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On Characterization of Smarandache Curves Constructed by Modified Orthogonal Frame

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Abstract
In this study, we investigate Smarandache curves constructed by a space curve with a modified orthogonal frame. Firstly, the relations between the Frenet frame and the modified orthogonal frame are summarized. Later, the Smarandache curves based on the modified orthogonal frame are obtained. Finally, the tangent, normal, binormal vectors and the curvatures of the Smarandache curves are determined. A special curve known as the Gerono lemniscate curve whose curvature is not differentiable, the principal normal and binormal vectors are discontinuous at zero point is considered as an example and the Smarandache curves of this curve are obtained by the aid of its modified orthogonal frame, and their graphics are given.

Keywords: *Gerono lemniscate curve, Modified orthogonal frame, Smarandache curves*

AMS Subject Classification (2020): *53A04; 14H50*

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1. Introduction

Curve theory is one of the most important and interesting research topics of differential geometry. Many studies have been done about curves in the scientific world and the characterizations of curves have been examined by considering different spaces. Even in prehistoric times, curves seem to have an important place in the fields of art and decoration. Curves are used frequently in many related fields such as computer graphics, animation, and modeling. In this study, we investigated the Smarandache curves using the modified orthogonal frame to give a new perspective to curves. The Smarandache curves are characterized using different frames in Euclidean and non-Euclidean spaces [1–9]. The Smarandache curves obtained from spacelike Salkowski and anti-Salkowski curves are given by Eren and Şenyurt in Minkowski space [10–13]. Also, the Smarandache curves are characterized using the positional adapted frame by Özen et al. [14, 15]. However, the Serret-Frenet frame is insufficient at points where the curvature of the space curve is zero. Because at points where the curvature is zero, the principal normal and binormal vector of a space curve becomes discontinuous. Sasai has defined the modified orthogonal frame as an alternative to the Frenet frame to solve this problem [16]. Then, the modified orthogonal frame was defined by

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Bükçü and Karaca for the curvature and the torsion of non-zero space curves in Minkowski 3-space [17]. This study aims to investigate the geometric properties of the Smarandache curves according to the modified orthogonal frame. First of all, the equations of the Smarandache curves according to the modified orthogonal frame are obtained. Then, the graphs of the obtained Smarandache curves are drawn. Therefore, it is aimed to contribute to the world of science with the newly obtained curves.

2. Preliminaries

In Euclidean 3-space, Euclidean inner product is given by $<,>= d\alpha_1^2 + d\alpha_2^2 + d\alpha_3^2$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in E^3$. Norm of a vector $\alpha \in E^3$ is $\|\alpha\| = \sqrt{<\alpha, \alpha>}$. For any the space curve α, if $\|\alpha'(s)\| = 1$, then the curve α is unit speed curve in Euclidean 3-space. Let α be a moving space curve with respect to the arc-length s in Euclidean 3-space E^3. t, n, and b are tangent, principal normal, and binormal unit vectors at $\alpha(s)$ point of the curve α, respectively. Then, there exists an orthogonal frame $\{t, n, b\}$ which satisfies the Frenet-Serret equation

$$\begin{aligned} t' &= \kappa n, \\ n' &= -\kappa t + \tau b, \\ b' &= -\tau n \end{aligned}$$

where κ and τ are the curvature and the torsion of the space curve α, respectively. For the reason that the principal normal and binormal vectors in the Frenet frame of a space curve are discontinuous at the points where the curvature is zero, the modified orthogonal frame was introduced by Sasai as an alternative to the Frenet frame. In this sense, we assume that the curvature $\kappa(s)$ of the space curve α is not zero and then we define the modified orthogonal frame $\{T, N, B\}$ as follow:

$$T = \frac{d\alpha}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \wedge N$$

where $T \wedge N$ is the vector product of T and N. The relations between the modified orthogonal frame $\{T, N, B\}$ and Serret-Frenet frame $\{t, n, b\}$ at non-zero points of κ are

$$T = t, \quad N = \kappa n, \quad B = \kappa b.$$

From these equations, it is known that the differentiation of the elements of the modified orthogonal frame $\{T, N, B\}$ satisfy

$$\begin{aligned} T'(s) &= N(s), \\ N'(s) &= -\kappa^2 T(s) + \frac{\kappa'}{\kappa} N(s) + \tau B(s), \\ B'(s) &= -\tau N(s) + \frac{\kappa'}{\kappa} B(s) \end{aligned}$$

where κ' denotes the differentiation of the curvature with respect to the arc-length parameter s and $\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of the space curve α. Moreover, the modified orthogonal frame $\{T, N, B\}$ satisfies

$$\langle T, T\rangle = 1,\ \langle N, N\rangle = \langle B, B\rangle = \kappa^2,\ \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0.$$

3. Smarandache curves constructed by modified orthogonal frame

In this section, we investigate the Smarandache curves according to the modified orthogonal frame $\{T, N, B\}$ in Euclidean 3-space. Let $\alpha = \alpha(s)$ be unit speed regular curve with arc-length parameter s.

Definition 3.1. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors T and N of the curve α can be defined as

$$\beta_1(s^*) = \frac{1}{\sqrt{2}} (T(s) + N(s)) \tag{3.1}$$

such that s^* is the arc-length parameter of the Smarandache curve β_1.

