CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 1, pp. 1-11
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

On some general integral formulae

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ABSTRACT. We repeat and reformulate some more or less known general integral formulae and deduce from them some applications in a concise way. We then present some general double integral formulae which play an essential role in the calculation of fundamental solutions to homogeneous elliptic operators. In particular, this yields generalizations of definite integrals found in standard integral tables. In the final section, the area of an ellipsoidal hypersurface in $\mathbf{R}^n$ is represented by a hyperelliptic integral.

Keywords: Leray's formula, elliptic integrals, definite double integrals.

2020 Mathematics Subject Classification: 26A42, 33E05, 35E05, 44A10, 46F10.

1. INTRODUCTION AND NOTATION

By "general integral formulae", we understand here integral formulae containing "arbitrary" functions, i.e., formulae that hold at least for functions in a space of infinite dimension. E.g., Frullani's formula

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = f(0) \log(\frac{b}{a}), \quad a > 0,\ b > 0$$

holds for each temperate test function $f \in \mathcal{S}(\mathbf{R}^1)$, but of course also in a much more general context, see [17]. In contrast, the special case

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x} \, \mathrm{d}x = \log\Big(\frac{b}{a}\Big), \quad a > 0,\ b > 0,$$

of Frullani's formula is just a special definite integral.

Besides the many integral representations (Cauchy, Bochner–Martinelli, Leray–Koppelmann etc.) in complex analysis, see, e.g., [1], there is a host of general integral formulae in real analysis contained in integral tables, see, e.g., [2, 13.2 Schlömilch's Transformation, p. 251], [6, pp. 7, 63, 117, 129, 227 307], [9, pp. 93, 96, 98, 102, 107,109, 110, 114, 119, 121, 123, 125, 126, 130], [12, pp. 6–8], [15, Thms. 1–6, pp. 125–134].

The aim of this article is to attract attention to some general integral formulae in real analysis, to their connection with integrals over δ-measures (see Section 2) and to some applications (see Section 3). In Section 4, we present a general integral formula for double integrals, which earlier enabled to represent fundamental solutions of the homogeneous elliptic operators $\partial_x^4 + \partial_y^4 + \partial_z^4 + 2a\partial_x 2\partial_y^2 + 2b\partial_x^2\partial_z^2 + 2c\partial_y^2\partial_z^2$, see [20]. Section 5 is devoted to the calculation of the (hypersurface) area of an ellipsoidal hypersurface in $\mathbf{R}^n$. In dimensions $n \geq 4$ and for generic diameters, this yields a hyper-elliptic integral not reducible to elliptic ones.

Received: 19.11.2023; Accepted: 01.02.2024; Published Online: 06.02.2024
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DOI: 10.33205/cma.1406998

Let us introduce some notation. The inner product of $x, \xi \in \mathbf{R}^n$ is denoted by $x\xi$. We employ the standard notation for distributions as in [18] and, in particular, we denote the Heaviside function by Y, see [18, p. 36]. We write δ_s for the delta distribution with support in $s \in \mathbf{R}$, i.e., $\delta_s = \frac{\mathrm{d}}{\mathrm{d}x} Y(x-s)$.

The Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \int_{\mathbf{R}^n} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\,\mathrm{d}x$$

for $f \in L^1(\mathbf{R}^n)$ and extended to the space of temperate distributions $\mathcal{S}'(\mathbf{R}^n)$ by continuity.

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution T in one variable t with respect to a submersive C^∞ function $h : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is defined as in [16, Def. 1.2.12, p. 19], i.e.,

$$\langle \phi, h^*T \rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_\Omega Y(t - h(x)) \phi(x)\,\mathrm{d}x \Big), T \Big\rangle, \quad \phi \in \mathcal{D}(\Omega).$$

2. An integral formula of W. Gröbner and N. Hofreiter and its companion

In [12, Equ. 031.13f], the formula

$$\int_0^\infty f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{x} = 2 \int_{2\sqrt{\xi\eta}}^\infty \frac{f(u)}{\sqrt{u^2 - 4\xi\eta}}\,\mathrm{d}u, \quad \xi > 0,\, \eta > 0 \tag{2.1}$$

is stated. A companion formula holds for ξ, η of opposite sign:

$$\int_0^\infty f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{x} = \int_{-\infty}^\infty \frac{f(u)}{\sqrt{u^2 - 4\xi\eta}}\,\mathrm{d}u, \quad \xi\eta < 0. \tag{2.2}$$

(Obviously, suitable conditions on the function f must be imposed in order to ensure the existence of the improper integrals.)

An important application of the above formulae is the Fourier transform of Riemann's singularity function $Y(x)x^{-1}\mathrm{e}^{-\mathrm{i}\eta/x}$, $\eta \in \mathbf{R}\backslash\{0\}$. In fact, if $f(u) = \mathrm{e}^{-\mathrm{i}u}$, then formulae (2.1) and (2.2) yield, by means of the well-known integral representations of the Bessel functions J_0, N_0, K_0,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\mathrm{i}(\xi x + \eta/x)} \frac{\mathrm{d}x}{x} &= 2Y(\xi\eta) \int_{2\sqrt{\xi\eta}}^\infty \frac{\mathrm{e}^{-\mathrm{i}u}}{\sqrt{u^2 - 4\xi\eta}}\,\mathrm{d}u + Y(-\xi\eta) \int_{-\infty}^\infty \frac{\mathrm{e}^{-\mathrm{i}u}}{\sqrt{u^2 - 4\xi\eta}}\,\mathrm{d}u \\ &= -\pi Y(\xi\eta)\big[N_0(2\sqrt{\xi\eta}) + \mathrm{i}J_0(2\sqrt{\xi\eta})\big] + 2Y(-\xi\eta)K_0(2\sqrt{-\xi\eta}),\end{aligned}$$

i.e.,

$$\begin{aligned}\mathcal{F}_x\big(Y(x)x^{-1}\mathrm{e}^{-\mathrm{i}\eta/x}\big)(\xi) &= \mathcal{F}_{xy}\big(Y(x)\delta(xy-1)\big)(\xi, \eta) \\ &= -\pi Y(\xi\eta)\big[N_0(2\sqrt{\xi\eta}) + \mathrm{i}J_0(2\sqrt{\xi\eta})\big] + 2Y(-\xi\eta)K_0(2\sqrt{-\xi\eta}).\end{aligned}$$

If we extend the integral formulae (2.1) and (2.2) to the negative axis by using the equation

$$-\int_{-\infty}^0 f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{x} = \int_0^\infty f\Big(-\xi x - \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{x},$$

we arrive at the following proposition.

