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On Some New Normed Narayana Sequence Spaces

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Research Article

Abstract — In this paper, we first establish the regular matrix N using Narayana numbers. Then, we create new normed sequence spaces $Z(N)$ using the matrix N and demonstrate that these spaces are linearly isomorphic to Z where $Z \in \{c_0, c, \ell_p, \ell_\infty\}$. Additionally, we provide inclusion relations for the spaces $c_0(N)$, $c(N)$, $\ell_p(N)$, and $\ell_\infty(N)$. Furthermore, we construct the Schauder bases of the $c_0(N)$, $c(N)$, and $\ell_p(N)$. Finally, we compute the α-, β-, and γ-duals of these spaces and characterize the classes $(Z(N), X)$ for the certain choice of the sequence space X.

Keywords *Narayana numbers, Narayana matrices, sequence spaces, matrix transformations*

Mathematics Subject Classification (2020) 40C05, 46A45

1. Introduction

Narayana studied the problem of a herd of cows and calves in the fourteenth century [1, 2]. This is analogous to the Fibonacci rabbit problem, which can be expressed as follows: Every year, a cow gives birth to one calf, and starting in the fourth year, each calf has its annual production. The question is, after 20 years, how many calves are there in total? This problem can be solved in the same way as the Fibonacci rabbit problem [3]. The Narayana issue can be modeled if s is the year by recurrence

$$n_{s+3} = n_{s+2} + n_s$$

with $s \geq 0$, $n_0 = 0$, $n_1 = 1$, and $n_2 = 1$ [1]. Thus,

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \cdots$$

are the first few terms in this sequence [4]. This sequence is known as the Narayana sequence, commonly called the Fibonacci-Narayana sequence or the Narayana's cows sequence. There has been a lot of interest in the Narayana sequence and its generalizations recently. For more details, see [1,5–11].

We can provide some fundamental details about sequence spaces and summability theory. Each Γ subset of ω is referred to as a sequence space, and ω denotes the space of all real or complex sequences. We use the symbols ℓ_∞, c, and c_0 to symbolize the spaces of all bounded, convergent, and null sequences. Moreover, we designate the spaces of all convergent, bounded, absolutely, and p-absolutely convergent series by cs, bs, ℓ_1, and ℓ_p, $1 < p < \infty$, respectively.

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A K-space is a sequence space with a linear topology where each of the mappings $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$, for all $i \in \mathbb{N}$, is continuous. An FK-space is a K-space and a complete linear metric space. An FK-space with a normable topology is a BK-space.

If there are real entries in an infinite matrix $A = (a_{rs})$, then A_r represents the r_{th} row, for all $r \in \mathbb{N}$. If the series is convergent, for all $r \in \mathbb{N}$, then the A-transform of $u = (u_s) \in \omega$ is provided as follows:

$$(Au)_r = \sum_{s=0}^{r} a_{rs} u_s$$

If $Au \in \Psi$, then it is stated that A is a matrix transformation from Υ to Ψ, for all $u \in \Upsilon$. The notation (Υ, Ψ) denotes the class of all the matrices that transform from Υ to Ψ. The matrix domain of A in Υ is the set of all the vectors $u = (u_s)$ in ω such that $Au \in \Upsilon$. The approach of constructing a sequence space using the domain of an infinite matrix was first made by Ng and Lee [12]. In the following years, many studies were published in the literature. The papers [13–17] and the books [18, 19] can be investigated to get detailed information about them.

If Υ and Ψ are two sequence spaces, then the multiplier set $\mathfrak{D}(\Upsilon : \Psi)$ is described as

$$\mathfrak{D}(\Upsilon : \Psi) = \{x = (x_s) \in \omega : xu = (x_s u_s) \in \Psi, \text{ for all } (u_s) \in \Upsilon\}$$

Thus, α-, β-, and γ-duals of Υ are described as

$$\Upsilon^\alpha = \mathfrak{D}(\Upsilon : \ell_1), \quad \Upsilon^\beta = \mathfrak{D}(\Upsilon : cs), \quad \text{and} \quad \Upsilon^\gamma = \mathfrak{D}(\Upsilon : bs)$$

Recently, special integer sequences have been intensively studied in the Fibonacci, Lucas, Padovan, Catalan, Bell, and Schröder sequence spaces theory. For example, in 2022, Dağlı defined a new regular matrix using Schröder numbers and constructed new sequence spaces with the help of this matrix. For related works, the reader can refer to [20–28]. In line with the works mentioned above, our article aims to use Narayana numbers in the theory of sequence spaces.

