ is a pseudo-absorbing R-submodule of $S_1 = S/(0 \oplus P_2)S$; hence $T + (0 \oplus P_2)S$ is a 2-absorbing R-submodule of S by Lemma 2.2. Thus $\mathrm{pabSpec}(S) \neq \emptyset$. □

Theorem 3.4. *Let $S = (S_1 \rightarrow \bar{S} \leftarrow S_2)$ be any non-zero separated module over a pullback ring as (1). Then S is a pseudo-absorbing comultiplication R-module if and only if each S_i is a pseudo-absorbing comultiplication R_i-module, $i = 1, 2$.*

Proof. Suppose that S is a pseudo-absorbing comultiplication R-module. If $\bar{S} = 0$, then by [7, Lemma 2.7 (i)], $S = S_1 \oplus S_2$; hence S_i is a pseudo-absorbing comultiplication R_i-module by Proposition 2.3 (i), for each $i = 1, 2$. So we may assume that $\bar{S} \neq 0$. In this case, we will show that $(0 :_R S) \neq 0$. Assume to the contrary, $(0 :_R S) = 0$. Then it is easy to verify that $(0 :_R PS) = 0$. Since $PS \neq S$ and $(PS :_R S) = P$, PS is a pseudo-absorbing submodule of S. It follows that $(0 :_R PS) = 0$. As S is a pseudo-absorbing comultiplication R-module, we have $PS = (0 :_S (0 :_R PS)) = S$ that is a contradiction. Thus $(0 :_R S) \neq 0$. Let T_1 be a nonzero pseudo-absorbing submodule of S_1. Then there exists a submodule $T = (T_1 \rightarrow \bar{T} \leftarrow T_2)$ of S such that $T' = T + (0 \oplus P_2)S$ is a pseudo-absorbing submodule of S (see Proposition 3.3 (b)). An inspection will show that $(0 :_R (0 \oplus P_2)S) = P_1 \oplus 0$. Since $0 \neq (0 :_R S) \subseteq (0 :_R T')$ and $0 \neq (0 :_R S) \subseteq (0 :_R T)$, $(0 :_R T') = (0 :_R T) \cap (0 :_R (0 \oplus P_2)S) = P_1^n \oplus 0$ for some positive integer n. Then S is a pseudo-absorbing comultiplication module gives $T' = (0 :_S P_1^n \oplus 0)$. It suffices to show that $T_1 = (0 :_{S_1} P_1^n)$. Let $t \in T_1$. There exists $t_2 \in T_2$ such that $(t_1, t_2) \in T \subseteq T'$; so $(P_1^n \oplus 0)(t_1, t_2) = 0$. It then follows that $T_1 \subseteq (0 :_{S_1} P_1^n)$. For the reverse inclusion let $s_1 \in (0 :_{S_1} P_1^n)$. Then there is an element $s_2 \in S_2$ such that $(s_1, s_2) \in S$ and $(P_1^n \oplus 0)(s_1, s_2) = 0$; hence $(s_1, s_2) \in T'$. Thus $s_1 \in T_1$ and we have equality. Therefore S_1 is pseudo-absorbing comultiplication. Similarly, S_2 is pseudo-absorbing comultiplication.

Conversely, assume that each S_i is a pseudo-absorbing comultiplication R_i-module and let T be a pseudo-absorbing submodule of S. By Lemma 3.1 (ii), T_1, T_2 are pseudo-absorbing submodules of S_1, S_2, respectively. By assumption, $T_1 = (0 :_{S_1} P_1^n)$ and $T_2 = (0 :_{S_2} P_2^m)$ for some integers n, m. An inspection will show that $T = (0 :_S P_1^n \oplus P_2^m)$. Thus S is a pseudo-absorbing comultiplication R-module. □

Lemma 3.5. *Let R be a pullback ring as in (1). Then, up to isomorphism, the following separated R-modules are indecomposable pseudo-absorbing comultiplication modules:*

(i) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$, *where $E(R_1/P_1)$ is the R_1-injective hull of R_1/P_1;*

(ii) $(0 \to 0 \leftarrow E(R_2/P_2))$, *where $E(R_2/P_2)$ is the R_2-injective hull of R_2/P_2;*

(iii) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ *for all positive integers n, m.*

Proof. By [7, Lemma 2.8], these modules are indecomposable. They are pseudo-absorbing comultiplication modules by Theorem 2.7 and Theorem 3.4. □

Theorem 3.6. *Let R be a pullback ring as in (1), and let S be a non-zero indecomposable separated pseudo-absorbing comultiplication R-module. Then S is isomorphic to one of the modules listed in Lemma 3.5.*

Proof. First suppose that $S = PS$. Then by [7, Lemma 2.7 (i)], $S = S_1$ or $S = S_2$ and so S_i is an indecomposable pseudo-absorbing comultiplication R_i-module for some i and, since $PS = S$, it is of type (i) or (ii) in the list Lemma 3.5. So we may assume that $S \neq PS$.

