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On Ricci Recurrent Almost Kenmotsu 3-manifolds

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ABSTRACT

In this paper, we prove first that for an almost Kenmotsu 3-manifold satisfying $\xi(tr\, h^2) = 0$, its Ricci operator is recurrent if and only if the manifold is locally symmetric. Next, we show that φ-Ricci symmetry and φ-Ricci recurrence are equivalent conditions in almost Kenmotsu 3-manifolds. Thus, an almost Kenmotsu 3-manifold is φ-Ricci symmetric if and only if it has dominantly η-parallel Ricci operator.

Keywords: *Almost Kenmotsu manifold, local symmetry, Ricci recurrent.*
AMS Subject Classification (2020): *Primary: 53D15 ; Secondary: 53C25; 53C30.*

1. Introduction

A Riemannian 3-manifold M is locally symmetric ($\nabla R = 0$) if and only if its Ricci operator is parallel ($\nabla S = 0$). Kenmotsu ([10]) proved that a locally symmetric Kenmotsu manifold is of constant curvature -1 (see also [5] for Kenmotsu 3-manifolds).

In [6], Inoguchi proved that an almost Kenmotsu 3-manifold is locally symmetric if and only if it is isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some γ. This classification result was announced in [2]. The example $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$ was emphasized in [[13], Theorem 1.2 Case (IV)].

As a generalization of local symmetry, the Ricci operator S is said to be *recurrent* if the Ricci operator S satisfies

$$(\nabla_X S)Y = A(X)SY,$$

where X, Y are any vector fields and A is a 1-form on M. The Ricci operator S is said to be *η-recurrent* if the Ricci operator S satisfies

$$g((\nabla_X S)Y, Z) = A(X)g(SY, Z),$$

where X, Y, Z are all vector fields orthogonal to ξ and A is a 1-form on M.

In [17], they studied Ricci recurrence in H-almost Kenmotsu 3-manifolds with $\nabla_\xi h = 0$.

On the other hands, in [8], for an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α, Inoguchi and the author studied η-parallel Ricci operator.

Also, Shukla and Shukla [16] proved that a $(2n+1)$-dimensional Kenmotsu manifold is φ-Ricci symmetric if and only if it is an Einstein manifold. In particular, a 3-dimensional Kenmotsu manifold which is φ-Ricci symmetric is an Einstein manifold and a Kenmotsu manifold of constant curvature -1.

In this paper, we study the conditions under which an almost Kenmotsu 3-manifold is Ricci recurrent. In Section 4, we find that a homogeneous almost Kenmotsu 3-manifold M is Ricci recurrent if and only if it is locally symmetric. Thus, M is η-Ricci recurrent if and only if it has η-parallel Ricci operator.

In Section 5, we prove that for an almost Kenmotsu 3-manifold M satisfying $\xi(tr\, h^2) = 0$, its Ricci operator is recurrent if and only if M is locally symmetric. Moreover, for a strictly H-almost Kenmotsu 3-manifold M

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satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α, M is η-Ricci recurrent if and only if it is locally isomorphic to the type II Lie group $G(\lambda, \alpha)$ for some λ and α.

In Section 6, we prove that φ-Ricci symmetry and φ-Ricci recurrence are equivalent condition in almost Kenmotsu 3-manifolds. Thus, an almost Kenmotsu 3-manifold is φ-Ricci symmetric if and only if it has dominantly η-parallel Ricci operator.

2. Almost contact metric manifolds

In this section, we recall fundamental ingredients of almost contact metric geometry. For general information on almost contact metric geometry, we refer to Blair's monograph [1].

2.1. Almost contact structure

An *almost contact metric structure* of a $(2n+1)$-manifold M is a quartet (φ, ξ, η, g) of structure tensor fields which satisfies:

$$\eta(\xi) = 1, \quad \eta \circ \varphi = 0,$$
$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A $(2n+1)$-manifold $M = (M, \varphi, \xi, \eta, g)$ equipped with an almost contact metric structure is called an *almost contact metric manifold*. The vector field ξ is called the *characteristic vector field* of M. The 2-form

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the *fundamental* 2*-form* of M. On an almost contact metric manifold M, we introduce an endomorphism field h which plays a prominent role in this study by

$$h = \frac{1}{2}\pounds_\xi \varphi,$$

where $\pounds_\xi$ denotes the Lie differentiation by ξ.

Definition 2.1. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. A tangent plane Π_p at $p \in M$ is said to be *holomorphic* if it is invariant under φ_p.

It is easy to see that a tangent plane Π_p is holomorphic if and only if ξ_p is orthogonal to Π_p. The sectional curvature $K(\Pi_p)$ of a holomorphic plane Π_p is called the *holomorphic sectional curvature* (also called φ-sectional curvature) of Π_p. In case $\dim M = 3$, the holomorphic sectional curvature $K(\Pi_p)$ is denoted by H_p and called the *holomorphic sectional curvature* at p.

Here we introduce the notion of η-parallelism :

Definition 2.2. An endomorphism field F on an almost contact metric manifold M is said to be *η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector fields X, Y and Z on M orthogonal to ξ.

Now let us concentrate our attention to Riemannian 3-manifolds. On a Riemannian 3-manifold $M = (M, g)$, the Riemannian curvature R is described as

$$R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)SX - g(Z, X)SY - \frac{r}{2}R_1(X, Y)Z,$$

where ρ is the Ricci tensor field and S is the Ricci operator of (M, g). Here the curvature-like tensor field R_1 is defined by

$$R_1(X, Y)Z = (X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$

The covariant derivative ∇R is computed as

$$\begin{aligned}(\nabla_W R)(X, Y)Z = \; & (\nabla_W \rho)(Y, Z)X - (\nabla_W \rho)(Z, X)Y \\ & + g(Y, Z)(\nabla_W S)X - g(Z, X)(\nabla_W S)Y - \frac{dr}{2}(W)R_1(X, Y)Z.\end{aligned}$$

Hence the covariant derivative ∇R satisfies the following formula:

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,V) = \; & g((\nabla_W S)Y,Z)g(X,V) - g((\nabla_W S)Z,X)g(Y,V) \\ & + g(Y,Z)g((\nabla_W S)X,V) - g(Z,X)g((\nabla_W S)Y,V) \\ & - \frac{dr}{2}(W)g(R_1(X,Y)Z,V). \end{aligned}$$

We know that the local symmetry ($\nabla R = 0$) implies the constancy of the scalar curvature, thus we confirm the following well-known fact:

Proposition 2.1. *A Riemannian 3-manifold M is locally symmetric if and only if its Ricci operator is parallel.*

2.2. Normality

On an almost contact metric manifold M, we define a torsion tensor field N by

$$N(X,Y) := [\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi, \quad X,Y \in \mathfrak{X}(M).$$

Here $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. An almost contact metric manifold M is said to be *normal* if $N(X,Y) = 0$ for all X, $Y \in \mathfrak{X}(M)$.