In this paper, we define Narayana sequence spaces and investigate some of their basic topological properties. Moreover, we derive Schauder bases and compute the alpha, beta, and gamma duals of Narayana sequence spaces. In addition, we characterize some matrix transformations.

2. Narayana Sequence Spaces

This section presents the Narayana sequence spaces' definitions and characteristics. The Narayana matrix $N = (n_{rs})$ is defined by the following equation:

$$n_{rs} = \begin{cases} \dfrac{n_s}{n_{r+3} - 1}, & 1 \leqslant s \leqslant r \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}$. Equivalently,

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The inverse of $N = (n_{rs})$ is given by:

$$n_{rs}^{-1} = \begin{cases} (-1)^{r-s}\dfrac{n_{s+3}-1}{n_r}, & r \leqslant s \leqslant r+1 \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}$. Lately, Soykan [11] has researched the basic properties of Narayana numbers and discovered several interesting identities, such as

$$\sum_{s=0}^{r} n_s = n_{r+3} - 1$$

$$\sum_{s=0}^{r} n_{2s} = \frac{1}{3}\left(n_{2r+2} + n_{2r+1} + 2n_{2r} - 2\right)$$

and

$$\sum_{s=0}^{r} n_{2s+1} = \frac{1}{3}\left(2n_{2r+2} + 2n_{2r+1} + n_{2r} - 1\right)$$

where $r \in \mathbb{N}_0$. In this study, we define Narayana sequence spaces using Narayana numbers. The definitions of the Narayana sequence spaces $c_0(N)$, $c(N)$, $\ell_p(N)$, and $\ell_\infty(N)$ are as follows, respectively:

$$c_0(N) = \left\{x = (x_s) \in \omega \ : \ \lim_{r\to\infty} \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s = 0\right\}$$

$$c(N) = \left\{x = (x_s) \in \omega \ : \ \lim_{r\to\infty} \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \text{ exists}\right\}$$

$$\ell_p(N) = \left\{x = (x_s) \in \omega \ : \ \sum_{r} \left| \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \right|^p < \infty\right\}$$

and

$$\ell_\infty(N) = \left\{x = (x_s) \in \omega \ : \ \sup_{r\in\mathbb{N}} \left| \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \right| < \infty\right\}$$

The previously mentioned sequence spaces can be redefined by

$$c_0(N) = (c_0)_N, \quad c(N) = (c)_N, \quad \ell_p(N) = (\ell_p)_N, \quad \text{and} \quad \ell_\infty(N) = (\ell_\infty)_N \tag{2.1}$$

using the notation of the matrix domain. The N-transform of a sequence $x = (x_s)$ is defined as $y = (y_r)$ by

$$y_r = (Nx)_r = \sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s \tag{2.2}$$

for each $r \in \mathbb{N}$. Throughout the remainder of the article, the sequences x and y are related to (2.2). Therefore, for all $s \in \mathbb{N}$,

$$x_s = (Ny)_s = \sum_{i=s-1}^{s} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} y_i \tag{2.3}$$

Theorem 2.1. The space $\ell_p(N)$ is a BK-space with the norm

$$\|(Nx)_r\|_p = \|x\|_{\ell_p(N)} = \left(\sum_{r} |(Nx)_r|^p\right)^{1/p}, \quad 1 \leqslant p < \infty$$

and the spaces $\ell_\infty(N)$, $c_0(N)$, and $c(N)$ are BK-spaces with the norm

$$\|(Nx)_r\|_{\ell_\infty} = \|x\|_{\ell_\infty(N)} = \|x\|_{c_0(N)} = \|x\|_{c(N)} = \sup_{r\in\mathbb{N}} |(Nx)_r|$$

