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NORM RETRIEVAL IN DYNAMICAL SAMPLING FORM

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Abstract. In this article, we study the construction of norm retrievable frames that have a dynamical sampling structure. For a closed subspace W of $\mathbb{R}^n$, we show that when the collection of subspaces $\{A^\ell W\}_{i\in I}$ is norm retrievable in $\mathbb{R}^n$ for a unitary or Jordan operator A, then there always exists a collection of norm retrievable frame vectors that have a dynamical sampling structure in $\mathbb{R}^n$.

1. Introduction

Given a signal $x \in \mathbb{H}$ in a separable Hilbert space with a given orthonormal basis $\{e_i\}_{i\in I}$ in $\mathbb{H}$, Parseval's identity allows us to reconstruct the signal x from the measurements$\{\langle x, e_i\rangle\}_{i\in I}$. The set of coefficients $\{\langle x, e_i\rangle\}_{i\in I}$ is unique. We are unable to recreate the signal x from the remaining data if a measurement is missing or damaged. We can see the need for a set of vectors that allows for some loss resilience while also having a reconstruction property similar to Parseval's identity. A frame $\{x_i\}_{i\in I}$ for $\mathbb{H}$ allows for redundancy while preserving a structure so that reconstruction is possible. Now, the set of measurements $\{\langle x, x_i\rangle\}_{i\in I}$ is not necessarily unique.

We can reconstruct the signal x from the measurements $\{\langle x, x_i\rangle\}$ using the frame vectors $\{x_i\}_{i\in I}$ in $\mathbb{H}$. But let's say that the measurements' phase was lost or was impossible to determine. These restrictions may apply in a setting like a tomography or crystallography. We are unable to create the exact signal x when we just have the phaseless measurements $\{|\langle x, x_i\rangle|\}$. The idea of phase retrieval for Hilbert space frames was first proposed by Casazza, Balan, and Edidin [11] in 2006 to extract the phase of a signal from a redundant linear system using the intensity measurements $\{|\langle x, x_i\rangle|\}$. They showed that we require a minimum $2n-1$ vectors to have phase retrieval in $\mathbb{R}^n$. Phase retrieval is a stronger condition than being a

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frame. A set of vectors does not meet the requirements for phase retrieval if it is not a frame. Norm retrieval is a different condition that is less strong than phase retrieval. The notion of norm retrieval is described in [10], a collection of vectors performs norm retrieval if two vectors in the Hilbert space have the same intensity measurements, then they have the same norm. The phase retrieval conditions are relaxed by the norm retrieval property. There exist norm retrievable sets that are not phase retrievable, but every phase retrievable set is also a norm retrievable set. Fewer vectors are needed for norm retrieval compared to phase retrieval. For instance, orthonormal bases are not phase retrievable but they are norm retrievable sets.

When $\Omega \subseteq \{1, 2, ..., n\}$ are the coarse sample points in $\mathbb{H}$, the measurements $\{\langle x, e_i\rangle : i \in \Omega\}$ have insufficient information in general to recover the original signal x. Given an operator A on $\mathbb{H}$, suppose the signal $x \in \mathbb{H}$ evolves through the operator A over time to become $A^\ell x$ at time ℓ. Now, we can have extra information $\{A^\ell x(i) : i \in \Omega\}$ about the signal x. In [6], Aldroubi and his collaborators recently showed that x can be recovered from the measurements of $\{\langle A^\ell x, e_i\rangle : \ell = 0, 1, \ldots, L;\ i \in \Omega\}$ if and only if the time-space samples is a set of frame vectors.

In this article, We will look at how these two most recent advancements in frame theory cross. We will attempt to demonstrate when norm retrieval is feasible under the unitary and the Jordan operators using samples obtained from the dynamical sampling structure. We consider the norm retrieval problem in the dynamical sampling setting in the finite-dimensional real Hilbert space $\mathbb{R}^n$.

2. Preliminaries

In this section, we provide some of the terminology and findings in frame theory, phase retrieval, norm retrieval, and dynamical sampling that are essential to understanding the conclusions we reach.

2.1. Frames.

Definition 1. *[23] A set of vectors $\{x_i\}_{i\in I}$ is said to be a **frame** in a Hilbert space $\mathbb{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that*

$$A||x||^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B||x||^2, \text{ for all } \quad x \in \mathbb{H}. \tag{1}$$

A and B are called upper and lower frame bounds of the frame $\{x_i\}_{i\in I}$, respectively. If $A = B$, then $\{x_i\}_{i\in I}$ is called a **tight frame**. The set of vectors $\{x_i\}_{i\in I}$ is called a **Parseval frame** if $A = B = 1$.

Let $\{e_i\}_{i\in I}$ be the standart orthonormal basis in $\ell^2(I)$. Given the set $\{x_i\}_{i\in I}$ in $\mathbb{H}$, the operator $\Phi : \mathbb{H} \to \ell^2(I)$, which is generated from the set $\{x_i\}_{i\in I}$,

$$\Phi(x) = \sum_{i\in I} \langle x, x_i\rangle e_i \quad \text{for all} \quad x \in \mathbb{H} \tag{2}$$

is called the **analysis operator** associated with the set $\{x_i\}_{i\in I}$.