By Theorem 3.4, S_i is a pseudo-absorbing comultiplication R_i-module, for each $i = 1, 2$. Therefore by the structure of pseudo-absorbing comultiplication modules over DVR (see Theorem 2.5), $S_i = M_i \oplus N_i$, where N_i is a direct sum of copies of R_i/P_i^n $(n \geq 1)$ and M_i is a direct sum of copies of $E(R_i/P_i)$. Then we have

$S = (N_1 \to \bar{S} \leftarrow N_2) \oplus (M_1 \to 0 \leftarrow 0) \oplus (0 \to 0 \leftarrow M_2)$. As S is indecomposable and $S \neq PS$, we find that $S = (N_1 \to \bar{S} \leftarrow N_2)$. We will show S is as in (iii) in the list of Lemma 3.5. There are positive integers u, v and w such that $P_1^u S_1 = 0$, $P_2^v S_2 = 0$ and $P^w S = 0$. Choose $s \in S_1 \cup S_2$ with $\bar{s} \neq 0$ and let $o(s)$ denote the least positive integer k such that $P^k s = 0$ if there is such k and if no such k exists, then $o(s) = \infty$ and $o(s)$ minimal among such s. Assume $s \in S_2$, and so write $s = s_2$ and $m = k = o(s_2)$. Now pick $s_1 \in S_1$ with $\bar{s}_1 = \bar{s}_2 = \bar{s}$ and $o(s_1) = n$ minimal (so $o(s_2) \neq \infty$ and $o(s_1) \neq \infty$). There exists an $s = (s_1, s_2)$ such that $o(s) = n_1$, $o(s_1) = m_2$ and $o(s_2) = k_1$. Then $R_i s_i$ is pure in S_i for $i = 1, 2$ (see [7, Theorem 2.9]). Therefore, $R_1 s_1 \cong R_1/P_1^{m_2}$ (resp. $R_2 s_2 \cong R_2/P_2^{k_1}$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i s_i$ is pure-injective. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{s}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 s_1 = M_1 \to \bar{M} \leftarrow M_2 = R_2 s_2)$. Then M is an R-submodule of S which is pseudo-absorbing comultiplication by Lemma 3.5 and is a direct summand of S; this implies that $S = M$, and S is as in (iii) in the list of Lemma 3.5 (see [7, Theorem 2.9]). □

Corollary 3.7. *Let R be a pullback ring as in (1).*

(i) *Every separated pseudo-absorbing comultiplication R-module S is of the form $S = M \oplus N$, where M is a direct sum of copies of the modules as in (i)-(ii), and N is a direct sum of copies of the modules as in (iii) of Lemma 3.5.*

(ii) *Every separated pseudo-absorbing comultiplication R-module is pure-injective.*

Proof. Apply Theorem 3.6 and [7, Theorem 2.9]. □

4. The non-separated pseudo-absorbing comultiplication modules

We continue to use the notation already established, so R is the pullback ring as in (1). In this section we find the indecomposable non-separated pseudo-absorbing comultiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable pseudo-absorbing comultiplication modules. We need the following proposition proved in [12, Lemma 4.1, Proposition 4.2 and Proposition 4.3].