Proposition 2.1. *Let f be a continuous function on $\mathbf{R}$ such that the integral $\int_{-\infty}^\infty f(u)\,\mathrm{d}u/(1+|u|)$ is convergent in the sense that*

$$\lim_{M \to -\infty} \lim_{N \to \infty} \int_M^N f(u) \frac{\mathrm{d}u}{1 + |u|}$$

converges. Set $t_+^{-1/2} = Y(t)t^{-1/2}$ *for* $t \in \mathbf{R} \setminus \{0\}$*. Then the formula*

$$\int_{-\infty}^{\infty} f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{|x|} = 2\int_{-\infty}^{\infty} f(u)(u^2 - 4\xi\eta)_+^{-1/2}\,\mathrm{d}u \tag{2.3}$$

holds for all $\xi, \eta \in \mathbf{R} \setminus \{0\}$*.*

Proof. The application

$$\mathbf{R} \setminus \{0\} \longrightarrow \mathbf{R} : x \longmapsto u = \xi x + \frac{\eta}{x}$$

has the range $\{u \in \mathbf{R};\ u^2 \geq 4\xi\eta\}$ and it covers this range twice. Furthermore,

$$\Big|\frac{\mathrm{d}u}{\mathrm{d}x}\Big| = \Big|\xi - \frac{\eta}{x^2}\Big| = \frac{1}{|x|}\Big|\xi x - \frac{\eta}{x}\Big| = \frac{\sqrt{u^2 - 4\xi\eta}}{|x|}$$

and hence formula (2.3) follows from substitution. We observe that the integral on the left-hand side of formula (2.3) has to be interpreted as the limit

$$\lim_{M,N\to\infty}\int_{M^{-1}<|x|<N} f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{|x|},$$

and this limit converges due to the conditional convergence of the integral $\int_{-\infty}^{\infty} f(u)\,\mathrm{d}u/(1 + |u|)$. $\square$

Let us remark that, vice versa, formula (2.3) implies the equations in (2.1) and (2.2). In fact, if ξ, η are positive, then we simply set $f(u) = 0$ for $u < 0$; if $\xi\eta < 0$, we first observe that the integral $\int_0^\infty f(\xi x + \eta/x)\mathrm{d}x/x$ depends only on the value of the product $\xi\eta$ as shown by applying the substitutions $x \mapsto cx$, $c > 0$, and $x \mapsto x^{-1}$, respectively, in this integral. Hence

$$\int_{-\infty}^{0} f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{|x|} = \int_0^\infty f\Big(-\xi x - \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{x} = \int_0^\infty f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{x}$$

holds for $\xi\eta < 0$.

Let us next explain the connection of the integral on the left-hand side of formula (2.3) with the measures $\delta_s(xy)$ supported by the hyperbolas $xy = s$ in $\mathbf{R}^2$, $s \in \mathbf{R} \setminus \{0\}$. As distributions, these measures are defined as

$$\begin{aligned}\langle \phi, \delta_s(xy)\rangle &= \frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbf{R}^2}\phi(x,y)Y(s - xy)\,\mathrm{d}x\mathrm{d}y\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\int_{-\infty}^{\infty}\Big[Y(x)\int_{-\infty}^{s/x}\phi(x,y)\,\mathrm{d}y + Y(-x)\int_{s/x}^{\infty}\phi(x,y)\,\mathrm{d}y\Big]\mathrm{d}x \qquad (2.4)\\ &= \int_{\mathbf{R}}\phi\Big(x, \frac{s}{x}\Big)\frac{\mathrm{d}x}{|x|}, \qquad \phi \in \mathcal{D}(\mathbf{R}^2),\ s \in \mathbf{R} \setminus \{0\}.\end{aligned}$$

Incidentally, we observe that the absolute value in $|x|$ is missing in the well-known textbook [10], which has so many merits and so few flaws, see [10, Ch. III, Section 1.3, Ex. 3, Equ. (4), p. 223]. Note that a different, ad hoc definition of the symbol $\delta_s(xy)$—while, strictly logically, being possible—is not in agreement with the usual definitions of the composition of functions and of the pull-back of distributions. In fact, for a different determination of $\delta_s(xy) \in \mathcal{D}'(\mathbf{R}^2_{xy})$, $s \in \mathbf{R} \setminus \{0\}$, the equation

$$\delta_s(xy) = \lim_{\epsilon\searrow 0}\frac{1}{2\epsilon}Y(\epsilon - |xy - s|), \qquad s \in \mathbf{R} \setminus \{0\} \tag{2.5}$$

does not hold in $\mathcal{D}'(\mathbf{R}^2)$ as it should due to $\delta_s = \lim_{\epsilon\searrow 0} Y(\epsilon - |t - s|)/(2\epsilon)$ in $\mathcal{D}'(\mathbf{R}^1_t)$. (Equation (2.5) also shows that $\delta_s(xy)$ must be a *positive* Radon measure, in contrast to the determination in [10].) Similarly, the definition $Y' := -\delta$ might, strictly logically, be correct, but it would not make much sense either. We finally observe that $\delta_0(xy) = \delta(xy)$ cannot be defined unambiguously since the mapping $h(x, y) = xy$ is not submersive for $xy = 0$, i.e., on $h^{-1}(\operatorname{supp} T)$ for $T = \delta \in \mathcal{D}'(\mathbf{R}^1)$. Also, the limit in (2.5) diverges in $\mathcal{D}'(\mathbf{R}^2)$ if $s = 0$.