The **synthesis operator** is the adjoint $\Phi^* : \ell^2(I) \to \mathbb{H}$ of the analysis operator Φ and is defined by

$$\Phi^* : \ell^2(I) \to \mathbb{H}, \quad \Phi^*((c_i)_{i\in I}) = \sum_{i\in I} c_i x_i. \tag{3}$$

The operator $S = \Phi^*\Phi : \mathbb{H} \to \mathbb{H}$ is called the **frame operator** of the frame $\{x_i\}_{i\in I}$ and is defined by

$$S(x) = \Phi^*\Phi(x) = \sum_{i\in I} \langle x, x_i\rangle x_i. \tag{4}$$

The operator S is a bounded, self adjoint, positive, and invertible operator that satisfies the operator inequality $AI \leq S \leq BI$ where A and B signify the upper and lower frame limits and I denotes the identity operator on $\mathbb{H}$. A frame is complete if it meets the lowest frame criteria. On the other hand, the upper frame condition requires a well-defined analysis operator.

Definition 2. *[25] Given a frame $\{x_i\}_{i\in I}$ in $\mathbb{H}$, if there are scalars $\{c_i\}_{i\in I}$ such that $\{c_i x_i\}_{i\in I}$ is a Parseval frame, then the frame $\{x_i\}_{i\in I}$ for a Hilbert space $\mathbb{H}$ is said to be scalable. If there is a value of $\delta > 0$, such that $c_i > \delta$ for all $i \in I$, then the set $\{x_i\}_{i\in I}$ is known as a strictly scalable frame.*

2.2. **Phase Retrieval and Norm Retrieval.** For the given set $\{x_i\}_{i\in I}$ in $\mathbb{H}$, the reconstruction of x up to a constant phase from the absolute value of the inner product of the coefficients measurements $\{\langle x, x_i\rangle\}_{i\in I}$ is called phase retrieval which defined by Balan, Casazza, and Edidin in [11].

Applications, where measurements of a signal can only identify by amplitude rather than the phase, are included in speech recognition [30], optics applications like X-ray crystallography [20, 29], quantum state tomography [28], and electron microscopy [27, 31]. Phase retrieval problem has been extensively studied in [10–13, 15–18, 24].

Definition 3. *[11] A collection of vectors $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ is called **phase retrieval** if for all $x, y \in \mathbb{R}^n$ which satisfies $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i = 1, .., M$, then $x = cy$ where $c = \pm 1$ in $\mathbb{R}^n$.*

Definition 4. *[10] A collection of vectors $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ is called **norm retrieval** if for all $x, y \in \mathbb{R}^n$ which satisfies $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i = 1, .., M$, then $||x|| = ||y||$.*

Lemma 1. *[17] In $\mathbb{R}^n$, if the number of n vectors $\{x_i\}_{i=1}^n$ do norm retrieval, they have to be orthogonal to each other.*

There is also the idea of phase and norm retrieval by projections, which is agree with our earlier definitions when the projections are one-dimensional.

Definition 5. *[10] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^M$ (or $\{P_i\}_{i=1}^M$) yields **phase retrieval** if for $x, y \in \mathbb{R}^n$ satisfying $||P_i x|| = ||P_i y||$ for all $i = 1, ..., M$, then $x = cy$ for some scalar c with $c = \pm 1$.*

Definition 6. *[10] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^M$ (or $\{P_i\}_{i=1}^M$) yields **norm retrieval** if for $x, y \in \mathbb{R}^n$ satisfying $||P_i x|| = ||P_i y||$ for all $i = 1, ..., M$, then $||x|| = ||y||$.*

Definition 7. *[11] A frame $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ satisfies the **complement property** if for any index set $I \subset \{1, \dots M\}$, either span$\{x_i\}_{i\in I} = \mathbb{R}^n$ or span$\{x_i\}_{i\in I^c} = \mathbb{R}^n$.*

Theorem 1. *[11] A frame $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ yields phase retrieval if and only if it has the complement property.*

2.3. **Dynamical Sampling.** Given a bounded operator A, a vector $b \in \mathbb{H}$ and $\ell \in \mathbb{N}$, we can get a collection of vectors $\{b, Ab, A^2b, ...A^\ell b\}$ by applying the operator A to the vector b. The dynamical sampling problem which defined by Aldroubi, Davis, and Krishtal in [7] is looking for the conditions on the set of vectors $\{b_i \in \mathbb{H} : i \in \Omega, |\Omega| < dim(\mathbb{H})\}$, the operator A and $\ell_i \in \mathbb{N}$ such that the collection of vectors

$$\{b_i, Ab_i, ..., A^{\ell_i} b_i\}_{\{i\in\Omega, \ell_i\in\mathbb{N}\}}$$

is a frame in $\mathbb{H}$. In 2012, Aldroubi and his collaborators created a mathematical system for a dynamical sampling structure with results appearing in [5, 6]. The dynamical sampling problem gets the attention of other researchers and has been recently studied by [1–4, 8, 9, 14, 22, 26].

Let A be a matrix that can be written as $A^* = B^{-1}DB$ where D is a diagonal and B is an invertible matrix. Let $\{\lambda_j\}_{j\in J}$ be distinct eigenvectors of D and $\{P_j\}_{j\in J}$ denote the orthogonal projections in $\mathbb{H}$ onto the eigenspaces $\{E_j\}_{j\in J}$ of D associated to the eigenvalues $\{\lambda_j\}_{j\in J}$. Then we have the following result.