PROOF. The matrices N are triangular since (2.1) is true. Then, from Wilansky's Theorem 4.3.12 of [29], the spaces $\ell_p(N)$ are BK-spaces with the given norms where $1 \leqslant p \leqslant \infty$. Moreover, the spaces $c_0(N)$ and $c(N)$ are BK-spaces with the given norms from Wilansky's Theorem 4.3.2 of [29]. □

We may state the theorems relating to the inclusion relations concerning the spaces $c_0(N)$, $c(N)$, $\ell_p(N)$, and $\ell_\infty(N)$.

Theorem 2.2. Let Z represent any one of the four standard sequence spaces (c_0, c, ℓ_p, or ℓ_∞). The inclusion $Z \subset Z(N)$ strictly holds.

PROOF. It is obvious that the inclusion $Z \subset Z(N)$ holds. Assume that $Z = c$. Besides, consider the sequence $g = (g_s)$ defined by $g = (1, 0, 1, 0, 1, \cdots)$. Then,

$$(Ng)_r = \sum_{s=1}^{r} \frac{n_s}{n_{r+3} - 1} g_s = \frac{n_1 + n_3 + \cdots + n_r}{n_{r+3} - 1} \tag{2.4}$$

is converges. Thus, we immediately observe that g is in $c(N)$ but not in c. Because at least one sequence in $c(N) \setminus c$, the inclusion $c(N) \subset c$ is strict. The proofs of the other inclusions are similar. □

Theorem 2.3. The inclusions $\ell_p(N) \subset c_0(N) \subset c(N) \subset \ell_\infty(N)$ strictly hold.

PROOF. Since the matrix N is regular and the inclusion $\ell_p \subset c_0 \subset c \subset \ell_\infty$ holds, the inclusion part holds. Consider the sequence $h = (h_s)$ defined by $h = (1, 1, 1, 1, \cdots)$. Then,

$$(Nh)_r = \sum_{s=1}^{r} \frac{n_s}{n_{r+3} - 1} h_s = 1$$

for all $r \in \mathbb{N}$. Then, $Nh \in c \setminus c_0$. That is, $h \in c(N) \setminus c_0(N)$. This verifies the strictness of the inclusion $c_0(N) \subset c(N)$. Similarly, the strictness of other inclusions can be established. □

Theorem 2.4. If $1 \leq p < q$, then $\ell_p(N) \subset \ell_q(N)$.

PROOF. Suppose $1 \leq p < q$. The matrix N is known to be regular, and the inclusion $\ell_p \subset \ell_q$ holds. These imply that the inclusion part holds. Then, consider a sequence $k = (k_s) \in \ell_q \setminus \ell_p$. Define a sequence $l = (l_s)$ by

$$l_s = \left(\frac{k_s(n_{s+3} - 1) - k_{s-1}(n_{s+2} - 1)}{n_s} \right)$$

such that $s \in \mathbb{N}$. Then,

$$\begin{aligned}
(Nl)_r &= \frac{1}{n_{r+3} - 1} \sum_{s=1}^{r} n_s l_s \\
&= \frac{1}{n_{r+3} - 1} \sum_{s=1}^{r} \left[k_s(n_{s+3} - 1) - k_{s-1}(n_{s+2} - 1) \right] \\
&= \frac{1}{n_{r+3} - 1} k_r (n_{r+3} - 1) \\
&= k_r
\end{aligned} \tag{2.5}$$

for each $r \in \mathbb{N}$ where the negative subscripted terms are considered to be zero and $k_0 = 0$. Therefore, we conclude that $Nl = k \in \ell_q \setminus \ell_p$, which means that $l \in \ell_q(N) \setminus \ell_p(N)$. Consequently, there is at least one sequence in $\ell_q(N)$ that is not in $\ell_p(N)$. □

Theorem 2.5. Let $Z \in \{c_0, c, \ell_p, \ell_\infty\}$. Then, $Z(N) \cong Z$.