2.3. Almost Kenmotsu structure

Definition 2.3 ([9]). An almost contact metric manifold M is said to be *almost Kenmotsu* if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. An almost Kenmotsu manifold is said to be *Kenmotsu* if it is normal. An almost Kenmotsu manifold is said to be a *strictly almost Kenmotsu* if it is non-normal.

It should be remarked that every almost Kenmotsu manifold satisfies $\operatorname{div}\xi = 2n$. Hence almost Kenmotsu manifolds can not be compact.

Proposition 2.2 ([4]). *If an almost Kenmotsu manifold is of constant curvature, then it is a Kenmotsu manifold of constant curvature -1.*

Kenmotsu [10] showed

Proposition 2.3. *Let M be a Kenmotsu manifold of dimension greater than 3. Then M is of constant holomorphic sectional curvature if and only if it is of constant curvature -1.*

Three dimensional case will be discussed in the next subsection. To close this subsection we introduce the following notion.

Definition 2.4. An almost Kenmotsu manifold whose characteristic vector field ξ is a harmonic unit vector field is called an *H-almost Kenmotsu manifold*.

Perrone showed the following fundamental fact ([12, Theorem 4.1], [14, Proposition 7]).

Proposition 2.4. *An almost Kenmotsu manifold M is H-almost Kenmotsu if and only if ξ is an eigenvector field of S.*

2.4. Kenmotsu 3-manifolds

Here we recall curvature properties of *Kenmotsu 3-manifolds*.

Proposition 2.5. *The Riemannian curvature R of a Kenmotsu 3-manifold M has the form*

$$R(X,Y)Z = \frac{r+4}{2}(X \wedge Y)Z + \frac{r+6}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$

The Ricci operator S has the form

$$S = \frac{r+2}{2}\,\mathrm{I} - \frac{r+6}{2}\,\eta \otimes \xi.$$

The principal Ricci curvatures are $(r+2)/2$, $(r+2)/2$ and -2. The Ricci operator S commutes with φ. For a unit vector $X \in TM$ orthogonal to ξ, the sectional curvatures of planes $X \wedge \varphi X$ and $X \wedge \xi$ are given by

$$H = K(X \wedge \varphi X) = \frac{r}{2} + 2, \quad K(X \wedge \xi) = -1.$$

Note that every Kenmotsu 3-manifold is H-almost Kenmotsu. Thus the notion of H-almost Kenmotsu manifold is intermediate notion between Kenmotsu and almost Kenmotsu manifold.

Proposition 2.6 ([5])**.** *The following three properties for a Kenmotsu* 3*-manifold M are mutually equivalent.*

- *M has constant holomorphic sectional curvature.*
- *M has constant scalar curvature.*
- *M is of constant curvature -1.*

2.5. Ricci recurrences

Definition 2.5 ([11])**.** As a generalization of local symmetry, the Ricci operator S is said to be *recurrent* if the Ricci operator S satisfies

$$(\nabla_X S)Y = A(X)SY, \tag{2.1}$$

where X, Y are any vector fields and A is a 1-form on M. Moreover, M is said to be *Ricci recurrent* if its Ricci tensor is recurrent.

In [8], Inoguchi and the author studied η-parallelism. Now, we introduce the notion of η-Ricci recurrence.

Definition 2.6. The Ricci operator S is said to be *η-recurrent* if the Ricci operator S satisfies

$$g((\nabla_X S)Y, Z) = A(X)g(SY, Z), \tag{2.2}$$

where X, Y, Z are all vector fields orthogonal to ξ and A is a 1-form on M. Moreover, M is said to be *η-Ricci recurrent* if its Ricci operator is η-recurrent.

Definition 2.7 ([8])**.** An endomorphism field F on an almost contact metric manifold M is said to be *dominantly η-parallel* if it satisfies

$$g((\nabla_X F)Y, Z) = 0 \tag{2.3}$$

for all vector fields X and Y on M and any vector field Z on M orthogonal to ξ.

Definition 2.8 ([16])**.** The Ricci operator S is said to be *φ-symmetric* if the Ricci operator S satisfies

$$\varphi^2((\nabla_X S)Y) = 0, \tag{2.4}$$

where X, Y are any vector fields. Moreover, M is said to be φ-Ricci symmetric if its Ricci operator is φ-symmetric.

Now, we introduce the notion of *φ-recurrence* as the following:

Definition 2.9. The Ricci operator S is said to be *φ-recurrent* if the Ricci operator S satisfies

$$\varphi^2((\nabla_X S)Y) = A(X)SY, \tag{2.5}$$

where X, Y are any vector fields and A is a 1-form on M. Moreover, M is said to be φ-Ricci recurrent if its Ricci operator is φ-recurrent.

3. Almost Kenmotsu 3-manifolds

3.1. Fundamental formulas

Let M be an almost Kenmotsu 3-manifold. Denote by $\mathcal{U}_1$ the open subset of M consisting of points p such that $h \neq 0$ around p. Next, let $\mathcal{U}_0$ the open subset of M consisting of points $p \in M$ such that $h = 0$ around p. Since h is smooth, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_0$ is an open dense subset of M. So any property satisfied in $\mathcal{U}$ is also satisfied in whole M. For any point $p \in \mathcal{U}$, there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ around p, where e_1 is an eigenvector field of h.