Theorem 2. *[6, Thm: 2.2] Let $\Omega \subseteq \{1, 2, \dots, n\}$ and $\{b_i : i \in \Omega\}$ be vectors in $\mathbb{R}^n$. Let D be a diagonal matrix and r_i be the degree of the D-annihilator of b_i. Then $\{D^j b_i : i \in \Omega; j = 0, 1, ..., l_i; l_i = r_i - 1\}$ is a frame of $\mathbb{R}^n$ if and only if $\{P_j(b_i) : i \in \Omega\}$ is a frame of E_j for all $j \in J$.*

The authors of [6] extended the Theorem 2 to non-diagonalizable operators as follows.

Theorem 3. *[6, Thm 2.6] Let J be a matrix in Jordan form as in 9. Let $\Omega \subseteq \{1, 2, \dots, n\}$ and $\{b_i : i \in \Omega\}$ be vectors in $\mathbb{R}^n$, r_i be the degree of the J-annihilator of the vector b_i and $l_i = r_i - 1$. Then the following propositions are equivalent.*

1. *The set of vectors $\{J^j b_i : i \in \Omega,\ j = 0, 1, ..., l_i,\}$ is a frame for $\mathbb{R}^n$.*
2. *For every $s = 1, .., n$, $\{P_s(b_i) : i \in \Omega\}$ is a frame for W_s.*

3. Results

We first start with creating a standard dynamical sampling system in $\mathbb{R}^n$ using a bounded linear operator A. Assume that the vector $b \in \mathbb{R}^n$ evolves through the operator A to become the vector $A^\ell b$ at time $\ell \in \mathbb{N}$. Let $\Omega \subseteq \{1, 2, ..., n\}$ be the sample points and define $A^\ell W = span\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega\}$.

In [14], we show the construction of norm retrievable sets $\{A^\ell b_i\}_{\{\ell=0,1,...M, i\in\Omega\}}$ that arise from a dynamical sampling system in a finite-dimensional real Hilbert space $\mathbb{R}^n$. In this paper, we show the relations between the norm retrievable set of $\{A^\ell b_i\}_{\{\ell=0,1,...M, i\in\Omega\}}$ in $\mathbb{R}^n$ and the set of projections $\{P_\ell\}_{\{\ell=0,1,...M\}}$ under the unitary and Jordan operator, where P_ℓ is the orthogonal projection onto the subspace $A^\ell W$.

First, we show that the collection of vectors $\{A^\ell b_i\}_{\{\ell=0,1,...M, i\in\Omega\}}$ is norm retrievable in $\mathbb{R}^n$ if the identity operator on $\mathbb{R}^n$ is in the spanning set of the rank one projection of the vectors $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$ as shown in the following Theorem 4.

Theorem 4. *Let A be a bounded linear operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$ be a collection of vectors in $\mathbb{R}^n$. The collection of vectors $\{A^\ell b_i\}_{\{0\leq\ell\leq M, i\in\Omega\}}$ accomplishes norm retrieval condition in $\mathbb{R}^n$ for some $M \in \mathbb{N}$ if there exists a solution $\{C_{\ell,i}\}_{\{0\leq\ell\leq M, i\in\Omega\}}$ to the following system of linear equations*

$$\sum_{\ell,i} C_{\ell,i} |\langle e_j, A^\ell b_i\rangle|^2 = 1 \tag{5}$$

$$\sum_{\ell,i} C_{\ell,i} \langle e_j, A^\ell b_i\rangle \langle e_k, A^\ell b_i\rangle = 0 \tag{6}$$

for all $j, k = 1, 2, ..n$ with $j \neq k$.

Proof. Assume that given the operator A on $\mathbb{R}^n$ and the collection of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega, |\Omega| < n\}$, the measurements, $|\langle x, A^\ell b_i\rangle| = |\langle y, A^\ell b_i\rangle| \quad \forall\, 0 \leq \ell \leq M, i \in \Omega$ for fixed $x, y \in \mathbb{R}^n$, are known. Then we have

$$\langle x - y, A^\ell b_i\rangle = 0 \quad \text{or} \quad \langle x + y, A^\ell b_i\rangle = 0 \quad \forall \ell, i$$

and

$$\langle x - y, \langle x + y, A^\ell b_i\rangle A^\ell b_i\rangle = \langle x - y, A^\ell b_i (A^\ell b_i)^*(x + y)\rangle = 0 \quad \forall \ell, i.$$

Given any scalar value $C_{\ell,i}$, we have $C_{\ell,i}\langle x - y, A^\ell b_i (A^\ell b_i)^*(x + y)\rangle = 0 \quad \forall \ell, i.$

If $I \in \text{span}\{A^\ell b_i (A^\ell b_i)^*\}_{\{0\leq\ell\leq M, i\in\Omega\}}$, then $\langle x - y, x + y\rangle = 0$ and $||x|| = ||y||$.