Proposition 4.1. *Let R be a pullback ring as in (1). Assume that M is any non-zero R-module and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.*

(i) *If N is a non-zero submodule of M, then $0 \to K \to \varphi^{-1}(N) = T \to N \to 0$ is a separated representation of N.*

(ii) *If M is a non-separated R-module, then $P^nM \neq 0$ and $K \subseteq P^nS$ for all positive integers n.*

(iii) *If S has a submodule T with $(T :_R S) = 0$ or $P_1 \oplus 0$ or $P_1^2 \oplus 0$ or $0 \oplus P_2$ or $0 \oplus P_2^2$, then M is separated.*

Proposition 4.2. *Let R be a pullback ring as in (1). Then $E(R/P)$, the injective hull of R/P, is a non-separated pseudo-absorbing comultiplication R-module.*

Proof. For each $i = 1, 2$, let E_i be the R_i-injective hull of R_i/P_i, regarded as an R-module (so E_1, E_2 are the modules listed under (i)-(ii) in Lemma 3.5). Set $A_n = (0 :_{E_1} P_1^n)$ and $B_n = (0 :_{E_1} P_2^n)$ $(n \geq 1)$. Then by [11, Proposition 3.1], the non-zero proper R-submodules of $E = E(R/P)$ are: E_1, E_2, A_n, B_m, $E_1 + B_n$, $A_m + E_2$ and $A_n + B_m$ for all $n, m \geq 1$. If L is a non-zero proper submodule of E, then $(L :_R E) = 0$ since E is divisible. Thus every non-zero proper submodule of E is a pseudo-absorbing R-submodule by Proposition 3.3 (a). Since the cases E_1, E_2, A_n, B_m are clear, we split the proof into two cases.

Case 1: Suppose that $L = A_n + B_m$ $(m, n \geq 1)$; we show that $L = (0 :_E P_1^n \oplus P_2^m)$. If $z \in (0 :_E P_1^n \oplus P_2^m)$, then there exist $z_1 \in E_1$ and $z_2 \in E_2$ such that $z = z_1 + z_2$ and $(P_1^n + P_2^m)(z) = 0$; so $P_1^n z = P_2^m = 0$ which implies that $P_1^n(z_1 + z_2) = P_1^n z_1 = 0$. Similarly, $P_2^m z_2 = 0$. Thus $z \in A_n + B_m$; hence $(0 :_E P_1^n \oplus P_2^m) \subseteq A_n + B_m$. The proof of the other inclusion is similar.

Case 2: Suppose that $L = A_n + E_2$ $(n \geq 1)$; we show that $L = (0 :_E P_1^n \oplus 0)$. If $z \in (0 :_E P_1^n \oplus 0)$, then there exist $z_1 \in E_1$ and $z_2 \in E_2$ such that $z = z_1 + z_2$ and $P_1^n z = 0$ which implies that $P_1^n(z_1 + z_2) = P_1^n z_1 = 0$. Thus $z \in A_n + E_2$; hence $(0 :_E P_1^n \oplus 0) \subseteq A_n + E_2$. The other implication is similar. Similarly we argue when $L = E_1 + B_m$ $(m \geq 1)$. Note that since $R/P = E_1 \cap E_2 = P_1E \cap P_2E$, E is a non-separated R-module (see [7, p. 4054]). □

Let $A = F[x, y]$ be the polynomial ring over a field F in two indeterminates x, y. Then $\bar{A} = A/(x^2, y^2)$ is a comultiplication $\bar{A}$-module. But $\bar{A}/\bar{A}\bar{x}\bar{y}$ is not a comultiplication $\bar{A}$-module. This shows that a homomorphic image of a comultiplication module need not be a comultiplication module [31], but we have the following theorem:

Theorem 4.3. *Assume that R is a pullback ring as in (1) and let M be a non-zero comultiplication non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. If N is a non-zero proper R-submodule of M, then M/N is a comultiplication R-module.*

Proof. Let L/N be a proper submodule of M/N. Then L is a proper submodule of M, so $L = (0 :_M (0 :_R L))$ with $(0 :_R L) \neq 0$. Then there exist integers t, s such that $L = (0 :_M P_1^t \oplus P_2^s)$. We show that $L/N = (0 :_{M/N} (P_1^t \oplus P_2^s))$. Let $x+N \in L/N$. Then $(P_1^t \oplus P_2^s)x = 0$ gives $(P_1^t \oplus P_2^s)(x+N) = 0$; so $x+N \in (0 :_{M/N} (P_1^t \oplus P_2^s))$. For the reverse inclusion, assume that $y + N \in (0 :_{M/N} (P_1^t \oplus P_2^s))$. Then $(P_1^t \oplus P_2^y)y \subseteq N \subseteq L$. We claim that $(P_1^t \oplus P_2^s)y = 0$. Assume to the contrary, $0 \neq (P_1^t \oplus P_2^s)y \subseteq L$. Then $(P_1^{2t} \oplus P_2^{2s})y = 0$. Let m be the least positive integer such that $P^m y = 0$ (so $P^{m-1}y \neq 0$). There exists $x \in S$ such that $y = \varphi(x)$ and $\varphi(P^m x) = 0$; so $\varphi(P_1^m x) = \varphi(P_2^m x) = 0$. By [19, Proposition 2.3], φ is one-to-one on $P_i S$ for each i, we find that $P_2^m x = P_1^m x = 0$; hence $P^m x = 0$. Set $U = P^{m-1}y$. Then $0 \to K \to \varphi^{-1}(U) = P^{s-1}x \to U \to 0$ is a separated representation of U by Proposition 4.1 (i) such that $K \subseteq P(P^{s-1}x) = 0$ which is a contradiction since M is non-separated. Thus $(P_1^n \oplus P_2^m)y = 0$, and so we have equality. □