Lemma 3.1 (*cf.* [3])**.** *Let M be an almost Kenmotsu* 3*-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ on $\mathcal{U}$ such that*

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi$$

for some locally defined smooth function λ. The Levi-Civita connection ∇ is described as

$$\nabla_{e_1}e_1 = -be_2 - \xi, \quad \nabla_{e_1}e_2 = be_1 + \lambda\xi, \quad \nabla_{e_1}e_3 = e_1 - \lambda e_2,$$
$$\nabla_{e_2}e_1 = ce_2 + \lambda\xi, \quad \nabla_{e_2}e_2 = -ce_1 - \xi, \quad \nabla_{e_2}e_3 = -\lambda e_1 + e_2,$$
$$\nabla_{e_3}e_1 = \alpha e_2, \quad \nabla_{e_3}e_2 = -\alpha e_1, \quad \nabla_{e_3}e_3 = 0,$$

where

$$b = -\frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1)), \quad c = -\frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2)),$$

and σ is the 1-form metrically equivalent to $S\xi$, that is,

$$\sigma = g(S\xi, \cdot) = \rho(\xi, \cdot).$$

The covariant derivative $\nabla_\xi h$ of h by ξ is given by

$$\nabla_\xi h = -2\alpha h\varphi + \frac{\xi(\lambda)}{\lambda}h,$$

for $h \neq 0$ on the open subset $\mathcal{U}$.

The commutation relations are

$$[e_1, e_2] = be_1 - ce_2, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2.$$

The Jacobi identity is described as

$$e_1(\alpha - \lambda) + \xi(b) + c(\alpha - \lambda) + b = 0, \quad e_2(\alpha + \lambda) - \xi(c) + b(\alpha + \lambda) - c = 0.$$

Remark 3.1. On an almost Kenmotsu 3-manifold M with $h \neq 0$,

$$\alpha = g(\nabla_\xi W, \varphi W)$$

is independent of the choice of unit eigenvector W of h.

The Riemannian curvature R is computed by the table of Levi-Civita connection in Lemma 3.1:

$$\begin{aligned}
&R(e_1, e_2)e_1 = -He_2 - \sigma(e_2)\xi, \quad R(e_1, e_2)e_2 = He_1 + \sigma(e_1)\xi,\\
&R(e_1, e_2)e_3 = \sigma(e_2)e_1 - \sigma(e_1)e_2, \qquad R(e_2, e_3)e_1 = \sigma(e_1)e_2 - \{\xi(\lambda) + 2\lambda\}\,\xi,\\
&R(e_2, e_3)e_2 = -\sigma(e_1)e_1 - K_{23}\xi, \qquad R(e_2, e_3)e_3 = \{\xi(\lambda) + 2\lambda\}\,e_1 + K_{23}e_2,\\
&R(e_3, e_1)e_1 = \sigma(e_2)e_2 + K_{13}\xi, \qquad R(e_3, e_1)e_2 = -\sigma(e_2)e_1 + \{\xi(\lambda) + 2\lambda\}\,\xi,\\
&R(e_3, e_1)e_3 = -K_{13}e_1 - \{\xi(\lambda) + 2\lambda\}\,e_2.
\end{aligned}$$

Here the sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ are given by

$$H = K_{12} = \frac{r}{2} + 2(\lambda^2 + 1), \quad K_{13} = -(\lambda^2 + 2\alpha\lambda + 1), \quad K_{23} = -(\lambda^2 - 2\alpha\lambda + 1).$$

Next, the Ricci operator S of an almost Kenmotsu 3-manifold M is described as :

$$\begin{aligned}
Se_1 &= \left(\frac{r}{2} + \lambda^2 - 2\alpha\lambda + 1\right) e_1 + \{\xi(\lambda) + 2\lambda\}e_2 + \sigma(e_1)\xi,\\
Se_2 &= \{\xi(\lambda) + 2\lambda\}e_1 + \left(\frac{r}{2} + \lambda^2 + 2\alpha\lambda + 1\right) e_2 + \sigma(e_2)\xi,\\
Se_3 &= \sigma(e_1)e_1 + \sigma(e_2)e_2 - 2(\lambda^2 + 1)\xi.
\end{aligned}$$

Note that the scalar curvature r is computed as

$$r = -2\{e_1(c) + e_2(b) + b^2 + c^2 + \lambda^2 + 3\}$$

and

$$\sigma(e_1) = -e_2(\lambda) - 2\lambda b, \quad \sigma(e_2) = -e_1(\lambda) - 2\lambda c.$$

Remark 3.2. Since M is 3-dimensional, we have the relations

$$\rho_{11} = H + K_{13}, \quad \rho_{22} = H + K_{23}, \quad \rho_{33} = K_{13} + K_{23},$$

where the Ricci tensor ρ is defined by $\rho_{ij} = \rho(e_i, e_j) = g(Se_i, e_j)$, $i, j = 1, 2, 3$.

4. Homogeneous almost Kenmotsu 3-manifolds

In this section, we provide explicit examples of almost Kenmotsu 3-manifolds as homogeneous spaces and study Ricci recurrence and η-Ricci recurrence.

4.1. *Two classes*

Perrone ([14]) proved that every simply connected homogeneous almost Kenmotsu 3-manifold is a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. There are two classes of 3-dimensional almost Kenmotsu Lie groups.

- The characteristic vector field ξ is orthogonal to the unimodular kernel (Type II Lie groups).
- The characteristic vector field ξ is transversal to the unimodular kernel but not orthogonal (Type IV Lie groups).

After performing normalization procedure, those Lie group has the Lie algebra determined by the following commutation relations:

- The type II Lie algebra $\mathfrak{g} = \mathfrak{g}(\lambda, \alpha)$ is generated by
$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2,$$
where λ, $\alpha \in \mathbb{R}$.
- The type IV Lie algebra $\mathfrak{g} = \mathfrak{g}[\alpha, \gamma]$ is generated by
$$[e_1, e_2] = \gamma e_1, \quad [e_2, e_3] = 2\alpha e_1, \quad [e_3, e_1] = -2e_1,$$
where α, $\gamma \in \mathbb{R}$ and $\gamma \neq 0$.

In this section, we exhibit these Lie groups in detail. For more information on these examples, we refer to [6, 7].