Now, we show that $I \in \text{span}\{A^\ell b_i (A^\ell b_i)^*\}_{\{0\leq\ell\leq M, i\in\Omega\}}$ if and only if the equations (5) and (6) have a solution. Let $\{e_j\}_{j=1}^n$ be the standard orthonormal bases in $\mathbb{R}^n$. Then, we can express any vector $A^\ell b_i \in \mathbb{R}^n$ as

$$A^\ell b_i = \begin{bmatrix} \langle e_1, A^\ell b_i\rangle \\ \langle e_2, A^\ell b_i\rangle \\ \vdots \\ \langle e_n, A^\ell b_i\rangle \end{bmatrix}.$$

Hence, we have

$$A^\ell b_i (A^\ell b_i)^* = \begin{bmatrix} |\langle e_1, A^\ell b_i\rangle|^2 & \langle e_1, A^\ell b_i\rangle\langle e_2, A^\ell b_i\rangle & \cdots \langle e_1, A^\ell b_i\rangle\langle e_n, A^\ell b_i\rangle \\ \langle e_2, A^\ell b_i\rangle\langle e_1, A^\ell b_i\rangle & |\langle e_2, A^\ell b_i\rangle|^2 & \cdots \langle e_2, A^\ell b_i\rangle\langle e_n, A^k b_i\rangle \\ \vdots & \vdots & \vdots \\ \langle e_n, A^\ell b_i\rangle\langle e_1, A^\ell b_i\rangle & \langle e_n, A^\ell b_i\rangle\langle e_2, A^{\ell_i} b_i\rangle & \cdots |\langle e_n, A^\ell b_i\rangle|^2 \end{bmatrix}.$$

The linear equation systems in (5) and (6) have a solution if and only if the identity operator $I \in \text{span}\{A^\ell b_i (A^\ell b_i)^*\}_{\{\ell=0,1,\ldots M\,,i\in\Omega\}}$. If so, we also have the collection of vectors $\{A^\ell b_i\}_{\{\ell=0,1,\ldots M,\, i\in\Omega\}}$ which does norm retrieval in $\mathbb{R}^n$ as demonstrated in the following example. □

Example 1. *Let*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then the set

$$F = \{b, Ab, A^2b, A^3b\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

contains an orthogonal basis. Hence, it does norm retrieval. Since the number of vectors is less than 5, it does not do phase retrieval in $\mathbb{R}^3$. Note that the span of the rank one operators generated by the vectors $\{b, Ab, A^2b, A^3b\}$ contains the identity operator.

$$bb^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Ab(Ab)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2b(A^2b)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A^3b(A^3b)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$I = bb^* + \frac{1}{2}A^2b(A^2b)^* + \frac{1}{2}A^3b(A^3b)^*.$$

When A is an $n \times n$ diagonal operator, the authors in [3, Thm.3] showed that the set of vectors $\{A^\ell b_i\}_{0\leq \ell \leq M, i\in\Omega\}}$ is a scalable frame if and only if there exists a positive solution $\{C_{\ell,i}\}_{0\leq \ell\leq M, i\in\Omega\}}$ to the system of equations in (5) and (6). Theorem 4 illustrates that if the solution $\{C_{\ell,i}\}_{0\leq\ell\leq M,i\in\Omega\}}$ to the system of linear equations in (5) and (6) is not a positive solution, there exist norm retrievable frames $\{A^\ell b_i\}_{0\leq\ell\leq M,i\in\Omega\}}$ which are not scalable frames.

Theorem 4 does not give the conditions on the operator A, the set of sample points $\{b_i \in \mathbb{R}^n : i \in \Omega,\ |\Omega| < n\}$ and the time increments ℓ but we show later sections how it works to obtain dynamical sampling frame which does norm retrieval.

Theorem 5. *[10] Given a collection of vectors $\{x_i\}_{i=1}^M$ in a Hilbert space $\mathbb{H}^n$. The following statements are equivalent to each other.*
(1) *The set of vectors $\{x_i\}_{i=1}^M$ is phase retrievable in $\mathbb{H}^n$*
(2) *The set of vectors $\{Ax_i\}_{i=1}^M$ is phase retrievable for all invertible operator A on $\mathbb{H}^n$*
(3) *The set of vectors $\{Ax_i\}_{i=1}^M$ is norm retrievable for all invertible operator A on $\mathbb{H}^n$.*

Given the collection of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ in $\mathbb{R}^n$. Let $W = span\{b_i \in \mathbb{R}^n; i \in \Omega\}$. For every $\ell \in \mathbb{N}$, the subspaces, which are generated by iteration of W under the operator A, can be defined as

$$A^\ell W = span\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\} \subset \mathbb{R}^n.$$

Let $\{P_\ell\}$ be the orthogonal projections from $\mathbb{R}^n$ onto $A^\ell W$ for each $\ell \in \mathbb{N}$. Theorem 5 states that if the collection of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ is phase retrievable in W, then the collection of vectors $\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ is phase retrievable in $A^\ell W$ for every $\ell \in \mathbb{N}$ when A is an invertible operator on $\mathbb{R}^n$. Assume there exists an $M \in \mathbb{N}$ such that $\mathbb{R}^n = span\{A^\ell b_i\}_{\{i\in\Omega,\ \ell=0,1,\ldots M\}}$. The collection of vectors $\{A^\ell b_i\}_{\{i\in\Omega\}}$ satisfies phase retrieval in $A^\ell W$ for every $\ell = 0, 1, \ldots M$ but it does not imply that $\{A^\ell b_i\}_{\{i\in\Omega\ \ell=0,1,\ldots M\}}$ is phase retrievable in $\mathbb{R}^n$.