Note that we can apply the measure $\delta_s(xy)$ not only to test functions $\phi \in \mathcal{D}(\mathbf{R}^2)$, but to each continuous function $\phi(x, y)$ such that $\phi(x, y)(|x| + |y|)^\epsilon$ is bounded on the hyperbola $xy = s$ for some positive ϵ. Therefore

$$\langle f(\xi x + \eta y), \delta_s(xy)\rangle = \int_{-\infty}^{\infty} f\Big(\xi x + \frac{\eta s}{x}\Big) \frac{\mathrm{d}x}{|x|} \tag{2.6}$$

holds, e.g., for $f \in \mathcal{S}(\mathbf{R})$ and $\xi, \eta, s \in \mathbf{R} \setminus \{0\}$. Upon replacing η by ηs in Proposition 2.1, formula (2.6) leads to the following proposition.

Proposition 2.2. *We set, as before, $t_+^{-1/2} = Y(t)t^{-1/2}$ for $t \in \mathbf{R} \setminus \{0\}$ and assume that $\xi, \eta \in \mathbf{R} \setminus \{0\}$. Then the equation*

$$\int_{\mathbf{R}^2} F(\xi x + \eta y, xy)\, \mathrm{d}x\mathrm{d}y = 2 \int_{\mathbf{R}^2} F(u, s)(u^2 - 4\xi\eta s)_+^{-1/2}\, \mathrm{d}u\mathrm{d}s \tag{2.7}$$

holds for each measurable function $F : \mathbf{R}^2 \to \mathbf{C}$ such that the integral on the right-hand side of (2.7) is absolutely convergent.

Proof. We first note that the substitution

$$\mathbf{R}^2 \longrightarrow \mathbf{R}^2 : (x, y) \longmapsto (u, s) = (\xi x + \eta y, xy)$$

covers twice its range $\{(u, s);\ u^2 \geq 4\xi\eta s\}$ and has the Jacobian $\xi x - \eta y = \pm\sqrt{u^2 - 4\xi\eta s}$. Hence equation (2.7) holds for $F \in \mathcal{D}(\{(u, s);\ u^2 \neq 4\xi\eta s\})$ and consequently, by density, also for all measurable F making one (and hence both) of the integrals in (2.7) absolutely convergent. $\square$

3. Generalization to $\mathbf{R}^{n+1}$. The formulae of J.Leray, J.Faraut and K. Harzallah

Let us generalize now Proposition 2.2 to $n + 1$ dimensions by considering the Lorentz form $t^2 - |x|^2$, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, instead of the form $(x, y) \mapsto xy$ on $\mathbf{R}^2$.

Proposition 3.3. *Let $\tau \in \mathbf{R}$, $\xi \in \mathbf{R}^{n+1}$ such that $\tau > |\xi|$ and set $\rho = \sqrt{\tau^2 - |\xi|^2}$ and $t_+^{n/2-1} = Y(t)t^{n/2-1}$ for $t \in \mathbf{R}$. We assume that $F : \mathbf{R}^2 \to \mathbf{C}$ is measurable and that the integral $\int_{\mathbf{R}^2} |F(u, s)| [u^2 - \rho^2 s]_+^{n/2-1} \mathrm{d}u\mathrm{d}s$ is finite. Then*

$$\int_{\mathbf{R}^{n+1}} F(\tau t + \xi x, t^2 - |x|^2)\, \mathrm{d}t\mathrm{d}x = \frac{\pi^{n/2}\rho^{1-n}}{\Gamma(\frac{n}{2})} \int_{\mathbf{R}^2} F(u, s)[u^2 - \rho^2 s]_+^{n/2-1} \mathrm{d}u\mathrm{d}s. \tag{3.8}$$

Proof. Upon using a Lorentz transformation (which automatically preserves volumes), we can replace (τ, ξ) by $(\rho, 0)$. We assume first that F belongs to $\mathcal{C}(\mathbf{R}^2)$ and has compact support. Using polar coordinates $x = r\omega$, $r > 0$, $\omega \in \mathbf{S}^{n-1}$, the substitutions $u = \rho t$ and $s = \rho^{-2}u^2 - r^2$, $\mathrm{d}s = -2r\mathrm{d}r$, $r = (\rho^{-2}u^2 - s)^{1/2}$, and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbf{R}^{n+1}} F(\rho t, t^2 - |x|^2)\, \mathrm{d}t\mathrm{d}x &= \frac{2\pi^{n/2}}{\rho\,\Gamma(\frac{n}{2})} \int_{-\infty}^{\infty} \Big[\int_0^{\infty} F\Big(u, \frac{u^2}{\rho^2} - r^2\Big) r^{n-1}\mathrm{d}r\Big] \mathrm{d}u \\
&= \frac{\pi^{n/2}}{\rho\,\Gamma(\frac{n}{2})} \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{u^2/\rho^2} F(u, s)\Big(\frac{u^2}{\rho^2} - s\Big)^{n/2-1} \mathrm{d}s\Big] \mathrm{d}u \\
&= \frac{\pi^{n/2}\rho^{1-n}}{\Gamma(\frac{n}{2})} \int_{\mathbf{R}^2} F(u, s)[u^2 - \rho^2 s]_+^{n/2-1} \mathrm{d}u\mathrm{d}s.
\end{aligned}$$

As in Proposition 2.2, the proof is completed by a density argument. $\square$

From equation (3.8) in Proposition 3.3, we can easily deduce Leray's formula for the Laplace transform of Lorentz invariant functions on the cone $C = \{(t, x) \in \mathbf{R}^{n+1};\ t \geq |x|\}$, see [14, Equ. (19.11), p. 41], [13, Thm. 1, p. 53], [19].