4.2. *Type II Lie groups*

Let $G(\lambda, \alpha)$ be a 3-dimensional non-unimodular Lie group with Lie algebra $\mathfrak{g}(\lambda, \alpha)$ generated by the orthonormal basis $\{e_1, e_2, e_3\}$ with commutation relations
$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2.$$
Then a left invariant almost contact structure (φ, ξ, η) compatible to the left invariant metric g is defined by
$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad \xi = e_3, \quad \eta = g(e_3, \cdot).$$
Then (φ, ξ, η, g) is a left invariant almost Kenmotsu structure. Note that $\{e_1, e_2, e_3\}$ is regarded as a global orthonormal frame field as in Lemma 3.1 under the choice $b = c = 0$. The Levi-Civita connection is described as
$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = \lambda e_3, \quad \nabla_{e_1} e_3 = e_1 - \lambda e_2,$$
$$\nabla_{e_2} e_1 = \lambda e_3, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -\lambda e_1 + e_2,$$
$$\nabla_{e_3} e_1 = \alpha e_2, \quad \nabla_{e_3} e_2 = -\alpha e_1, \quad \nabla_{e_3} e_3 = 0.$$
The Riemannian curvature R of $G(\lambda, \alpha)$ is given by
$$\begin{aligned} &R(e_1, e_2)e_1 = -K_{12}e_2, \quad R(e_1, e_2)e_2 = K_{12}e_1, \quad R(e_1, e_2)e_3 = 0, \\ &R(e_2, e_3)e_1 = -2\lambda\xi, \quad R(e_2, e_3)e_2 = -K_{23}\xi, \quad R(e_2, e_3)e_3 = 2\lambda e_1 + K_{23}e_2, \\ &R(e_3, e_1)e_1 = K_{13}\xi, \quad R(e_3, e_1)e_2 = 2\lambda\xi, \quad R(e_3, e_1)e_3 = -K_{13}e_1 - 2\lambda e_2, \end{aligned}$$
where
$$K_{12} = -(1 - \lambda^2), \quad K_{13} = -(\lambda^2 + 2\lambda\alpha + 1), \quad K_{23} = -(\lambda^2 - 2\lambda\alpha + 1).$$
The Ricci operator S is given by
$$Se_1 = -2(1 + \lambda\alpha)e_1 + 2\lambda e_2, \quad Se_2 = 2\lambda e_1 - 2(1 - \lambda\alpha)e_2, \quad Se_3 = -2(1 + \lambda^2)\xi.$$

Thus every $G(\lambda,\alpha)$ is H-almost Kenmotsu. The scalar curvature r is computed as

$$r = -2(3+\lambda^2).$$

The principal Ricci curvatures are

$$\rho_1 = -2 + 2\lambda\sqrt{1+\alpha^2}, \quad \rho_2 = -2 - 2\lambda\sqrt{1+\alpha^2}, \quad \rho_3 = -2(1+\lambda^2).$$

Direct computations show that

$$(\nabla_{e_1}S)e_1 = 2\alpha\lambda\xi, \quad (\nabla_{e_1}S)e_2 = 2\lambda(\lambda^2+\alpha\lambda-1)\xi,$$
$$(\nabla_{e_1}S)e_3 = 2\alpha\lambda e_1 + 2\lambda(\lambda^2+\alpha\lambda-1)e_2, \quad (\nabla_{e_2}S)e_1 = 2\lambda(\lambda^2-\alpha\lambda-1)\xi,$$
$$(\nabla_{e_2}S)e_2 = -2\alpha\lambda\xi, \quad (\nabla_{e_2}S)e_3 = 2\lambda(\lambda^2-\alpha\lambda-1)e_1 - 2\alpha\lambda e_2,$$
$$(\nabla_{e_3}S)e_1 = -4\alpha\lambda e_1 - 4\alpha^2\lambda e_2, \quad (\nabla_{e_3}S)e_2 = -4\alpha^2\lambda e_1 + 4\alpha\lambda e_2, \quad (\nabla_{e_3}S)e_3 = 0.$$

From these we obtain

$$\nabla S = 0 \Longleftrightarrow \lambda = 0 \quad or \quad (\ \alpha = 0 \quad and \quad \lambda = \pm 1\ ).$$

One can see that the locally symmetric Lie group $G(\lambda,\alpha)$ of type II are

- $G(0,\alpha)$ for any α. The Lie group $G(0,\alpha)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ of constant curvature equipped with a left invariant Kenmotsu structure.
- $G(\pm 1, 0)$ which is isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$ equipped with a left invariant strictly H-almost Kenmotsu structure.

Moreover we obtain the following classification:

Proposition 4.1 ([8]). *Every almost Kenmotsu Lie group $G(\lambda,\alpha)$ has η-parallel Ricci operator.*

Now, let's consider Ricci recurrent condition. Since $g((\nabla_{e_i}S)\xi,\xi) = A(e_i)g(S\xi,\xi),\ i=1,2,3$ holds if and only if $A(e_i)=0$, for $i=1,2,3$, we have

Proposition 4.2. *On a 3-dimensioanl non-unimodular Lie group $G(\lambda,\alpha)$ of type II equipped with a left-invariant almost Kenmotsu structure, the following properties are mutually equivalent:*

- *recurrent Ricci operator.*
- *recurrent φ-Ricci operator.*
- *local symmetry.*

Moreover, the recurrent η-Ricci operator of the Lie group $G(\lambda,\alpha)$ of type II is η-parallel Ricci operator.

4.3. Type IV Lie groups

Let us consider a 3-dimensional non-unimodular Lie group $G = G[\alpha,\gamma]$ of type IV equipped with a left invariant almost Kenmotsu structure. The Lie algebra $\mathfrak{g} = \mathfrak{g}[\alpha,\gamma]$ is determined by the commutation relations:

$$[e_1,e_2] = \gamma e_1, \quad [e_2,e_3] = 2\alpha e_1, \quad [e_3,e_1] = -2e_1, \quad \alpha\in\mathbb{R}, \quad \gamma\neq 0.$$

Then the Levi-Civita connection is described as

$$\begin{aligned} \nabla_{e_1}e_1 = -\gamma e_2 - 2e_3, \quad \nabla_{e_1}e_2 = \gamma e_1 - \alpha e_3, \quad \nabla_{e_1}e_3 = 2e_1 + \alpha e_2,\\ \nabla_{e_2}e_1 = -\alpha e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = \alpha e_1,\\ \nabla_{e_3}e_1 = \alpha e_2, \quad \nabla_{e_3}e_2 = -\alpha e_1, \quad \nabla_{e_3}e_3 = 0.\end{aligned} \tag{4.1}$$

The Lie group is strictly almost Kenmotsu. The unimodular kernel is spanned by

$$e_1, \quad \xi + \frac{2}{\gamma}(e_1 - e_2).$$

The operators h and $h'(=h\varphi)$ are given by

$$he_1 = -\alpha e_1 + e_2, \quad he_2 = e_1 + \alpha e_2, \quad h'e_1 = e_1 + \alpha e_2, \quad h'e_2 = \alpha e_1 - e_2.$$