PROOF. Define the mapping $\tau : \ell_p(N) \to \ell_p$ by $\tau x = y = Nx$, for all x in $\ell_p(N)$. It is obvious that τ is linear and one-to-one. Assume that $x = (x_s)$ is defined as (2.3) such that $y = (y_r)$ is any sequence in ℓ_p. Moreover, let $1 \leqslant p < \infty$. Since $y \in \ell_p$,

$$
\begin{aligned}
\|x\|_{\ell_p(N)} &= \left(\sum_{r=1}^{\infty} |(Nx)_r|^p\right)^{1/p} \\
&= \left(\sum_{r=1}^{\infty} \left|\sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} x_s\right|^p\right)^{1/p} \\
&= \left(\sum_{r=1}^{\infty} \left|\sum_{s=1}^{r} \frac{n_s}{n_{r+3}-1} \sum_{i=s-1}^{s} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} y_i\right|^p\right)^{1/p} \\
&= \left(\sum_{r} |y_r|^p\right)^{1/p} \\
&= \|y\|_p < \infty
\end{aligned}
$$

and

$$\|x\|_{\ell_\infty(N)} = \sup_{r\in\mathbb{N}} |(Nx)_r| = \|y\|_\infty < \infty$$

Consequently, x is a sequence in $\ell_p(N)$, and the mapping τ is onto and norm preserving. The proofs for the other spaces are similar. □

The result that ℓ_p space is not a Hilbert space for $p \neq 2$ is also valid for $\ell_p(N)$ space for $p \neq 2$ (see Theorem 2.6).

Theorem 2.6. The sequence space $\ell_p(N)$ is not a Hilbert space for $p \neq 2$.

Proof. We first prove that the space $\ell_2(N)$ is the only Hilbert space among the $\ell_p(N)$ spaces for $1 < p < \infty$. Consider the sequences $u = (u_s)$ and $v = (v_s)$ provided by

$$(u_s) = (1, 1, 0, 0, \cdots) \quad \text{and} \quad (v_s) = (1, -3, 0, 0, \cdots)$$

Thereby, $Nu = (1, 1, 0, 0, 0, \cdots)$ and $Nv = (1, -1, 0, 0, 0, \cdots)$. Since N is linear, so $N(u+v) = (2, 0, 0, 0, \cdots)$ and $N(u-v) = (0, 2, 0, 0, \cdots)$. Hence,

$$\|u+v\|^2_{\ell_p(N)} + \|u-v\|^2_{\ell_p(N)} = 8 \neq 2^{2(1+(1/p))} = 2\left(\|u\|^2_{\ell_p(N)} + \|v\|^2_{\ell_p(N)}\right) \tag{2.6}$$

Thus, it is evident that the norm $\|\cdot\|_{\ell_p(N)}$ violates the parallelogram identity when $p \neq 2$. □

It is understood that a matrix domain Z_A where A is triangular has a basis only if Z has a basis [30]. Thus, from Theorem 2.5, the following outcome can be deduced:

Theorem 2.7. Consider the sequence $b^{(s)} = \left(b^{(s)}\right)_{s\in\mathbb{N}}$ of the elements of the space $\ell_p(N)$ by

$$b_r^{(s)} = \begin{cases} (-1)^{r-s} \dfrac{n_{s+3}-1}{n_r}, & r \leqslant s \leqslant r+1 \\ 0, & s > r \end{cases}$$

for every fixed $s \in \mathbb{N}$ and $1 \leqslant p < \infty$. The following claims are accurate:

i. The sequence $\left(b^{(s)}\right)_{s\in\mathbb{N}_0}$ is a basis for the spaces $c_0(N)$ and $\ell_p(N)$, and any $x \in c_0(N)$ and $x \in \ell_p(N)$ has a unique representation of the form

$$x = \sum_s y_s b^{(s)}$$

ii. The sequence $\left(e, b^{(s)}\right)_{s\in\mathbb{N}}$ is a basis for the space $c(N)$, and any $x \in c(N)$ has a unique representation of the form

$$x = le + \sum_s [y_s - l] b^{(s)}$$

where $y_s = (N(x))_s \to l$ as $s \to \infty$.

iii. The space $\ell_\infty(N)$ does not have a basis.