Corollary 4.4. *Assume that R is a pullback ring as in (1) and let M be a non-zero pseudo-absorbing comultiplication non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. If N is a non-zero R-submodule of M, then M/N is a pseudo-absorbing comultiplication R-module.*

Proof. Let L/N be a pseudo-absorbing submodule of M/N. Then by Lemma 2.2, L is a pseudo-absorbing submodule of M, so $L = (0 :_M (0 :_R L))$ with $0 \neq (0 :_R L) = P_1^n \oplus P_2^m$. By an argument like that in Theorem 4.3, we find that $L/N = (0 :_{M/N} (P_1^n \oplus P_2^m))$. □

Theorem 4.5. *Let R be a pullback ring as in (1) and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pseudo-absorbing comultiplication if and only if M is pseudo-absorbing comultiplication.*

Proof. Assume that S is a pseudo-absorbing comultiplication R-module. Then $S/K \cong M$ is pseudo-absorbing comultiplication by Corollary 4.4. Conversely, suppose that M is a pseudo-absorbing comultiplication R-module and let T be a non-zero pseudo-absorbing submodule of S. Since M is non-separated, $(T :_R S) \in \{P, P_1 \oplus P_2^2, P_1^2 \oplus P_2, P_1^2 \oplus P_2^2\}$ by Proposition 4.1 (iii) and Lemma 3.1 (iii). If $(T :_R S) = P$, then $K \subseteq PS \subseteq T$ by Proposition 4.1 (ii). Now by Lemma 2.2, T/K is a pseudo-absorbing submodule of $S/K \cong M$. If $(T :_R S) = P_1^2 \oplus P_2$, then $K \subseteq P^2 S \subseteq (P_1^2 \oplus P_2)S \subseteq T$ which implies that T/K is a pseudo-absorbing submodule of S/K. Similarly, we argue when $(T :_R S) = P_1 \oplus P_2^2$ or $P_1^2 \oplus P_2^2$. Since S/K is pseudo-absorbing comultiplication, we have $T/K = (0 :_{S/K} P_1^n \oplus P_2^m)$

for some integers m, n. We show that $T = (0 :_S P_1^n \oplus P_2^m)$. Let $s \in T$. Then $(P_1^n \oplus P_2^m)(s + K) = 0$; so $(P_1^n \oplus 0)s \subseteq (P_1^n \oplus P_2^m)s \subseteq K$. Thus $(P_1^n \oplus P_2^m)s = 0$ since $P_iS \cap K = 0$ by [19, Proposition 2.3]; so $s \in (0 :_S P_1^n \oplus P_2^m)$ which implies that $T \subseteq (0 :_S P_1^n \oplus P_2^m)$. The other implication is clear. □

Corollary 4.6. *Assume that R is a pullback ring as in (1) and let $0 \to K \to S \to M \to 0$ be a separated representation of a pseudo-absorbing comultiplication non-separated R-module M with M/PM finite dimensional over $\bar{R}$. Then* $\mathrm{pabSpec}_R(M) \neq \emptyset$.