The eigenvalues of h are 0, λ and $-\lambda$ where
$$\lambda = \sqrt{\alpha^2+1}.$$
The covariant derivative $\nabla_\xi h$ is computed as
$$\nabla_\xi h = -2\alpha h\varphi.$$
Hence $\nabla_\xi h = 0$ holds when and only when $\alpha = 0$.
The Riemannian curvature R and the Ricci operator S are described as
$$R(e_1,e_2)e_1 = \left(\gamma^2-\alpha^2\right)e_2 + 2\gamma e_3, \quad R(e_1,e_2)e_2 = \left(\alpha^2-\gamma^2\right)e_1 + 2\alpha\gamma e_3, \quad R(e_1,e_2)e_3 = -2\gamma(e_1+\alpha e_2),$$
$$R(e_1,e_3)e_1 = 2\gamma e_2 + \left(4-\alpha^2\right)e_3, \quad R(e_1,e_3)e_2 = -2\gamma e_1 + 4\alpha e_3, \quad R(e_1,e_3)e_3 = \left(\alpha^2-4\right)e_1 - 4\alpha e_2,$$
$$R(e_2,e_3)e_1 = 2\alpha\gamma e_2 + 4\alpha e_3, \quad R(e_2,e_3)e_2 = -2\alpha\gamma e_1 + 3\alpha^2 e_3, \quad R(e_2,e_3)e_3 = -4\alpha e_1 - 3\alpha^2 e_2.$$
$$\rho_{11} = 2\alpha^2-\gamma^2-4, \quad \rho_{12} = -4\alpha, \quad \rho_{13} = 2\alpha\gamma,$$
$$\rho_{22} = -2\alpha^2-\gamma^2, \quad \rho_{23} = -2\gamma, \quad \rho_{33} = -2\alpha^2-4.$$
The principal Ricci curvatures are
$$-4-2\alpha^2-\gamma^2, \quad -4-2\alpha^2-\gamma^2, \quad 2\alpha^2.$$
Direct computation show that
$$(\nabla_{e_1}S)e_1 = -4\alpha^2\left\{\frac{1}{2}\gamma e_2 + e_3\right\}, \quad (\nabla_{e_1}S)e_2 = 2\alpha\left\{-\alpha\gamma e_1 + 2\gamma e_2 - (\frac{1}{2}\gamma^2-2)e_3\right\},$$
$$(\nabla_{e_1}S)e_3 = 2\alpha\left\{(-2\alpha)e_1 + (2-\frac{1}{2}\gamma^2)e_2 - 2\gamma e_3\right\},$$
$$(\nabla_{e_2}S)e_1 = -2\alpha\left\{(-2\alpha)\gamma e_1 + \gamma e_2 + \frac{1}{2}(4\alpha^2-\gamma^2)e_3\right\}, \quad (\nabla_{e_2}S)e_2 = -2\alpha(\gamma e_2 - 2\alpha e_3),$$
$$(\nabla_{e_2}S)e_3 = -2\alpha\left\{\frac{1}{2}(4\alpha^2-\gamma^2)e_1 - 2\alpha e_2 + 2\alpha\gamma e_3)\right\},$$
$$(\nabla_{e_3}S)e_1 = -2\alpha\left\{-4\alpha e_1 - 2(\alpha^2-1)e_2 - \gamma e_3\right\}, \ (\nabla_{e_3}S)e_2 = 2\alpha\left\{2(\alpha^2-1)e_1 - 4\alpha e_2 + \alpha\gamma e_3\right\},$$
$$(\nabla_{e_3}S)e_3 = 2\alpha\gamma(e_1+\alpha e_2).$$
From these we obtain
$$\nabla S = 0 \Longleftrightarrow (\nabla_\xi S)\xi = 0 \Longleftrightarrow \alpha = 0. \tag{4.2}$$
Hence $G[\alpha,\gamma]$ has η-parallel Ricci operator when and only when $\alpha = 0$. Moreover, the η-parallelism of S is equivalent to the local symmetry of $G[\alpha,\gamma]$. The strictly almost Kenmotsu Lie group $G[0,\gamma]$ is not H-almost Kenmotsu. Moreover $G[0,\gamma]$ is isometric to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ and satisfying $\nabla_\xi h = 0$.
As a generalization of local symmetry, let's consider Ricci recurrent condition.

Proposition 4.3. *On a 3-dimensioanl non-unimodular Lie group $G[\alpha,\gamma]$ of type IV equipped with a left-invariant almost Kenmotsu structure, the following properties are mutually equivalent:*

- *recurrent Ricci operator.*
- *recurrent η-Ricci operator .*
- *recurrent φ-Ricci operator.*
- *local symmetry.*

Proof. Since $g((\nabla_{e_2}S)e_2,e_1) = A(e_2)g(Se_2,e_1)$ holds if and only if
$$\alpha = 0 \quad or \quad A(e_2) = 0.$$
First, if $\alpha = 0$, then, from (4.2), $G[\alpha,\gamma]$ is locally symmetric. Next, if $\alpha \neq 0$ and $A(e_2) = 0$, then $g((\nabla_{e_2}S)e_2,e_2) = A(e_2)g(Se_2,e_2)$ holds if and only if $\alpha\gamma = 0$, in the sequel, we treat both situations. In $G[\alpha,\gamma]$, we supposed that $\gamma \neq 0$, so it is a contradiction. Hence the recurrent η-Ricci operator of the Lie group $G[\alpha,\gamma]$ of type IV is locally symmetric. The recurrent φ-Ricci operator can also be proven in a similar way. □

From Proposition 4.1, Proposition 4.2 and Proposition 4.3, we have

Theorem 4.1. *A homogeneous almost Kenmotsu 3-manifold M is Ricci recurrent if and only if it is locally symmetric and one of the following instances happens:*

1. *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1, or locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R}$.*
2. *If M is non H-almost Kenmotsu, then M is locally isomorphic to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$.*

Moreover, M is η-Ricci recurrent if and only if it has η-parallel Ricci operator and it is isomorphic to the following spaces:

1. *If M is H-almost Kenmotsu, then M is a Lie group $G(\lambda, \alpha)$ of type II.*
2. *If M is non H-almost Kenmotsu, then M is locally isomorphic to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$.*

5. Local symmetry and Ricci recurrence

In this section, we study recurrent and η-recurrent Ricci operator in almost Kenmotsu 3-manifolds.