Example 2. *Define* $W = span\{e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_1 + e_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}\}$.

Let A be an invertible operator on $\mathbb{R}^3$ such that $Ae_1 = e_2$ and $Ae_2 = e_3$. The iteration of the subspace W under A can be shown as

$$AW = span\{e_2, e_3, e_2 + e_3\}.$$

The collection of vectors in $\{e_1, e_2, e_1+e_2\}$ and $\{e_2, e_3, e_2+e_3\}$ is phase retrievable in W and AW, respectively. On the other hand, the collection of vectors $\{e_1, e_2, e_3, e_1+e_2, e_2+e_3\}$ is not phase retrievable because when we get the partion $\{e_1, e_2, e_1+e_2\}$ and $\{e_3, e_2+e_3\}$ of the set $\{e_1, e_2, e_3, e_1+e_2, e_2+e_3\}$, neither of these sets spans $\mathbb{R}^3$. This implies that the collection of vectors $\{e_1, e_2, e_3, e_1+e_2, e_2+e_3\}$ does not satisfy the complementary property (Definition 7) and fails the phase retrieval condition in R^3.

Theorem 6. *Let the set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ is phase retrievable in $W \subset \mathbb{R}^n$ and A is an invertible operator on $\mathbb{R}^n$. The collection of vectors $\{A^\ell b_i\}_{\{i\in\Omega\, \ell=0,1,\ldots M\}}$ is norm retrievable in $\mathbb{R}^n$ if the set of projections $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$, where P_ℓ is the orthogonal projection onto the subspace $A^\ell W = span\{\{A^\ell b_i\}_{i\in\Omega}\}$.*

Proof. For $x, y \in \mathbb{R}^n$, assume $|\langle x, A^\ell b_i\rangle| = |\langle y, A^\ell b_i\rangle|$ for all $i \in \Omega$, $\ell = 0, 1, \ldots M$. Let P_ℓ be the orthogonal projection onto the subspace $A^\ell W$ for each ℓ. Thus,

$$P_\ell A^\ell b_i = A^\ell b_i \quad \text{and} \quad |\langle P_\ell x, P_\ell A^\ell b_i\rangle| = |\langle P_\ell y, P_\ell A^\ell b_i\rangle|, \text{ for all}\, i \in \Omega.$$

According to Theorem 5, the set of vectors $\{A^\ell b_i\}_{i\in\Omega}$ is phase retrievable (and consequently norm retrievable) in $A^\ell W$ for all ℓ since A is an invertible operator and the collection of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ performs phase retrieval in W. This states that $||P_\ell x|| = ||P_\ell y||$ for all $\ell = 0, 1, \ldots M$. By our supposition, $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$ and we have $||x|| = ||y||$.

□

3.1. **Iteration of Subspaces Under the Unitary and Jordan Operator.** We can do norm retrieval more smoothly if our dynamical sampling operator is unitary.

Given the index set $\Omega \subset \{1, 2, \ldots, n\}$ and the orthonormal bases $\{e_i\}_{i=1}^n$ of R^n. Suppose U is a unitary operator on R^n. Let $W = span\{e_i : i \in \Omega, |\Omega| < n\}$ and $U^\ell W = span\{U^\ell e_i : i \in \Omega, |\Omega| < n\}$ for any $\ell \in \mathbb{N}$. Given any $\ell \in \mathbb{N}$, since U is a unitary operator, it preserves the inner product. Thus, we have $\langle U^\ell e_i, U^\ell e_k\rangle = \langle e_i, e_k\rangle = 0$ for all $i \neq k$. Which says that $\{U^\ell e_i\}_{i\in\Omega}$ is an orthonormal basis for $U^\ell W$ for each ℓ.

Lemma 2. *Suppose $W = span\{e_i : i \in \Omega, |\Omega| \leq n\}$ and $U^\ell W = span\{U^\ell e_i : i \in \Omega, |\Omega| < n\}$ for $\ell \geq 0$, where U is a unitary operator on $\mathbb{R}^n$ and $\{e_i\}_{i=1}^n$ is an orthonormal bases of R^n. Let P_ℓ be the orthogonal projection onto $U^\ell W$ for each ℓ. If the collection of projections $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$, then the collection of vectors $\{U^\ell e_i\}_{\{i\in\Omega,\ell=0,1,..M\}}$ is norm retrievable in $\mathbb{R}^n$.*

Proof. For $x, y \in \mathbb{R}^n$, assume that $|\langle x, U^\ell e_i\rangle| = |\langle y, U^\ell e_i\rangle|$ for any $i \in \Omega$ and $\ell = 0, 1, ..M$. Since $U^\ell e_i \in U^\ell W$ for any $\ell = 0, 1, ..M$, we see that $P_\ell U^\ell e_i = U^\ell e_i$ and

$$\begin{aligned} |\langle x, U^\ell e_i\rangle| = |\langle y, U^\ell e_i\rangle| &\Longrightarrow |\langle x, P_\ell U^\ell e_i\rangle| = |\langle y, P_\ell U^\ell e_i\rangle| \\ &\Longrightarrow |\langle P_\ell x, U^\ell e_i\rangle| = |\langle P_\ell y, U^\ell e_i\rangle|. \end{aligned}$$

For each fixed ℓ, since P_ℓ is a projection on $U^\ell W$ and $\{U^\ell e_i\}_{i\in\Omega}$ is an orthonormal basis in $U^\ell W$, we have

$$||P_\ell x|| = \sum_{i\in\Omega} | < P_\ell x, U^\ell e_i > |^2 = \sum_{i\in\Omega} | < P_\ell y, U^\ell e_i > |^2 = ||P_\ell y|| \tag{7}$$

By assumption, since the collection of projections $\{P_\ell\}_{\ell=0}^M$ is norm retrievable in $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

In Lemma 2, the collection of vectors $\{U^\ell e_i : i \in \Omega,\ |\Omega| < n\}$ in $U^\ell W$ is orthonormal because U is a unitary operator. Obtaining orthonormal bases as sample sets is a strong condition but we can reduce this presumption as the following lemma.