3. Dual Spaces

In this section, the alpha, beta, and gamma duals of our novel sequence spaces are determined. In the following, the set of all the bounded subsets of $\mathbb{N}$ is denoted by N and assume that p^* denotes the conjugate of p, i.e., $p^{-1} + p^{*-1} = 1$. In this section, the lemmas used to prove the theorems are presented.

Lemma 3.1. [31] The following claims are accurate:

i. $A = (a_{rs}) \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ if and only if

$$\sup_{K\in N} \sum_{r=0}^{\infty} \left| \sum_{s\in K} a_{rs} \right| < \infty \tag{3.1}$$

ii. $A = (a_{rs}) \in (c_0 : c) = (c : c)$ if and only if

$$\exists \alpha_s \in \mathbb{C} \ni \lim_{r\to\infty} a_{rs} = \alpha_s, \quad \text{for all } s \in \mathbb{N}_0 \tag{3.2}$$

and

$$\sup_{r\in\mathbb{N}} \sum_{s=1}^{\infty} |a_{rs}| < \infty \tag{3.3}$$

iii. $A = (a_{rs}) \in (\ell_\infty : c)$ if and only if (3.2) holds and

$$\lim_{r\to\infty} \sum_{s=1}^{\infty} |a_{rs}| = \sum_{s=1}^{\infty} |\lim_{r\to\infty} a_{rs}| \tag{3.4}$$

iv. $A = (a_{rs}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ if and only if (3.3) holds.

Lemma 3.2. [31] The followings are valid:

i. Let $1 < p < \infty$. Then, $A = (a_{rs}) \in (\ell_p : \ell_\infty)$ if and only if

$$\sup_{r\in\mathbb{N}} \sum_{s=1}^{\infty} |a_{rs}|^{p^*} < \infty \tag{3.5}$$

ii. Let $1 < p < \infty$. Then, $A = (a_{rs}) \in (\ell_p : c)$ if and only if (3.2) and (3.5) hold.

iii. $A = (a_{rs}) \in (\ell_p : \ell_1)$ if and only if

$$\sup_{\mathsf{N}\in N} \sup_{s\in\mathbb{N}} \left| \sum_{r\in\mathsf{N}} a_{rs} \right|^{p} < \infty, \quad 0 < p \leq 1 \tag{3.6}$$

and

$$\sup_{\mathsf{N}\in N} \sum_{s=1}^{\infty} \left| \sum_{r\in\mathsf{N}} a_{rs} \right|^{p^*} < \infty, \quad 1 < p < \infty \tag{3.7}$$

iv. Let $0 < p \leq 1$. Then, $A = (a_{rs}) \in (\ell_p : \ell_\infty)$ if and only if

$$\sup_{r,s\in\mathbb{N}} \left| a_{rs} \right|^{p} < \infty \tag{3.8}$$

v. Let $0 < p \leq 1$. Then, $A = (a_{rs}) \in (\ell_p : c)$ if and only if (3.2) and (3.8) hold.

Theorem 3.3. Let $t = (t_r) \in \omega$. Define the matrix $T = (t_{rs})$ by

$$t_{rs} = \begin{cases} (-1)^{r-s} \dfrac{n_{s+3} - 1}{n_r} t_r, & 0 \leqslant s \leqslant r \\ 0, & s > r \end{cases}$$

for all $s, r \in \mathbb{N}$. Then, $\{c_0(N)\}^\alpha = \{c(N)\}^\alpha = \{\ell_\infty(N)\}^\alpha = \mathfrak{c}_1$ where $\mathfrak{c}_1$ defined by

$$\mathfrak{c}_1 \quad = \quad \left\{ t = (t_s) \in w \ : \ \sup_{K \in N} \sum_{r=0}^{\infty} \left| \sum_{s \in K} t_{rs} \right| < \infty \right\}$$