5.1. The system of local symmetry

Now we start our study on the η-parallelism of the Ricci operator. First of all, we recall the table of covariant derivative ∇S over $\mathcal{U}_1$ obtained in [6, 7].

Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1, then direct computation shows the following results.

$$
\begin{aligned}
(\nabla_{e_1} S)e_1 = \ & \{e_1(\rho_{11}) + 2b\rho_{12} + 2\rho_{13}\}e_1 + \{e_1(\rho_{12}) - b(\rho_{11} - \rho_{22}) - \lambda\rho_{13} + \rho_{23}\}e_2 \\
& + \{e_1(\rho_{13}) - (\rho_{11} - \rho_{33}) + \lambda\rho_{12} + b\rho_{23}\}e_3, \\
(\nabla_{e_1} S)e_2 = \ & \{e_1(\rho_{12}) - b(\rho_{11} - \rho_{22}) - \lambda\rho_{13} + \rho_{23}\}e_1 + \{e_1(\rho_{22}) - 2b\rho_{12} - 2\lambda\rho_{23}\}e_2 \\
& + \{e_1(\rho_{23}) - \rho_{12} - b\rho_{13} + \lambda(\rho_{22} - \rho_{33})\}e_3, \\
(\nabla_{e_1} S)e_3 = \ & \{e_1(\rho_{13}) - (\rho_{11} - \rho_{33}) + \lambda\rho_{12} + b\rho_{23}\}e_1 + \{e_1(\rho_{23}) - \rho_{12} - b\rho_{13} + \lambda(\rho_{22} - \rho_{33})\}e_2 \\
& + \{e_1(\rho_{33}) - 2\rho_{13} + 2\lambda\rho_{23}\}e_3, \\
(\nabla_{e_2} S)e_1 = \ & \{e_2(\rho_{11}) - 2c\rho_{12} - 2\lambda\rho_{13}\}e_1 + \{e_2(\rho_{12}) + c(\rho_{11} - \rho_{22}) + \rho_{13} - \lambda\rho_{23}\}e_2, \\
& + \{e_2(\rho_{13}) + \lambda(\rho_{11} - \rho_{33}) - \rho_{12} - c\rho_{23}\}e_3, \\
(\nabla_{e_2} S)e_2 = \ & \{e_2(\rho_{12}) + c(\rho_{11} - \rho_{22}) + \rho_{13} - \lambda\rho_{23}\}e_1 + \{e_2(\rho_{22}) + 2c\rho_{12} + 2\rho_{23}\}e_2 \\
& + \{e_2(\rho_{23}) + \lambda\rho_{12} + c\rho_{13} - (\rho_{22} - \rho_{33})\}e_3, \\
(\nabla_{e_2} S)e_3 = \ & \{e_2(\rho_{13}) + \lambda(\rho_{11} - \rho_{33}) - \rho_{12} - c\rho_{23}\}e_1 + \{e_2(\rho_{23}) + \lambda\rho_{12} + c\rho_{13} - (\rho_{22} - \rho_{33})\}e_2 \\
& + \{e_2(\rho_{33}) + 2\lambda\rho_{13} - 2\rho_{23}\}e_3, \\
(\nabla_{e_3} S)e_1 = \ & \{e_3(\rho_{11}) - 2\alpha\rho_{12}\}e_1 + \{e_3(\rho_{12}) + \alpha(\rho_{11} - \rho_{22})\}e_2 + \{e_3(\rho_{13}) - \alpha\rho_{23}\}e_3, \\
(\nabla_{e_3} S)e_2 = \ & \{e_3(\rho_{12}) + \alpha(\rho_{11} - \rho_{22})\}e_1 + \{e_3(\rho_{22}) + 2\alpha\rho_{12})\}e_2 + \{e_3(\rho_{23}) + \alpha\rho_{13}\}e_3 \\
(\nabla_{e_3} S)e_3 = \ & \{e_3(\rho_{13}) - \alpha\rho_{23}\}e_1 + \{e_3(\rho_{23}) + \alpha\rho_{13}\}e_2 + e_3(\rho_{33})e_3,
\end{aligned}
$$

where the Ricci tensor ρ is defined by $\rho_{ij} = \rho(e_i, e_j) = g(Se_i, e_j)$, $i, j = 1, 2, 3$.
From these, we have

Proposition 5.1 ([8]). *If an almost Kenmotsu 3-manifold M is locally symmetric, then $\nabla_\xi h = 0$ holds on M.*

The converse statement of this Proposition does not hold. In fact, the Lie group $G(\lambda, 0)$ with $\lambda^2 \neq 1$ in Type ll Lie groups satisfies $\nabla_\xi h = 0$ but it is not locally symmetric.
Locally symmetric almost Kenmotsu 3-manifolds are classified as follows:

Theorem 5.1 ([6]). *Let M be an almost Kenmotsu 3-manifold. Then M is locally symmetric if and only if M is one of the following spaces:*

1. *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1 or locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R}$ or*

2. *If M is non H-almost Kenmotsu, then M is locally isomorphic to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ for some $\gamma\neq 0$.*

This classification result was announced in [2]. The example $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ for some $\gamma\neq 0$ was discovered in [[13], Theorem 1.2 Case (IV)].

Corollary 5.1 ([8])**.** *Every complete locally symmetric almost Kenmotsu 3-manifold is realized as a Lie group equipped with a left invariant almost Kenmotsu structure.*

5.2. Recurrent Ricci operators

In this subsection, we study recurrent Ricci operators in almost Kenmotsu 3-manifolds.
Using (2.1), $(\nabla_{e_1}S)=A(e_1)S$ holds if and only if

$$e_1(\rho_{11})+2b\rho_{12}+2\rho_{13}=\rho_{11}A(e_1), \tag{5.1}$$
$$e_1(\rho_{12})-b(\rho_{11}-\rho_{22})-\lambda\rho_{13}+\rho_{23}=\rho_{12}A(e_1), \tag{5.2}$$
$$e_1(\rho_{13})-(\rho_{11}-\rho_{33})+\lambda\rho_{12}+b\rho_{23}=\rho_{13}A(e_1), \tag{5.3}$$
$$e_1(\rho_{22})-2b\rho_{12}-2\lambda\rho_{23}=\rho_{22}A(e_1), \tag{5.4}$$
$$e_1(\rho_{23})-\rho_{12}-b\rho_{13}+\lambda(\rho_{22}-\rho_{33})=\rho_{23}A(e_1), \tag{5.5}$$
$$e_1(\rho_{33})-2\rho_{13}+2\lambda\rho_{23}=\rho_{33}A(e_1). \tag{5.6}$$