Proposition 3.4. *Let $(\tau, \xi) \in C$ and set $\rho = \sqrt{\tau^2 - |\xi|^2}$. We assume that $g : [0, \infty) \longrightarrow \mathbf{C}$ is measurable such that $\int_0^\infty |g(s)| K_{(n-1)/2}(\rho\sqrt{s}) s^{(n-1)/4}\,\mathrm{d}s$ is finite. Then*

$$\int_C \mathrm{e}^{-(\tau t+\xi x)} g(t^2 - |x|^2)\,\mathrm{d}t\mathrm{d}x = \int_0^\infty K_{(n-1)/2}(\rho\sqrt{s}) \Big(\frac{2\pi\sqrt{s}}{\rho}\Big)^{(n-1)/2} g(s)\,\mathrm{d}s. \tag{3.9}$$

Proof. We set $g(s) = 0$ for $s < 0$ and $F(u, s) = Y(u)\mathrm{e}^{-u} g(s)$. Then the function $F(\tau t + \xi x, t^2 - |x|^2)$ coincides with $\mathrm{e}^{-(\tau t - \xi x)} g(t^2 - |x|^2)$ on C and it vanishes on $\mathbf{R}^{n+1} \setminus C$. Hence we can apply Proposition 3.3, and [12, Equ. 313.23] implies

$$\begin{aligned}\int_C \mathrm{e}^{-(\tau t+\xi x)} g(t^2 - |x|^2)\,\mathrm{d}t\mathrm{d}x &= \frac{\pi^{n/2}\rho^{1-n}}{\Gamma(\frac{n}{2})} \int_0^\infty g(s) \Big[\int_{\rho\sqrt{s}}^\infty \mathrm{e}^{-u}(u^2 - \rho^2 s)^{n/2-1}\mathrm{d}u\Big]\mathrm{d}s \\ &= \int_0^\infty K_{(n-1)/2}(\rho\sqrt{s}) \Big(\frac{2\pi\sqrt{s}}{\rho}\Big)^{(n-1)/2} g(s)\,\mathrm{d}s.\end{aligned}$$

This completes the proof. $\square$

We remark that Leray's formula is the analogue of Poisson–Bochner's formula for the Fourier transform of radially invariant distributions, see [18, Equ. (VII, 7; 22), p. 259].

Examples. We can derive Faraut–Harzallah's formula for the Laplace transform of powers of Lorentz distances [7, Prop. III.9, p. 43] from formula (3.9) above by setting $g(s) = s^{(\mu-n-1)/2}$, $\mu \in \mathbf{C}$, $\operatorname{Re}\mu > n - 1$. This yields

$$\begin{aligned}\int_C \mathrm{e}^{-(\tau t+\xi x)} (t^2 - |x|^2)^{(\mu-n-1)/2}\,\mathrm{d}t\mathrm{d}x &= \Big(\frac{2\pi}{\rho}\Big)^{(n-1)/2} \int_0^\infty K_{(n-1)/2}(\rho\sqrt{s}) s^{(2\mu-n-3)/4}\,\mathrm{d}s \\ &= 2\Big(\frac{2\pi}{\rho}\Big)^{(n-1)/2} \int_0^\infty K_{(n-1)/2}(\rho\sigma)\sigma^{\mu-(n+1)/2}\,\mathrm{d}\sigma \\ &= \frac{2^{\mu-1}\pi^{(n-1)/2}\Gamma(\frac{\mu}{2})\Gamma(\frac{1+\mu-n}{2})}{(\tau^2 - |\xi|^2)^{\mu/2}}, \qquad \tau > |\xi|,\end{aligned}$$

by [11, Equ. 6.561.16]. Let us remark that the special case of $\mu = n + 1$ furnishes Exercise 1 in [4, p. 174].

Let us also explain how Proposition 3.3 is connected with a formula in [3]. If we set $n = 2$ and apply formula (3.8), using a limit process, to the distribution $F(u, s) = Y(u)f(u)\delta_1(s)$ for $f \in \mathcal{C}(\mathbf{R})$ with compact support, then we obtain

$$\int_0^\infty \langle f(\tau t + \xi x), \delta(t^2 - |x|^2 - 1)\rangle\,\mathrm{d}t = \frac{\pi}{\rho} \int_\rho^\infty f(u)\,\mathrm{d}u \tag{3.10}$$

for $(\tau, \xi) \in \mathbf{R}^3$ with $\tau > |\xi|$ and $\rho = \sqrt{\tau^2 - |\xi|^2}$.

Due to

$$Y(t)\delta(t^2 - |x|^2 - 1) = \frac{1}{2\sqrt{1 + |x|^2}}\,\delta\big(t - \sqrt{1 + |x|^2}\big),$$

we infer that

$$\int_{\mathbf{R}^2} f\big(\tau\sqrt{1 + |x|^2} + \xi x\big) \frac{\mathrm{d}x}{\sqrt{1 + |x|^2}} = \frac{2\pi}{\rho} \int_\rho^\infty f(u)\,\mathrm{d}u.$$

Finally, employing the parametrization $x_1 = \cosh\alpha\sinh\beta$, $x_2 = \sinh\alpha$, $t = \sqrt{1+|x|^2} = \cosh\alpha\cosh\beta$ of the upper shell $t > 0$ of the hyperboloid $t^2 = 1 + |x|^2$ and taking account of $\mathrm{d}x = \cosh^2\alpha\cosh\beta\,\mathrm{d}\alpha\mathrm{d}\beta$, we arrive at

$$\int_{\mathbf{R}^2} f\big(\tau\cosh\alpha\cosh\beta+\xi_1\cosh\alpha\sinh\beta+\xi_2\sinh\alpha\big)\cosh\alpha\,\mathrm{d}\alpha\mathrm{d}\beta$$

$$= \frac{2\pi}{\rho}\int_\rho^\infty f(u)\,\mathrm{d}u, \quad \tau > |\xi|,\ \rho = \sqrt{\tau^2-|\xi|^2},$$

which is formula 3.1.4.1 in [3].