PROOF. We give the proof only for the sequence space $c_0(N)$. Thus,

$$t_r x_r = \sum_{s=r-1}^{r} (-1)^{r-s} \frac{n_{s+3} - 1}{n_r} t_r y_s = (Tx)_r, r \in \mathbb{N} \tag{3.9}$$

It follows from (3.9), $tx = (t_r x_r) \in \ell_1$, for $x \in c_0(N)$, if and only if $Ty \in \{c_0(N)\}$, for $y \in c_0$. Hence, by Lemma 3.1 from (3.1), it is concluded that $\{c_0(N)\}^\alpha = \mathfrak{c}_1$. □

Theorem 3.4. Let the sets $\mathfrak{c}_2$ and $\mathfrak{c}_3$ be as follows:

$$\mathfrak{c}_2 = \left\{ t = (t_s) \in \omega \ : \ \sup_{\mathsf{N} \in N} \sup_{s \in \mathbb{N}} \left| \sum_{r \in \mathsf{N}} t_{rs} \right|^p < \infty \right\}$$

and

$$\mathfrak{c}_3 = \left\{ t = (t_s) \in \omega \ : \ \sup_{\mathsf{N} \in N} \sum_{s=1}^{\infty} \left| \sum_{r \in \mathsf{N}} t_{rs} \right|^{p^*} < \infty \right\}$$

respectively. Then,

$$\{\ell_p(N)\}^\alpha = \begin{cases} \mathfrak{c}_2, & 0 < p \leq 1 \\ \mathfrak{c}_3, & 1 < p < \infty \end{cases}$$

PROOF. This is accomplished by using the same procedure as in the proof of Theorem 3.3 but substituting (3.6) and (3.7) of Lemma 3.2 (*iii*) for (3.1) of Lemma 3.1 (*i*) with t_{rs} rather than a_{rs}. □

Theorem 3.5. Consider the definition of $D = (d_{rj})$ using the sequence $a = (a_j)$ by

$$d_{rs} = \begin{cases} \displaystyle\sum_{i=s}^{r} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} a_i, & 0 \leqslant s \leqslant r \\ 0, & s > r \end{cases} \tag{3.10}$$

and define the following sets

$$\mathfrak{d}_1 = \left\{ a = (a_s) \in \omega \ : \ \sup_{r \in \mathbb{N}} \sum_{s=1}^{\infty} |d_{rs}| < \infty \right\}$$

$$\mathfrak{d}_2 = \left\{ a = (a_s) \in \omega \ : \ \lim_{r \to \infty} d_{rs} = \alpha_s \right\}$$

$$\mathfrak{d}_3 = \left\{ a = (a_s) \in \omega \ : \ \lim_{r \to \infty} \sum_{s=1}^{\infty} |d_{rs}| = \sum_{s=1}^{\infty} \left| \lim_{r \to \infty} d_{rs} \right| \right\}$$

$$\mathfrak{d}_4 = \left\{ a = (a_s) \in \omega \ : \ \lim_{r \to \infty} \sup_{s \in \mathbb{N}} \sum_{s=1}^{\infty} |d_{rs}| < \infty \right\}$$

and

$$\mathfrak{d}_5 = \left\{ a = (a_s) \in \omega \ : \ \sup_{r, s \in \mathbb{N}} |d_{rs}|^p < \infty \right\}$$

Then,

i. $\{c_0(N)\}^\beta = \mathfrak{b}_1 \cap \mathfrak{b}_2$ and $\{c_0(N)\}^\gamma = \mathfrak{b}_1$

ii. $\{c(N)\}^\beta = \mathfrak{b}_1 \cap \mathfrak{b}_2$ and $\{c(N)\}^\gamma = \mathfrak{b}_1$

iii. $\{\ell_\infty(N)\}^\beta = \mathfrak{b}_2 \cap \mathfrak{b}_3$ and $\{\ell_\infty(N)\}^\gamma = \mathfrak{b}_1$

iv. $\{\ell_p(N)\}^\beta = \begin{cases} \mathfrak{b}_2 \cap \mathfrak{b}_4, & 0 \leqslant p < 1 \\ \mathfrak{b}_2 \cap \mathfrak{b}_5, & 1 \leqslant p < \infty \end{cases}$ and $\{\ell_p(N)\}^\gamma = \begin{cases} \mathfrak{b}_4, & 0 \leqslant p < 1 \\ \mathfrak{b}_5, & 1 \leqslant p < \infty \end{cases}$