Next, $(\nabla_{e_2}S)=A(e_2)S$ holds if and only if

$$e_2(\rho_{11})-2c\rho_{12}-2\lambda\rho_{13}=\rho_{11}A(e_2), \tag{5.7}$$
$$e_2(\rho_{12})+c(\rho_{11}-\rho_{22})+\rho_{13}-\lambda\rho_{23}=\rho_{12}A(e_2), \tag{5.8}$$
$$e_2(\rho_{13})+\lambda(\rho_{11}-\rho_{33})-\rho_{12}-c\rho_{23}=\rho_{13}A(e_2), \tag{5.9}$$
$$e_2(\rho_{22})+2c\rho_{12}+2\rho_{23}=\rho_{22}A(e_2), \tag{5.10}$$
$$e_2(\rho_{23})+\lambda\rho_{12}+c\rho_{13}-(\rho_{22}-\rho_{33})=\rho_{23}A(e_2), \tag{5.11}$$
$$e_2(\rho_{33})+2\lambda\rho_{13}-2\rho_{23}=\rho_{33}A(e_2). \tag{5.12}$$

Lastly, $(\nabla_{\xi}S)=A(\xi)S$ holds if and only if

$$\xi(\rho_{11})-2\alpha\rho_{12}=\rho_{11}A(\xi), \tag{5.13}$$
$$\xi(\rho_{12})+\alpha(\rho_{11}-\rho_{22})=\rho_{12}A(\xi), \tag{5.14}$$
$$\xi(\rho_{13})-\alpha\rho_{23}=\rho_{13}A(\xi), \tag{5.15}$$
$$\xi(\rho_{22})+2\alpha\rho_{12}=\rho_{22}A(\xi), \tag{5.16}$$
$$\xi(\rho_{23})+\alpha\rho_{13}=\rho_{23}A(\xi), \tag{5.17}$$
$$\xi(\rho_{33})=\rho_{33}A(\xi). \tag{5.18}$$

Since $\rho_{33}=-2(\lambda^2+1)$, applying this to (5.18), we have

$$2\lambda\xi(\lambda)=(\lambda^2+1)A(\xi).$$

Since $\lambda^2+1>0$, we have that $A(\xi)=0$ if and only if

$$\lambda=0 \quad \textit{or} \quad \xi(\lambda)=0. \tag{5.19}$$

So, for the case satisfying (5.19), we study Ricci recurrence in an almost Kenmotsu 3-manifold and hence we have

Theorem 5.2. *Let M be an almost Kenmotsu 3-manifold satisfying $\xi(tr\, h^2)=0$. Then its Ricci operator is recurrent if and only if M is locally symmetric.*

Proof. First, on $\mathcal{U}_0$, from (5.6), (5.12), and (5.18), we have $A(e_i)=0,\ i=1,2,3$. Hence M is locally symmetric and it is locally isomorphic to the hyperbolic 3-space $\mathbb{H}^3(-1)$ of curvature -1.
Next, on $\mathcal{U}_1$, if $\xi(\lambda)=0$, then from (5.18), we get

$$A(\xi)=0, \tag{5.20}$$

and applying (5.20) to (5.14), then we have

$$\alpha = 0. \tag{5.21}$$

From now on, we consider the case $\alpha = 0$ and $\xi(\lambda) = 0$ on $\mathcal{U}_1$. From (5.6) and (5.12), we have

$$\begin{aligned} 3\lambda e_1(\lambda) - e_2(\lambda) - 2b\lambda + 2c\lambda^2 &= (\lambda^2+1)A(e_1), \\ 3\lambda e_2(\lambda) - e_1(\lambda) - 2c\lambda + 2b\lambda^2 &= (\lambda^2+1)A(e_2). \end{aligned} \tag{5.22}$$

From (5.15) and (5.17), we have

$$\xi e_2(\lambda) + 2\lambda\xi(b) = 0, \quad \xi e_1(\lambda) + 2\lambda\xi(c) = 0. \tag{5.23}$$

On the other hands, using the formula $divS = \frac{1}{2}grad(r)$, we get

$$\begin{aligned} \xi e_2(\lambda) + 2\lambda\xi(b) &= 3\lambda e_1(\lambda) - e_2(\lambda) - 2b\lambda + 2c\lambda^2, \\ \xi e_1(\lambda) + 2\lambda\xi(c) &= 3\lambda e_2(\lambda) - e_1(\lambda) - 2c\lambda + 2b\lambda^2. \end{aligned} \tag{5.24}$$

From (5.23) and (5.24), we have

$$\begin{aligned} 3\lambda e_1(\lambda) - e_2(\lambda) - 2b\lambda + 2c\lambda^2 &= 0, \\ 3\lambda e_2(\lambda) - e_1(\lambda) - 2c\lambda + 2b\lambda^2 &= 0. \end{aligned} \tag{5.25}$$

Comparing (5.22) and (5.25), we have

$$A(e_1) = A(e_2) = 0. \tag{5.26}$$

Hence, from (5.20) and (5.26) we have that M is locally symmetric. □

Remark 5.1. In [17], the authors studied Ricci recurrent in H-almost Kenmotsu 3-manifolds with $\nabla_\xi h = 0$.

5.3. *η-Ricci recurrence in H-almost Kenmotsu 3-manifolds*

In this subsection, we study η-Ricci recurrence in H-almost Kenmotsu 3-manifolds.