4. Algebraic double integrals and “elliptic arctan-integrals"

In [16], we employed the formula

$$\partial_3 E(x_1,1,x_3) = -\frac{1}{4\pi^2}\int_0^{x_3}\mathrm{d}\lambda\int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{P(\alpha,-\lambda-x_1\alpha,1)} \tag{4.11}$$

in order to represent the (uniquely determined) even and homogeneous fundamental solution E of the homogeneous elliptic operator $P(\partial)$ of degree four and in three variables, see [16, Prop. 5.2.7, p. 357, and p. 359, line two from below].

Using formula (4.11), we calculated E in the cases of $P(\partial) = \partial_1^4+\partial_2^4+\partial_3^4$, see [16, Ex. 5.2.9, p. 359], and of $P(\partial) = \partial_1^4+\partial_2^4+\partial_3^4+2a\partial_1^2\partial_2^2$, $a > -1$, see [16, Ex. 5.2.11, p. 362]. For the operator $P(\partial) = \partial_1^4+\partial_2^4+\partial_3^4$, the fundamental solution E was first obtained in [8, p. 350]; for elliptic operators of the general form $P(\partial) = \sum_{j=1}^3\sum_{k=1}^3 c_{jk}\partial_j^2\partial_k^2$, this was done in [20, Prop. 3, p. 1198]. All these fundamental solutions can explicitly be represented by the complete elliptic integral of the first kind.

In the following, let us repeat some steps in these calculations starting from formula (4.11). We assume that $x_3 > 0$. Substitution of the variables

$$\alpha = t\sqrt[4]{\mu},\ \lambda = \sqrt[4]{\mu},\ \frac{\partial(\alpha,\lambda)}{\partial(t,\mu)} = \begin{pmatrix}\sqrt[4]{\mu} & \frac{t}{4}\mu^{-3/4}\\ 0 & \frac14\mu^{-3/4}\end{pmatrix}$$

leads to

$$\partial_3 E(x_1,1,x_3) = -\frac{1}{16\pi^2}\int_0^{x_3^4}\frac{\mathrm{d}\mu}{\sqrt{\mu}}\int_{-\infty}^\infty\frac{\mathrm{d}t}{P(t\sqrt[4]{\mu},-(1+tx_1)\sqrt[4]{\mu},1)}.$$

In the case of the operator $P(\partial) = \partial_1^4+\partial_2^4+\partial_3^4+2a\partial_1^2\partial_2^2$, $a > -1$, we obtain

$$P(t\sqrt[4]{\mu},-(1+tx_1)\sqrt[4]{\mu},1) = Q(t)\mu+1,$$

where $Q(t)$ is a polynomial of degree four fulfilling $Q(t) > 0$ for $t \in \mathbf{R}$. (In the notation, we suppressed the dependence of the coefficients of Q on x_1.) Inverting the order of integrations and substituting $u = \sqrt{Q(t)\mu}$ results in

$$\partial_3 E(x_1,1,x_3) = -\frac{1}{8\pi^2}\int_{-\infty}^\infty\frac{\arctan\big(x_3^2\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t.$$

By these considerations, we want to motivate our treatment of integrals of the form

$$\begin{aligned} I := \int_{-\infty}^\infty \frac{\arctan\big(\gamma\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t &= \frac12\int_0^{\gamma^2}\frac{\mathrm{d}\mu}{\sqrt{\mu}}\int_{-\infty}^\infty\frac{\mathrm{d}t}{Q(t)\mu+1}\\ &= \frac12\int_{\gamma^{-2}}^\infty\frac{\mathrm{d}\mu}{\sqrt{\mu}}\int_{-\infty}^\infty\frac{\mathrm{d}t}{Q(t)+\mu}, \quad \gamma > 0.\end{aligned} \tag{4.12}$$

As will be seen in Corollary 4.2 below, I in formula (4.12) can be expressed as an elliptic integral of the first kind and we shall call it therefore an "elliptic arctan-integral".

Let us first explain the basic idea of the evaluation of I in the simpler case of the biquadratic $Q(t) = t^4 + pt^2 + r$. We shall assume that $r > 0$ and $p > -2\sqrt{r}$, which are the conditions that the polynomial Q is positive on the real axis. If, additionally, $0 < r \le p^2/4$ and if we set $\lambda = \sqrt{r}$, we can write Q in the form $Q(t) = (t^2 + a^2)(t^2 + b^2)$ with $a > 0$, $b > 0$ and hence $ab = \lambda$ and $a + b = \sqrt{a^2 + b^2 + 2ab} = \sqrt{p + 2\lambda}$. Therefore [12, Equ. 141.14] yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t^4 + pt^2 + \lambda^2} = \frac{\pi}{\lambda\sqrt{p + 2\lambda}}, \tag{4.13}$$

and this equation persists for all $\lambda > 0$ and $p > -2\lambda$ by analytic continuation.