Proof. We give the proof only for the $\beta-$dual of the sequence space $\ell_p(N)$. Consider the equation

$$\begin{aligned}\sum_{s=1}^{r} d_s x_s &= \sum_{s=1}^{r} \left[\sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} y_i\right] d_s \\ &= \sum_{s=1}^{r} \left[\sum_{i=s}^{r} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} d_i\right] y_s \\ &= (Dy)_r\end{aligned}$$

for any $r \in \mathbb{N}_0$. This equation states that if x is an element of $\ell_p(N)$, then dx is an element of cs if and only if Dy is an element of c, for x in ℓ_p. This means that D is an element of $(\ell_p : c)$. As a consequence, by Lemma 3.2 from (3.2) and (3.5), it is deduced that

$$\{\ell_p(N)\}^\beta = \begin{cases} \mathfrak{b}_2 \cap \mathfrak{b}_4, & 0 \leqslant p < 1 \\ \mathfrak{b}_2 \cap \mathfrak{b}_5, & 1 \leqslant p < \infty \end{cases}$$

□

4. Matrix Transformations

In this section, let $\lambda \in \{c_0(N), c(N), \ell_p(N), \ell_\infty(N)\}$ and $\mu \in \{c_0, c, \ell_\infty, \ell_1\}$. We provide necessary and sufficient conditions for matrix mappings from the spaces λ to any one of the spaces μ and from the spaces μ to the spaces λ.

Theorem 4.1. Define, for all $s, r \in \mathbb{N}_0$, $\mathcal{Z}^{(r)} = (z_{ms}^{(r)})$ and $\mathcal{Z} = (z_{rs})$ by

$$z_{ms}^{(r)} = \begin{cases} \sum_{i=s}^{m} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} a_{ri}, & 0 \leqslant s \leqslant m \\ 0, & s > m \end{cases}$$

and

$$z_{rs} = \sum_{i=s}^{m} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} a_{ri}$$

Then, $\mathcal{A} = (a_{rs}) \in (\ell_p(N) : \mu)$ if and only if $\mathcal{Z}^{(r)} \in (\ell_p : c)$, for all $r \in \mathbb{N}_0$, and $\mathcal{Z} \in (\ell_p : \mu)$.

Proof. Let $\mathcal{A} \in (\ell_p(N) : \mu)$ and $y = (y_s) \in \ell_p(N)$. Thus,

$$\sum_{s=1}^{\infty} a_{rs} x_s = \sum_{s=1}^{\infty} \left[\sum_{i=s}^{r} (-1)^{s-i} \frac{n_{i+3}-1}{n_s} a_{ri}\right] y_s \tag{4.1}$$

for all $m, r \in \mathbb{N}$. $\mathcal{A}y$ exists, therefore $Z^{(r)} \in (\ell_p : c)$. Moreover, $\mathcal{A}y = \mathcal{Z}x$ by using $m \to \infty$ as in (4.1). Given that $\mathcal{A}y \in \mu$, $\mathcal{Z}x \in \mu$ follows, with the result that $\mathcal{Z} \in (\ell_p : \mu)$.