Proof. By Theorem 4.5, S is a pseudo-absorbing comultiplication R-module, so $\mathrm{pabSpec}_R(S) \neq \emptyset$ by Proposition 3.3 (b). Thus S has a pseudo-absorbing submodule T. By an argument like that Theorem 4.5, we get $K \subseteq T$. Now by Lemma 2.2, T/K is a pseudo-absorbing submodule of $S/K \cong M$; hence $\mathrm{pabSpec}_R(M) \neq \emptyset$. □

Proposition 4.7. *Let R be a pullback ring as in (1), and let M be an indecomposable pseudo-absorbing comultiplication non-separated R-module with M/PM finite-dimensional top over $\bar{R}$. If $0 \to K \to S \to M \to 0$ is a separated representation of M, then S has finite-dimensional top and is pure-injective.*

Proof. Since $S/PS \cong M/PM$ by [7, Proposition 2.6 (i)], we find that S has finite-dimensional top. Pure-injectivity of S follows from Theorem 4.5 and Corollary 3.7. □

Let R be a pullback ring as in (1) and let M be an indecomposable pseudo-absorbing comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 4.7, S is pure-injective. So in the proofs of [7, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [7, Proposition 2.6 (ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by 'M is an indecomposable pseudo-absorbing comultiplication non-separated R-module": because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following result:

Corollary 4.8. *Let R be a pullback ring as in (1) and let M be an indecomposable pseudo-absorbing comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then the following hold:*

(i) *S is a direct sum of finitely many indecomposable pseudo-absorbing comultiplication modules.*

(ii) *At most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Before embarking on the proof of the next result let us explain its idea. Let R be a pullback ring as in (1). Let M be any R-module and let $0 \to K \to S \to M \to 0$ be a separated representation of M. We have shown already that if M is indecomposable pseudo-absorbing comultiplication with M finite-dimensional top, then S is a direct sum of just finitely many indecomposable separated pseudo-absorbing comultiplication modules and these are known by Theorem 3.6. In any separated representation $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ the kernel of the map φ to M is annihilated by P, hence is contained in the socle of the separated module S. Thus M is obtained by amalgamation in the socles of the various direct summands of S. So the questions are: does this provide any further condition on the possible direct summands of S? How can these summands be amalgamated in order to form M? For the case of finitely generated R-modules M these questions are answered by Levy's description [20], see also [21, Section 11]. Levy shows that the indecomposable finitely generated R-modules are of two non-overlapping types which he calls deleted cycle and block cycle types. It is the modules of deleted cycle type which are most relevant to us. Such a module is obtained from a direct summand, S, of indecomposable separated modules by amalgamating the direct summands of S in pairs to form a chain but leaving the two ends unamalgamated. Reflecting the fact that the dimension over $\bar{R}$ of the socle of any finitely generated indecomposable separated module is ≤ 2 each indecomposable summand of S may be amalgamated with at most two other indecomposable summands. Consider the indecomposable separated R-modules $S(n,m) = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ with $n, m \geq 2$ (it is generated over R by $(1 + P_1^n, 1 + P_2^m)$). Actually, separated indecomposable R-modules also include R_1/P_1^n for $n \geq 2$, which can be regarded up to isomorphism as $S(n,1) = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2)$. Similarly, for $m \geq 2$, $S(1,m) = (R_1/P_1 \to \bar{R} \leftarrow R_2/P_2^m)$ is a separated indecomposable R-module. Moreover, R_1, R_2 and R themselves can be viewed as separated indecomposable R-modules, corresponding to the cases $n = \infty$ and $m = 1$, $n = 1$ and $m = \infty$, $n = m = \infty$. Deleted cycle indecomposable R-modules are introduced as follows: Let S be a direct sum of finitely many modules $S(i) = S(n_{i,1}, n_{i,2})$ (with $i < s$ a non-negative integer). Here $n_{i,j} \geq 2$ for every $j < s$ and $j = 1, 2$, with two possible exceptions $i = 0$, $j = 1$ and $i = s - 1$ and $j = 2$, where the values $n_{i,j} = 1$ or

∞ are allowed. Then amalgamate the direct summands in S by identifying the P_2-part of the socle of $S(i)$ and the P_1-part of the socle $S(i+1)$ for every $i < s-1$. For instance, given the separated modules $S_1 = (R_1 \to \bar{R} \leftarrow R_2/P_2^3) = Ra$ with $P_2^3 a = 0$ and $S_2 = (R_1/P_1^7 \to \bar{R} \leftarrow R_2/P_2^2) = Ra'$ with $P_1^7 a' = 0 = P_2^2 a'$. Then one can form the non-separated module $(S_1 \oplus S_2)/(R(p_2^2 a - p_1^6 a') = Rc + Rc'$ where $c = a + R(p_2^2 a - p_1^6 a')$, $c' = a' + R(p_2^2 a - p_1^6 a')$, $P_2^3 c = 0 = P_1^7 c' = P_2^2 c$ and $P_2^2 c = P_1^6 c'$ which is obtained by identifying the P_2-part of the socle of S_1 with the P_1-part of the socle of S_2. We will use that same description, but with pseudo-absorbing comultiplication separated modules in place of the finitely generated ones, gives us the non-zero indecomposable pseudo-absorbing comultiplication non-separated R-modules. As a consequence, any non-zero indecomposable pseudo-absorbing comultiplication separated module with 1-dimensional socle may occur only at one of the ends of the amalgamation chain (see [7, Proposition 3.4]). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable pseudo-absorbing comultiplication. We do that now and thus complete the classification of the indecomposable pseudo-absorbing comultiplication non-separated modules with finite-dimensional top.