Corollary 1. *Let U be a unitary operator on $\mathbb{R}^n$. For a collection of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega,\ |\Omega| < n\}$ in $\mathbb{R}^n$, define $W = span\{b_i \in \mathbb{R}^n : i \in \Omega,\ |\Omega| < n\}$ and$U^\ell W = span\{U^\ell b_i : i \in \Omega,\ |\Omega| <\}$ for $\ell \in \mathbb{N}$. Let P_ℓ be the orthogonal projection onto $U^\ell W$ for $\ell \in \mathbb{N}$. If the collection of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ is norm retrievable in W and the set of projections $\{P_\ell\}_{\{\ell=0,1,\ldots M\}}$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$, then the collection of vectors $\{U^\ell b_i\}_{\{i\in\Omega,\ell=0,1,\ldots M\}}$ is norm retrievable in $\mathbb{R}^n$.*

Proof. For $x, y \in \mathbb{R}^n$, assume that $|\langle x, U^j b_i\rangle| = |\langle y, U^j b_i\rangle| \quad \forall i \in \Omega,\ \ell = 0, 1, ..M$. For each fixed ℓ, since the unitary operators preserve norm retrieval condition, the collection of vectors $\{U^\ell b_i : i \in \Omega,\ |\Omega| < n\}$ is norm retrievable in $U^\ell W$. This says that for any given $x, y \in \mathbb{R}^n$ and $\ell \in \mathbb{N}$, $|\langle x, U^j b_i\rangle| = |\langle y, U^j b_i\rangle|$, $\forall i \in \Omega$ implies that $||P_\ell x|| = ||P_\ell y||$. Since we assumed that the set of projections $\{P_\ell\}_{\{\ell=0,1,\ldots M\}}$ is norm retrievable in $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

In Corollary 1 and Lemma 2, we assumed that the set of projections $\{P_\ell\}_{\{\ell=0,1,\ldots M\}}$ is norm retrievable in $\mathbb{R}^n$ for some $M \in \mathbb{N}$. In general, we do not know whether such an $M \in \mathbb{N}$ exists or not. Now that we have a condition, we can guarantee that the projection set $\{P_\ell\}_{\{\ell=0,1,\ldots M\}}$ performs norm retrieval on $\mathbb{R}^n$. We need the definition of fusion frames defined in [19].

Definition 8. *[19] Let I be an index set and $\{v_i\}_{i\in I}$ be a family of weights. That is $v_i > 0$ for all $i \in I$. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space $\mathbb{H}$ and P_{Wi} is the orthogonal projection onto the subspace Wi for each $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a **fusion frame** for $\mathbb{H}$, if there exists constants $0 < A \leq B < \infty$ such that*

$$A||x||^2 \leq \sum_{i\in I} {v_i}^2 ||P_{W_i}(x)||^2 \leq B||x||^2, \text{ for all } \quad x \in \mathbb{H}. \tag{8}$$

*A and B are called the fusion frame bounds. The family (W_i, v_i) is called a **Parseval fusion frame** if $A = B = 1$ and a **tight fusion frame** if $A = B$.*

Theorem 7. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ be a set of orthonormal vectors in $\mathbb{R}^n$. The set of vectors $\{U^\ell b_i : i \in \Omega, \quad \ell = 0, 1, \ldots M\}$ is a tight frame in $\mathbb{R}^n$ if and only if the set of orthogonal projections $\{P_\ell\}_{\{\ell=0,1,\ldots M\}}$ is a tight fusion frame with weights $v_\ell = 1$ for all ℓ, where P_ℓ is the orthogonal projection onto $U^\ell W$ for each ℓ.*

Proof. ($\Longrightarrow$) Suppose the set of vectors $\{U^\ell b_i : i \in \Omega, \ell = 0, 1, ...M\}$ does tight frame in $\mathbb{R}^n$ with frame bound $A > 0$. Then, given any $x \in \mathbb{R}^n$, we can write

$$||x||^2 = \frac{1}{A} \sum_{i\in\Omega,\ell=0,1,\dots,M} |\langle x, U^\ell b_i\rangle|^2.$$

Since $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ is a set of orthonormal vectors in $\mathbb{R}^n$ and U is a unitary operator on $\mathbb{R}^n$, $\{U^\ell b_i : i \in \Omega\}$ is also orthonormal set of vectors in $U^\ell W$ for each ℓ. Hence, the orthogonal projection P_ℓ onto the subspace $U^\ell W = span\{U^\ell\ b_i : i \in \Omega\}$ can be written as

$$P_\ell(x) = \sum_{i\in\Omega} \langle x, U^\ell b_i\rangle U^\ell b_i.$$

Thus,

$$||x||^2 = \frac{1}{A} \sum_{i\in\Omega,\ell=0,1,\dots,M} |\langle x, U^\ell b_i\rangle|^2 = \frac{1}{A} \sum_{\ell=0,1,\dots,M} ||P_\ell(x)||^2$$

and the set of orthogonal projections $\{P_\ell\}_{\{\ell=0,1,\dots,M\}}$ is a A -tight fusion frame with weights $v_\ell = 1$.