Conversely, suppose that $\mathcal{Z}^{(r)} \in (\ell_p : c)$, for all $r \in \mathbb{N}$, and that $\mathcal{Z} \in (\ell_p : \mu)$. Let $y = (y_s) \in \ell_p(N)$. Consequently, for all $r \in \mathbb{N}$, $\{a_{rs}\}_{s=1}^\infty \in \ell_p^\beta$, which means that $\{a_{rs}\}_{s=1}^\infty \in (\ell_p(N)^\beta$, for all $r \in \mathbb{N}$. From (4.1), $\mathcal{A}y = \mathcal{Z}x$ by as $m \to \infty$. Hence, $\mathcal{A} \in (\ell_p(N) : \mu)$. □

Theorem 4.2. Let $A = (a_{rs})$ be an infinite matrix and define the matrix $B = (b_{rs})$ by

$$b_{rs} = \sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} a_{ri} \tag{4.2}$$

for all $s, r \in \mathbb{N}$, and μ be a sequence space. Then, $A \in (\mu : \ell_p(N))$ if and only if $B \in (\mu : \ell_p)$.

PROOF. Let $z = (z_s) \in \mu$. Then,

$$\begin{aligned} \sum_{s=1}^{r} b_{rs} z_s &= \sum_{s=1}^{r} \left(\sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} a_{ri} \right) z_s \\ &= \sum_{i=0}^{s} (-1)^{s-i} \frac{n_{i+3} - 1}{n_s} \left(\sum_{s=1}^{r} a_{is} z_s \right) \end{aligned}$$

for all $m, r \in \mathbb{N}$. Since $r \to \infty$, $(Bz)_m = (\Delta(Az))_m$, for all $m \in \mathbb{N}$. Thus, $z \in \mu$. Hence, $Az \in \ell_p(N)$ if and only if $Bz \in \ell_p$. $\square$

Combining Theorem 4.1 and the matrix mapping characterization findings presented in [31], we arrive at the following conclusions:

Corollary 4.3. The following claims are accurate:

i. $\mathcal{A} \in (\ell_p(N) : c_0)$ if and only if

$$\sup_{m \in \mathbb{N}_0} \sum_{s=1}^{\infty} \left| z_{ms}^{(r)} \right|^{p^*} < \infty \tag{4.3}$$

$$\lim_{m \to \infty} z_{ms}^{(r)} \text{ exists, for all } s \in \mathbb{N}_0 \tag{4.4}$$

and

$$\lim_{r \to \infty} z_{rs} = 0, \text{ for all } s \in \mathbb{N}_0$$

ii. $\mathcal{A} \in (\ell_p(N) : c)$ if and only if (4.3) and (4.4) hold,

$$\sup_{r \in \mathbb{N}_0} \sum_{s=1}^{\infty} |z_{rs}|^{p^*} < \infty \tag{4.5}$$

and

$$\lim_{r \to \infty} z_{rs} \text{ exists, for all } s \in \mathbb{N}_0$$

iii. $\mathcal{A} \in (\ell_p(N) : \ell_\infty)$ if and only if (4.3)-(4.5) hold.

iv. $\mathcal{A} \in (\ell_p(N) : \ell_1)$ if and only if (4.3) and (4.4) hold and

$$\sup_{N} \sum_{s=1}^{\infty} \left| \sum_{r \in N} z_{rs} \right|^{p^*} < \infty$$

Then, combining Theorem 4.2 and the matrix mapping characterization findings presented in [31], we arrive at the following conclusions:

Corollary 4.4. The following claims are accurate:

i. $\mathcal{A} \in (c_0 : \ell_p(N)) = (c : \ell_p(N)) = (\ell_\infty : \ell_p(N))$ if and only if $\sup_K \sum_{r=0}^{\infty} \left| \sum_{s \in K} b_{rs} \right|^p < \infty$.

ii. $\mathcal{A} \in (\ell_1 : \ell_p(N))$ if and only if $\sup_s \sum_{r=0}^{\infty} |b_{rs}|^p < \infty$.

5. Conclusion

Fibonacci, Lucas, Padovan, Catalan, Bell, and Schröder sequence spaces have been studied by many authors in the past and today. Hence, an introduction to sequence spaces was made within this study with the help of Narayana numbers. For this reason, this study is expected to serve as a guide for studies on the additivity theory. By benefiting from this study, researchers can consider similar results regarding Narayana difference sequence spaces, paranormed Narayana sequence spaces, the compactness measure of $\ell_p(N)$ space, and the convergence field of the Narayana matrix.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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