In [8], for an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α, we got the following :

Proposition 5.2 ([8]). *Let M be an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. Then M is a Kenmotsu 3-manifold or it is locally isomorphic to one of the type II Lie group $G(\lambda, \alpha)$ for some λ and α.*

Moreover, for H-almost Kenmotsu 3-manifolds with *η-parallel Ricci operator* satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α, we obtained the following partial classification:

Proposition 5.3 ([8]). *Let M be an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. Then M has η-parallel Ricci operator if and only if it is locally isomorphic to one of the following spaces:*

1. *the warped product $I \times_{ce^t} \overline{M}$. Here I is an open interval with coordinate t, $\overline{M}$ is a Riemannian 2-manifold of constant curvature, c is a positive constant, or*
2. *the type II Lie group $G(\lambda, \alpha)$ for some λ and α.*

From (2.2), we have

Theorem 5.3. *Let M be a strictly H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. Then M has recurrent η-Ricci operator if and only if it is locally isomorphic to the type II Lie group $G(\lambda, \alpha)$ for some λ and α.*

Proof. Since we assume that M is a H-almost Kenmotsu 3-manifold, $\rho_{13} = \rho_{23} = 0$. Using the definition (2.2), the η-Ricci recurrent condition in H-almost Kenmotsu 3-manifolds is (5.1), (5.2), (5.4), (5.7), (5.8) and (5.10) such that $\rho_{13} = \rho_{23} = 0$. If we apply the condition that α is a constant and $\xi(\lambda) = 0$ to the these equations, we get the following:

$$\tfrac{1}{2}e_1(r) + 2(\lambda-\alpha)e_1(\lambda) + 4b\lambda = (\tfrac{r}{2} + \lambda^2 - 2\alpha\lambda + 1)A(e_1), \tag{5.27}$$

$$e_1(\lambda) + 2\alpha b\lambda = \lambda A(e_1), \tag{5.28}$$

$$\tfrac{1}{2}e_1(r) + 2(\lambda+\alpha)e_1(\lambda) - 4b\lambda = (\tfrac{r}{2} + \lambda^2 + 2\alpha\lambda + 1)A(e_1), \tag{5.29}$$

$$\tfrac{1}{2}e_2(r) + 2(\lambda-\alpha)e_2(\lambda) - 4c\lambda = (\tfrac{r}{2} + \lambda^2 - 2\alpha\lambda + 1)A(e_2), \tag{5.30}$$

$$e_2(\lambda) - 2\alpha c\lambda = \lambda A(e_2), \tag{5.31}$$

$$\tfrac{1}{2}e_2(r) + 2(\lambda+\alpha)e_2(\lambda) + 4c\lambda = (\tfrac{r}{2} + \lambda^2 + 2\alpha\lambda + 1)A(e_2). \tag{5.32}$$

From (5.27) and (5.29), we have

$$\alpha e_1(\lambda) - 2b\lambda = \alpha\lambda A(e_1). \tag{5.33}$$

From (5.30) and (5.32), we have

$$\alpha e_2(\lambda) + 2c\lambda = \alpha\lambda A(e_2). \tag{5.34}$$

Applying (5.28) to (5.33) and applying (5.31) to (5.34) we have

$$b\lambda = 0 \quad \textit{and} \quad c\lambda = 0. \tag{5.35}$$

Hence on $\mathcal{U}_1$, we have $b = c = 0$ and $e_1(\lambda) = e_2(\lambda) = 0$. Since we suppose that $\xi(\lambda) = 0$, we get λ is non-zero constant. From (5.28) and (5.31), we get $A(e_1) = A(e_2) = 0$ and the manifold has η-parallel Ricci operator. Hence from Proposition 5.3, M is locally isomorphic to the type II Lie group $G(\lambda, \alpha)$ for some λ and α. □

Remark 5.2. For a Kenmotsu 3-manifold M (i.e. with $\lambda = 0$), its Ricci operator is η-recurrent if and only if

$$e_i(r) = (r + 2)A(e_i), \quad i = 1, 2,$$

where r is the scalar curvature and A is a 1-form on M. From Proposition 4.2, we can see that the η-recurrent Ricci operator for Lie group $G(0, \alpha)$ of type II (i.e. Lemma 1 under $\lambda = 0, \alpha \in \mathbb{R}$ and $b = c = 0$) is η-parallel.

Remark 5.3. If $A = \eta$, then an almost Kenmotsu 3-manifold M has η-Ricci recurrent if and only if it has η-parallel Ricci operator.

6. φ-Ricci recurrence

Shukla and Shukla [16] proved that a $(2n + 1)$-dimensional Kenmotsu manifold has φ-Ricci symmetric operator if and only if it is an Einstein manifold. In particular, a 3-dimensional Kenmotsu manifold which is φ-Ricci symmetric is an Einstein manifold and a Kenmotsu manifold of constant curvature -1. In this section, we study φ-Ricci symmetry and φ-Ricci recurrence in almost Kenmotsu 3-manifolds.

6.1. φ-Ricci symmetry

Using (2.3), we have

Proposition 6.1 ([8])**.** *An almost Kenmotsu 3-manifold M has dominantly η-parallel Ricci operator with $\xi(tr\, h^2) = 0$ if and only if it is locally symmetric.*

By direct calculation, we have

Proposition 6.2. *An almost Kenmotsu 3-manifold M is φ-Ricci symmetric if and only if it has dominantly η-parallel Ricci operator.*

Thus, we have

Corollary 6.1. *An almost Kenmotsu 3-manifold M is φ-Ricci symmetric with $\xi(tr\, h^2) = 0$ if and only if it is locally symmetric.*

6.2. φ-Ricci recurrence

From (2.4) and (2.5), we have

Proposition 6.3. *An almost Kenmotsu 3-manifold M is φ-Ricci recurrent if and only if it is φ-Ricci symmetric.*

Proof. From $g(\varphi^2((\nabla_{e_i} S)\xi), \xi) = g(A(X)S\xi, \xi)$, we have $A(e_i) = 0$, $i = 1, 2, 3$. Hence if an almost Kenmotsu 3-manifold M is φ-Ricci recurrent then it is φ-Ricci symmetric. □

Hence, from Proposition 6.2 and Proposition 6.3, we have

Corollary 6.2. *On an almost Kenmotsu 3-manifold M, the following properties are mutually equivalent:*

- *φ-Ricci recurrence.*
- *φ-Ricci symmetry.*

- *Dominantly η-parallelism of the Ricci operator.*

Moreover, from Proposition 4.2, Proposition 4.3 and Corollary 6.2, we have

Corollary 6.3. *On a homogeneous almost Kenmotsu 3-manifold M, the following properties are mutually equivalent:*

- *φ-Ricci recurrence.*
- *φ-Ricci symmetry.*
- *Dominantly η-parallelism of the Ricci operator.*
- *local symmetry.*

In [8], we found that for a homogeneous almost Kenmotsu 3-manifold M, the dominantly η-parallel Ricci operator is equivalent to the local symmetry of M.

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