($\Longleftarrow$) These follow from the definition of a tight fusion frame with weights $v_\ell = 1$ for all ℓ.

□

If $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ is a set of vectors that are orthogonal but not orthonormal in $\mathbb{R}^n$, then the set $\{U^\ell b_i : i \in \Omega, \quad \ell = 0, 1, ...M\}$ is not necessarily a tight frame in $\mathbb{R}^n$ anymore. In this case, we have the following corollary that follows from Theorem 5, Lemma 1 and Corollary 1.

Corollary 2. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ consists of orthogonal vectors in $\mathbb{R}^n$. The set of vectors $\{U^\ell b_i : i \in \Omega, \ell = 0, 1, ...M\}$ is norm retrievable in $\mathbb{R}^n$ if $x \in span\{P_\ell(x)\}_{\ell=0}^M$, for any $x \in \mathbb{R}^n$.*

Now, we are interested in the linear operator A on $\mathbb{R}^n$ that has all real eigenvalues and is unitarily similar to the Jordan form. We want to construct subspaces $A^\ell W$ in $\mathbb{R}^n$ which are not necessarily orthogonal to each other and show the relations between the norm retrievable set of vectors $\{A^\ell b_i\}_{\{\ell=0,1,\dots M, i\in\Omega\}}$ in $\mathbb{R}^n$ and the set of projections $\{P_\ell\}_{\{\ell=0,1,\dots M\}}$, where P_ℓ is the orthogonal projection onto the subset $A^\ell W$. To set up the following construction, we apply the notation from [6].

Suppose $J \in M_n(\mathbb{R})$ is a Jordan matrix that has all real eigenvalues, then we have

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix}. \tag{9}$$

For each $j = 1, 2, ..s$, $J_j = \lambda_j I_j + N_j$ where I_j is an $r_j \times r_j$ identity matrix and N_j is a $r_j \times r_j$ nilpotent block-matrix of the form

$$N_j = \begin{pmatrix} N_{j_1} & 0 & \cdots & 0 \\ 0 & N_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{j_i} \end{pmatrix}. \tag{10}$$

Each N_{ji} is a $r_j^i \times r_j^i$ cyclic nilpotent matrix of the form

$$N_{ji} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{11}$$

with $r_j^1 \geq r_j^2 \geq ... \geq r_j^i$ and $r_j^1 + r_j^2 + ... + r_j^i = r_j$. The matrix J has distinct eigenvalues λ_j, $j = 1, 2, ..s$ and $r_1 + r_2 + ... + r_s = n$.

Before we state our theorem related to the Jordan form, we would like to give an illustrative example to interested readers.

Example 3. *Let $J = \lambda I + N \in \mathbb{R}^{4\times 4}$ and assume that*

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

where

$$N_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for $i = 1, 2$. Then, we have the subspaces

$$\begin{aligned} W &= span\{e_1, e_3\} \\ JW &= span\{\lambda e_1 + e_2, \lambda e_3 + e_4\} \\ J^2W &= span\{\lambda^2 e_1 + 2\lambda e_2, \lambda^2 e_3 + 2\lambda e_4\} \end{aligned}$$

Let P_ℓ be the orthogonal projection onto the subspace $J^\ell W$ for each $\ell = 0, 1, 2$. For fixed ℓ,

$$||J^\ell e_1||^2 = \lambda^{2\ell} + \ell^2 \lambda^{2(\ell-1)} = ||J^\ell e_3||^2.$$

Let $c_\ell = \lambda^{2\ell} + \ell^2\lambda^{2(\ell-1)}$ *for* $\ell = 0, 1, 2$*, then the orthogonal projection* P_ℓ *onto the subspace* $J^\ell W$ *can be written as*

$$P_\ell(x) = \frac{1}{c_\ell}\sum_{i=1,3}\langle x, J^\ell e_i\rangle J^\ell e_i \quad \textit{and} \quad ||P_\ell(x)||^2 = \frac{1}{c_\ell}\sum_{i=1,3}|\langle x, J^\ell e_i\rangle|^2.$$

If $\lambda = 0$*, then* $P_0 + P_1 = I$ *and the set of vectors* $\{J^\ell e_i\}_{i=1,3,\ell=0,1,2} = \{e_1, e_2, e_3, e_4\}$ *is an orthonormal bases and it does norm retrieval in* $\mathbb{R}^n$*. Assume* $\lambda \neq 0$*.*

$$\textit{For any } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^n, c_0P_0(x) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix}$$

$$c_1P_1(x) = \begin{bmatrix} \lambda^2 x_1 + \lambda x_2 \\ \lambda x_1 + x_2 \\ \lambda^2 x_3 + \lambda x_4 \\ \lambda x_3 + x_4 \end{bmatrix} \textit{ and } c_2P_2(x) = \begin{bmatrix} \lambda^4 x_1 + 2\lambda^3 x_2 \\ 2\lambda^3 x_1 + 4\lambda^2 x_2 \\ \lambda^4 x_3 + 2\lambda^3 x_4 \\ 2\lambda^3 x_3 + 4\lambda^2 x_4 \end{bmatrix}.$$