Inserting formula (4.13) into (4.12) and substituting $\lambda = \sqrt{\mu + r}$, then implies

$$\begin{aligned} I &= \frac{1}{2}\int_{\gamma^{-2}}^{\infty} \frac{\mathrm{d}\mu}{\sqrt{\mu}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{Q(t) + \mu} = \frac{\pi}{2}\int_{\gamma^{-2}}^{\infty} \frac{\mathrm{d}\mu}{\sqrt{\mu}\sqrt{\mu + r}\sqrt{p + 2\sqrt{\mu + r}}} \\ &= \frac{\pi}{\sqrt{2}}\int_{\sqrt{r+\gamma^{-2}}}^{\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda^2 - r}\sqrt{\lambda + p/2}}\,. \end{aligned} \tag{4.14}$$

By using formula 3.131.8 in [11], we can then represent I by an elliptic integral of the first kind, i.e., by

$$F(\varphi, k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{\sqrt{1 - k^2\sin^2\alpha}}, \quad 0 \le k < 1,\ \varphi \in \mathbf{R}.$$

This implies the following proposition.

Proposition 4.5. *Let $\gamma > 0$, $r > 0$, $p > -2\sqrt{r}$ and set $Q(t) = t^4 + pt^2 + r$. Then*

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\arctan\bigl(\gamma\sqrt{Q(t)}\bigr)}{\sqrt{Q(t)}}\,\mathrm{d}t \\ &= \begin{cases} \dfrac{\pi}{\sqrt[4]{r}}\, F\left(\arcsin\sqrt{\dfrac{2\sqrt{r}}{\sqrt{r} + \sqrt{r + \gamma^{-2}}}}, \sqrt{\dfrac{\sqrt{r} - p/2}{2\sqrt{r}}}\right) & : -2\sqrt{r} < p \le 2\sqrt{r}, \\[2ex] \dfrac{2\pi}{\sqrt{p + 2\sqrt{r}}}\, F\left(\arcsin\sqrt{\dfrac{p + 2\sqrt{r}}{p + 2\sqrt{r + \gamma^{-2}}}}, \sqrt{\dfrac{p - 2\sqrt{r}}{p + 2\sqrt{r}}}\right) & : p \ge 2\sqrt{r}. \end{cases} \end{aligned} \tag{4.15}$$

Let us observe that the limit case $\gamma \to \infty$ yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{t^4 + pt^2 + r}} = \begin{cases} \dfrac{2}{\sqrt[4]{r}}\,\mathbf{K}\left(\sqrt{\dfrac{\sqrt{r} - p/2}{2\sqrt{r}}}\right) & : -2\sqrt{r} < p \le 2\sqrt{r}, \\[2ex] \dfrac{4}{\sqrt{p + 2\sqrt{r}}}\,\mathbf{K}\left(\sqrt{\dfrac{p - 2\sqrt{r}}{p + 2\sqrt{r}}}\right) & : p \ge 2\sqrt{r}. \end{cases}$$

(As usual the function $\mathbf{K}$ denotes the complete elliptic integral, i.e., $\mathbf{K}(k) = F(\frac{\pi}{2}, k)$, $0 \le k < 1$.) The upper formula is in accordance with [12, Equ. 222.2c] upon using the substitution $x = t^2$.

More generally as in Proposition 4.5, we can replace the integrand $\mu^{-1/2}$ in formula (4.14) by a function $f(\mu)$ and use formula (4.13) in order to represent the double integral

$$\int_{\mu_1}^{\mu_2} f(\mu)\,\mathrm{d}\mu \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t^4 + pt^2 + r + \mu}, \quad 0 < \mu_1 < \mu_2,$$

by a simple one. If we substitute $\lambda = \sqrt{\mu + r}$ as before and set $z = p + 2\lambda$, then we obtain the following proposition.

Proposition 4.6. *Let $r > 0$, $p > -2\sqrt{r}$, $0 < \mu_1 < \mu_2$ and $f \in L^1([\mu_1, \mu_2])$. Then*

$$\int_{\mu_1}^{\mu_2} f(\mu)\,\mathrm{d}\mu \int_{-\infty}^{\infty} \frac{dt}{t^4 + pt^2 + r + \mu} = \pi \int_{p+2\sqrt{r+\mu_1}}^{p+2\sqrt{r+\mu_2}} f\Big(\frac{(z-p)^2}{4} - r\Big)\,\frac{\mathrm{d}z}{\sqrt{z}}. \tag{4.16}$$

Proposition 4.6 can be generalized to general positive quartics $Q(t) = t^4 + pt^2 + qt + r$. The corresponding result, i.e., Equ. (5) in [20, p. 1197], is a special case of [20, Prop. 2, p. 1196] and we just quote it in the next proposition.

Proposition 4.7. *Let $p, q, r \in \mathbf{R}$ such that the quartic $Q(t) = t^4 + pt^2 + qt + r$ is positive for each real t. Let $0 < \mu_1 < \mu_2 \le \infty$ and $f : (\mu_1, \mu_2) \to \mathbf{C}$ such that $\mu^{-3/4} f(\mu) \in L^1((\mu_1, \mu_2))$. Then*

$$\int_{\mu_1}^{\mu_2} f(\mu)\,\mathrm{d}\mu \int_{-\infty}^{\infty} \frac{dt}{Q(t) + \mu} = \pi \int_{z_1}^{z_2} f\big(\mu(z)\big)\,\frac{\mathrm{d}z}{\sqrt{z}}, \tag{4.17}$$

where

$$\mu(z) = \frac{(z-p)^2}{4} - r + \frac{q^2}{4z}$$

and $z_{1,2}$ denote the largest real roots of $\mu(z) = \mu_{1,2}$, respectively.

If we use the function $f(\mu) = \mu^{-1/2}$ in Proposition 4.7, we come back to elliptic arctan-integrals and we can generalize in this way Proposition 4.5.