Theorem 4.9. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two DVRs R_1, R_2 with common factor field $\bar{R}$. Then the indecomposable non-separated pseudo-absorbing comultiplication modules with finite-dimensional top, up to isomorphism, are the following:*

(i) *$M = E(R/P)$, the injective hull of R/P;*

(ii) *The indecomposable modules of finite length (apart from R/P which is separated), that is, $M = \sum_{i=1}^{s} Ra_i$ with*

$$p_1^{n_s} a_s = 0 = p_2^{m_1} a_1, p_1^{n_i - 1} a_i = p_2^{m_{i+1}-1} a_{i+1} (1 \leq i \leq s-1)$$

$m_i, n_i \geq 2$ except that $m_1 \geq 1$, $n_s \geq 1$.

(iii) *$M = E_1 + \sum_{i=1}^{s} Ra_i + E_2$ with*

$$a_0 = p_2^{m_1 - 1} a_1, b_0 = p_1^{n_s - 1} a_s, p_1 a_0 = 0 = p_2 b_0,$$

and $p_1^{n_i - 1} a_i = p_2^{m_{i+1}-1} a_{i+1}$ for all $1 \leq i \leq s-1$, where $E_1 \cong E(Ra_0) \cong E(R_1/P_1)$, $E_2 \cong E(Rb_0) \cong E(R_2/P_2)$ and $m_i, n_i \geq 2$ except that $m_1 \geq 1$, $n_s \geq 1$.

(iv) *$M = E_1 + \sum_{i=1}^{s} Ra_i$ with*

$$p_1^{n_s} a_s = 0, a_0 = p_2^{m_1 - 1} a_1, p_1 a_0 = 0,$$

and $p_1^{n_i-1}a_i = p_2^{m_{i+1}-1}a_{i+1}$ *for all* $1 \leq i \leq s-1$*, where* $E_1 \cong E(Ra_0) \cong E(R_1/P_1)$*, and* $m_i, n_i \geq 2$ *except that* $n_s \geq 1$*,*

(v) $M = \sum_{i=1}^{s} Ra_i + E_2$ *with*

$$p_2^{m_s}a_s = 0, b_0 = p_1^{n_1-1}a_1, p_2b_0 = 0,$$

and $p_2^{m_i-1}a_i = p_1^{n_{i+1}-1}a_{i+1}$ *for all* $1 \leq i \leq s-1$*, where* $E_2 \cong E(Rb_0) \cong E(R_2/P_2)$*, and* $m_i, n_i \geq 2$ *except that* $m_s \geq 1$*.*

Proof. Let M be an indecomposable non-separated pseudo-absorbing comultiplication R-module with finite-dimensional top and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. By Corollary 4.8 (ii), S is a direct sum of finitely many indecomposable pseudo-absorbing comultiplication separated modules. We know already that every indecomposable pseudo-absorbing comultiplication non-separated module has one of these forms so it remains to show that the modules obtained by these amalgamation are, indeed, indecomposable pseudo-absorbing comultiplication modules. (i) follows from Proposition 4.2. Since a quotient of any pseudo-absorbing comultiplication R-module is pseudo-absorbing comultiplication by Corollary 4.4, they are pseudo-absorbing comultiplication. The indecomposability follows from [20, 1.9] and [7, Theorem 3.5]. □

Remark 4.10. (i) Let R be the pullback ring as described in Theorem 4.9. Then by [7, Theorem 3.5] and Theorem 4.9, every indecomposable pseudo-absorbing comultiplication R-module with finite-dimensional top is pure-injective.

(ii) This paper includes the classification of indecomposable pseudo-absorbing comultiplication modules with finite-dimensional top over k-algebra $k[x, y : xy = 0]_{(x,y)}$.

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