This states that $\frac{\lambda^4+1}{2\lambda^2}c_0P_0 - c_1P_1 + \frac{1}{\lambda^2}c_2P_2 = I$ *and the set of projections* $\{P_\ell\}_{\ell=0,1,2}$ *does norm retrieval in* $\mathbb{R}^n$ *since the coefficients* $\{c_\ell\}_{\ell=0,1,2}$ *are independent from choice of* x*. This implies that the set of vectors* $\{J^\ell e_i\}_{i=1,3,\ell=0,1,2}$ *does norm retrieval in* $\mathbb{R}^n$*.*

Theorem 8. *Let* $J \in M_n(\mathbb{R})$ *be a Jordan matrix in the form of Equation* (9) *that has all real eigenvalues and* $W_j = span\{e_{k_{ji}} : j = 1, 2, ..., s\}$*, where* s *is the number of distinct eigenvalues in* J *and* $e_{k_{ji}}$ *is the standard orthonormal bases vector of* $\mathbb{R}^n$ *corresponding to the first row of the cyclic nilpotent matrix* N_{ji} *in* (11)*. Let* $P_{\ell j}$ *be the orthogonal projections onto the subsets* $J^\ell W_j$*. Suppose the order* r_j^i *of* N_{ji} *is the same for all* i, j*. Then the collection of vectors* $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\le i\le k(j),\ell=0,1,...r_j^i\}}$ *is norm retrievable in* $\mathbb{R}^n$*, where* $k(j)$ *is the number of cyclic nilpotent matrices* N_{ji} *in* N_j *if the set of projections is norm retrievable* $\mathbb{R}^n$*.*

Proof. By choice of $e_{k_{ji}}$ as a standard orthonormal basis vector corresponding to the first row of N_{ji}, the set of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\le i\le k(j)\}}$ forms an orthogonal bases in $J^\ell W_j$ for each ℓ. As shown in Example 3, for fixed ℓ, the norm of the vectors $J^\ell e_{k_{ij}}$ is the same for all i, j. Suppose $||J^\ell e_{k_{ij}}|| = c_{\ell j}$ for some $c_{\ell j} \in \mathbb{R}$. Since the set of vectors $\{J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\le i\le k(j)\}}$ forms an orthogonal basis in $J^\ell W_j$ for each ℓ, the set of vectors $\{\frac{1}{c_{\ell j}}J^\ell e_{k_{ji}}\}_{\{j=1,2,...,s,1\le i\le k(j)\}}$ forms an orthonormal bases in $J^\ell W_j$ for each ℓ. For fixed ℓ, the orthogonal projection $P_{\ell j}$ onto $J^l W_j$ can be defined by

$$P_{\ell j}(x) = \sum_{i,j}\langle x, \frac{1}{c_{\ell j}}J^\ell e_{k_{ji}}\rangle \frac{1}{c_{\ell i}}J^\ell e_{k_{ji}}.$$

This implies $\{J^l e_{k_{ij}}\}$ does norm retrieval in $\mathbb{R}^n$ if and only if $I = \sum_{\ell} c_{\ell,i} P^i_{\ell}$. Since the constants $c_{\ell j}$ is same for fixed ℓ, for any $x \in \mathbb{R}^n$, we have

$$||P_{\ell j}(x)||^2 = \frac{1}{c_{\ell j}^2} \sum_{i,j} |\langle x, J^{\ell} e_{k_{ji}} \rangle|^2.$$

To show that the set of vectors $\{J^{\ell} e_{k_{ji}}\}_{\{j=1,2,\ldots,s, 1\le i\le k(j), \ell=0,1,\ldots r^i_j\}}$ is norm retrievable in $\mathbb{R}^n$, assume $|\langle x, J^{\ell} e_{k_{ji}}\rangle| = |\langle y, J^{\ell} e_{k_{ji}}\rangle|$ for all ℓ, j, i for any given $x, y \in \mathbb{R}^n$. Since the constants $c_{\ell j}$ are independent of the choice of x and y, we have

$$||P_{\ell j}(x)||^2 = \frac{1}{c_{\ell j}^2} \sum_{i,j} |\langle x, J^{\ell} e_{k_{ji}} \rangle|^2 = \frac{1}{c_{\ell j}^2} \sum_{i,j} |\langle y, J^{\ell} e_{k_{ji}} \rangle|^2 = ||P_{\ell j}(y)||^2.$$

We assumed that the set of orthogonal projections $\{P^{\ell} e_{k_{ji}}\}_{\{j=1,2,\ldots,s, \ell=0,1,\ldots r^i_j\}}$ is norm retrievable in $\mathbb{R}^n$. This implies that $||x|| = ||y||$ and the collection of vectors $\{J^{\ell} e_{k_{ji}}\}_{\{j=1,2,\ldots,s, 1\le i\le k(j), \ell=0,1,\ldots r^i_j\}}$ is norm retrievable in $\mathbb{R}^n$.

□

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