Corollary 4.1. *Let $\gamma > 0$ and $p, q, r \in \mathbf{R}$ such that the quartic $Q(t) = t^4 + pt^2 + qt + r$ is positive for each real t. Then*

$$I = \int_{-\infty}^{\infty} \frac{\arctan\big(\gamma\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t = \pi \int_{z_1}^{\infty} \frac{\mathrm{d}z}{\sqrt{(z-p)^2 z - 4rz + q^2}}, \tag{4.18}$$

where z_1 is the largest real root of the cubic $(z-p)^2 z - 4(r + \gamma^{-2})z + q^2$.

Note that the right-hand side of equation (4.18) is an elliptic integral in Weierstraß' normal form. In particular, if the quartic Q has the form

$$Q(t) = \big[(t-t_1)^2 + u_1^2\big]\big[(t-t_2)^2 + u_2^2\big], \quad t_1, t_2 \in \mathbf{R},\ u_1 > 0, u_2 > 0, \tag{4.19}$$

then we can represent the integral I by the elliptic integral $F(\varphi, k)$ of the first kind.

Corollary 4.2. *Let $\gamma > 0$ and Q be as in equation (4.19). Then*

$$\begin{aligned} I = \int_{-\infty}^{\infty} \frac{\arctan\big(\gamma\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t &= \frac{2\pi}{\sqrt{(t_1-t_2)^2 + (u_1+u_2)^2}} \\ &\times F\Bigg(\arcsin\sqrt{\frac{(t_1-t_2)^2 + (u_1+u_2)^2}{(t_1-t_2)^2 + z_1}}, \sqrt{\frac{(t_1-t_2)^2 + (u_1-u_2)^2}{(t_1-t_2)^2 + (u_1+u_2)^2}}\Bigg), \end{aligned} \tag{4.20}$$

where z_1 is the largest real root of the equation

$$\big[z + (t_1-t_2)^2\big]\big[z - (u_1-u_2)^2\big]\big[z - (u_1+u_2)^2\big] = 4\gamma^{-2} z.$$

Proof. By translation the integral I depends only on the difference $t_1 - t_2$ and hence we can assume that $t_2 = -t_1$. Then $Q(t) = t^4 + pt^2 + qt + r$, where $p = -2t_1^2 + u_1^2 + u_2^2$, $q = 2t_1(u_1^2 - u_2^2)$, $r = (t_1^2 + u_1^2)(t_1^2 + u_2^2)$. This implies that the cubic

$$\begin{aligned}(z-p)^2 z - 4rz + q^2 &= z^3 + 2(2t_1^2 - u_1^2 - u_2^2)z^2 \\ &\quad + \left[(u_1^2 - u_2^2)^2 - 8t_1^2(u_1^2 + u_2^2)\right]z + 4t_1^2(u_1^2 - u_2^2)^2 \\ &= \left[z + 4t_1^2\right]\left[z - (u_1 - u_2)^2\right]\left[z - (u_1 + u_2)^2\right]\end{aligned}$$

has the three real roots $(u_1 + u_2)^2 > (u_1 - u_2)^2 > -(t_1 - t_2)^2$. Hence, similarly as in the proof of Proposition 4.1, formula 3.131.8 in [11] implies the result. □

We remark that Corollary 4.2 generalizes Proposition 4.5. In fact, if $Q(t) = (t^2 + u_1^2)(t^2 + u_2^2)$, i.e., if $t_1 = t_2 = 0$ in (4.19), then $p = u_1^2 + u_2^2, q = 0, r = u_1^2 u_2^2$ and formula (4.20) yields the lower formula on the right-hand side of (4.17). On the other hand, if $Q(t) = [(t-t_1)^2 + u_1^2][(t+t_1)^2 + u_1^2]$, i.e., if $t_2 = -t_1$ and $u_1 = u_2$ in (4.19), then $p = 2(u_1^2 - t_1^2), q = 0, r = (t_1^2 + u_1^2)^2$ and formula (4.20) yields the upper formula on the right-hand side of (4.17).

As before, the limit $\gamma \to \infty$ yields a *complete* elliptic integral since $z_1 \to (u_1 + u_2)^2$ for $\gamma \to \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{Q(t)}} = \frac{4}{\sqrt{(t_1 - t_2)^2 + (u_1 + u_2)^2}} \mathbf{K}\left(\sqrt{\frac{(t_1 - t_2)^2 + (u_1 - u_2)^2}{(t_1 - t_2)^2 + (u_1 + u_2)^2}}\right). \tag{4.21}$$

Note that the representation of $\int_{\mathbf{R}} \mathrm{d}t/\sqrt{Q(t)}$ in [12, Equ. 223.2e] is more complicated.

5. Representation of hypersurface areas by volume integrals

If the hypersurface M in $\mathbf{R}^n$ is given by $M = f^{-1}(1)$ for a homogeneous function f, then the area of M can be represented by a volume integral:

Proposition 5.8. *Let $f : \mathbf{R}^n \setminus \{0\} \longrightarrow (0, \infty)$ be C^1 and homogeneous of degree $\lambda > 0$ and set $M = f^{-1}(1)$. Then the hypersurface area $\Sigma(M)$ of M is given by*

$$\Sigma(M) = \frac{\lambda + n - 1}{\lambda} \int_{\{x \in \mathbf{R}^n;\, f(x) < 1\}} |\nabla f(x)|\, \mathrm{d}x. \tag{5.22}$$

Proof. Let $\mathrm{d}\sigma$ denote the surface measure on M and $\nu = \nabla f / |\nabla f|$ the outward unit normal. Due to Euler's equation, we have $x \cdot \nabla f(x) = \lambda f(x) = \lambda$ if $x \in M$ and $x \cdot \nabla |\nabla f|(x) = (\lambda - 1)|\nabla f|(x)$ for $x \in \mathbf{R}^n \setminus \{0\}$. Hence

$$\operatorname{div}(x|\nabla f|) = n|\nabla f| + x \cdot \nabla |\nabla f| = (n + \lambda - 1)|\nabla f|.$$

Therefore Gauß' divergence theorem yields

$$\begin{aligned}\Sigma(M) &= \int_M \mathrm{d}\sigma = \frac{1}{\lambda}\int_M x \cdot \nabla f\, \mathrm{d}\sigma = \frac{1}{\lambda}\int_M x|\nabla f| \cdot \nu\, \mathrm{d}\sigma \\ &= \frac{1}{\lambda}\int_{f(x)<1} \operatorname{div}(x|\nabla f|)\, \mathrm{d}x = \frac{\lambda + n - 1}{\lambda}\int_{f(x)<1} |\nabla f(x)|\, \mathrm{d}x.\end{aligned}$$

□

We shall apply formula (5.22) in order to show that the area of an ellipsoidal hypersurface in $\mathbf{R}^n$ can be represented by a hyperelliptic integral.

Proposition 5.9. *Let $n \geq 2$ and a_i, $i = 1, \dots, n$, be positive numbers and set*

$$M = \Big\{x \in \mathbf{R}^n; \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} = 1\Big\}.$$

Then its hypersurface area is given by

$$\Sigma(M) = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \Big(\prod_{j=1}^{n-1} a_j^2\Big) \int_0^\infty \Big(\sum_{j=1}^{n-1} \frac{1}{s + a_j^2}\Big) \frac{\sqrt{s + a_n^2}\, \mathrm{d}s}{\sqrt{s \prod_{j=1}^{n-1}(s + a_j^2)}}. \tag{5.23}$$

Proof. The function $f(x) = \sum_{i=1}^n x_i^2/a_i^2$ is homogeneous of degree $\lambda = 2$ and $|\nabla f| = 2(\sum_{i=1}^n x_i^2/a_i^4)^{1/2}$. Hence formula (5.22) in Proposition 5.8 implies, upon substituting $y_i = a_i x_i$, $i = 1, \dots n$,

$$\Sigma(M) = (n+1) \int_{f(x)<1} \sqrt{\sum_{i=1}^{n} \frac{x_i^2}{a_i^4}}\, \mathrm{d}x = (n+1)\Big(\prod_{i=1}^{n} a_i\Big) \int_{|y|<1} \sqrt{\sum_{i=1}^{n} \frac{y_i^2}{a_i^2}}\, \mathrm{d}y.$$

With the further substitution $y_n = t(\sum_{j=1}^{n-1} y_j^2/a_j^2)^{1/2}$, we then obtain

$$\Sigma(M) = 2(n+1)\Big(\prod_{i=1}^{n} a_i\Big) \int_0^\infty \sqrt{1 + \frac{t^2}{a_n^2}}\, \mathrm{d}t \int_{E_t} \sum_{j=1}^{n-1} \frac{y_j^2}{a_j^2}\, \mathrm{d}y', \tag{5.24}$$

where the inner integral runs over the ellipsoid

$$E_t = \Big\{y' \in \mathbf{R}^{n-1}; \sum_{j=1}^{n-1} \frac{y_j^2}{A_j^2} \leq 1\Big\}, \quad A_j = \frac{a_j}{\sqrt{t^2 + a_j^2}},\ j = 1, \dots, n-1,$$

and represents a sum of moments of second order thereof.

The calculation of such moments is quite straight-forward. We present it here just for completeness. Evidently, it suffices to consider the summand y_{n-1}^2/a_{n-1}^2 in the inner integral on the right-hand side of formula (5.24). Substituting $y_j = A_j u_j$, $j = 1, \dots, n-1$, and setting $u'' = (u_1, \dots, u_{n-2})$ we obtain

$$\int_{E_t} \frac{y_{n-1}^2}{a_{n-1}^2}\, \mathrm{d}y' = \frac{A_{n-1}^2}{a_{n-1}^2} \Big(\prod_{j=1}^{n-1} A_j\Big) \int_{|u'|<1} u_{n-1}^2\, \mathrm{d}u'$$

and

$$\begin{aligned}
\int_{|u'|<1} u_{n-1}^2\, \mathrm{d}u' &= 2\int_0^1 u_{n-1}^2\, \mathrm{d}u_{n-1} \int_{|u''|^2<1-u_{n-1}^2} \mathrm{d}u'' \\
&= \frac{2\pi^{n/2-1}}{\Gamma(\frac{n}{2})} \int_0^1 u_{n-1}^2 (1 - u_{n-1}^2)^{n/2-1}\, \mathrm{d}u_{n-1} \\
&= \frac{2\pi^{n/2-1}}{\Gamma(\frac{n}{2})} \cdot \frac{1}{2} B\Big(\frac{3}{2}, \frac{n}{2}\Big) = \frac{\pi^{(n-1)/2}}{2\,\Gamma(\frac{n+3}{2})}.
\end{aligned}$$

Altogether this yields

$$\Sigma(M) = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \Big(\prod_{j=1}^{n-1} a_j^2\Big) \int_0^\infty \Big(\sum_{j=1}^{n-1} \frac{1}{t^2 + a_j^2}\Big) \frac{\sqrt{t^2 + a_n^2}\, \mathrm{d}t}{\prod_{j=1}^{n-1} \sqrt{t^2 + a_j^2}}.$$

The final substitution $s = t^2$ then leads to formula (5.23) and thus concludes the proof. $\square$

We remark that the integral in formula (5.23) is an elliptic integral for $n = 2$ and for $n = 3$, but is hyperelliptic and not elliptic in dimensions $n \geq 4$ if the diameters $2a_i$, $i = 1, \ldots, n$, are generic positive real numbers. The representation of the length of an ellipse $(n = 2)$ and of the surface area of an ellipsoid $(n = 3)$, respectively, by elliptic integrals is known since the times of Legendre, see [5, Problem 1, p. 265, Problem 15, p. 279].

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