Now, we investigate the Frenet apparatus of the Smarandache curve β_1 obtained from the curve α. Taking the differential of the equation (3.1) according to s, we get

$$\beta_1' = \frac{d\beta_1}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(T'(s) + N'(s)\right)$$

and

$$T_{\beta_1}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa^2 T + \left(1 + \frac{\kappa'}{\kappa}\right)N + \tau B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\kappa^4 + \left(1 + \frac{\kappa'}{\kappa}\right)^2 + \tau^2}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\rho_1, \quad \rho_1 = \sqrt{\kappa^4 + \left(1 + \frac{\kappa'}{\kappa}\right)^2 + \tau^2}. \tag{3.2}$$

So, the tangent vector of the Smarandache curve β_1 is written as follows:

$$T_{\beta_1} = \frac{\left(-\kappa^2 T + \left(1 + \frac{\kappa'}{\kappa}\right)N + \tau B\right)}{\rho_1}. \tag{3.3}$$

By differentiating the equation (3.3) with respect to s, we obtain

$$\frac{dT_{\beta_1}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_1 T + \eta_1 N + \mu_1 B}{\kappa {\rho_1}^2} \tag{3.4}$$

where

$$\begin{aligned}
\lambda_1 &= \kappa^2\left(-\rho_1\left(\kappa + 3\kappa'\right) + \kappa {\rho'}_1\right), \\
\eta_1 &= -\kappa^3\rho_1 - \kappa' {\rho'}_1 - \kappa\left(\rho_1\tau^2 + {\rho'}_1\right) + \rho_1\left(\kappa' + \kappa''\right), \\
\mu_1 &= 2\rho_1\tau\kappa' + \kappa\left(-\tau {\rho'}_1 + \rho_1\left(\tau + \tau'\right)\right).
\end{aligned}$$

Substituting the equation (3.2) into the equation (3.4), we get

$$T'_{\beta_1} = \frac{\sqrt{2}}{\kappa {\rho_1}^3}\left(\lambda_1 T + \eta_1 N + \mu_1 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_1 are

$$\kappa_{\beta_1} = \|{T'}_{\beta_1}\| = \frac{\sqrt{2\left({\lambda_1}^2 + {\eta_1}^2 + {\mu_1}^2\right)}}{\kappa {\rho_1}^3}$$

and

$$N_{\beta_1} = \frac{\lambda_1 T + \eta_1 N + \mu_1 B}{\sqrt{{\lambda_1}^2 + {\eta_1}^2 + {\mu_1}^2}}, \tag{3.5}$$

respectively. From the equations (3.3) and (3.5), the binormal vector of the Smarandache curve β_1 is found as

$$B_{\beta_1} = \frac{1}{\rho_1 q_1}\left(\left(-\eta_1\tau + \mu_1\left(1 + \frac{\kappa'}{\kappa}\right)\right)T + \left(\lambda_1\tau + \mu_1\kappa^2\right)N - \left(\lambda_1\left(1 + \frac{\kappa'}{\kappa}\right) + \eta_1\kappa^2\right)B\right),$$

where $q_1 = \sqrt{{\lambda_1}^2 + {\eta_1}^2 + {\mu_1}^2}$. To calculate the torsion of the curve, we differentiate the curve β_1'

$$\beta_1'' = \frac{\vartheta_1 T + \sigma_1 N + \omega_1 B}{\sqrt{2}\kappa}$$

where

$$\begin{aligned}
\vartheta_1 &= -\kappa^2\left(\kappa + 3\kappa'\right), \\
\sigma_1 &= -\kappa\left(\kappa^2 + \tau^2\right) + \kappa' + \kappa'', \\
\omega_1 &= 2\tau\kappa' + \kappa\left(\tau + \tau'\right)
\end{aligned}$$

and similarly

$$\beta_1''' = \frac{1}{\sqrt{2}\kappa}\left(\varsigma_1 T + \xi_1 N + \zeta_1 B\right)$$

where

$$\begin{aligned}
\varsigma_1 &= \kappa\left(\kappa^4 + \kappa^2\tau^2 - 3\kappa'^2 - \kappa\left(3\kappa' + 4\kappa''\right)\right),\\
\xi_1 &= -\kappa^3\tau + \kappa'\left(2\tau + 3\tau'\right) + 3\tau\kappa'' + \kappa\left(-\tau^3 + \tau' + \tau''\right),\\
\zeta_1 &= -\left(\kappa^3 + 6\kappa^2\kappa' + 3\tau^2\kappa' + \kappa\tau\left(\tau + 3\tau'\right) - \kappa'' - \kappa^3\right).
\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_1 as follows:

$$\tau_{\beta_1} = \frac{\sqrt{2}\left(\left(\omega_1\left(1 + \frac{\kappa'}{\kappa}\right) - \sigma_1\tau\right)\varsigma_1 + \left(\omega_1\kappa^2 - \vartheta_1\tau\right)\xi_1 - \left(\sigma_1\kappa^2 + \vartheta_1\left(1 + \frac{\kappa'}{\kappa}\right)\right)\zeta_1\right)}{\left(\omega_1\left(1 + \frac{\kappa'}{\kappa}\right) - \sigma_1\tau\right)^2 + \left(\omega_1\kappa^2 - \vartheta_1\tau\right)^2 + \left(\sigma_1\kappa^2 + \vartheta_1\left(1 + \frac{\kappa'}{\kappa}\right)\right)^2}.$$

Definition 3.2. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors T and B of the curve α can be defined as

$$\beta_2(s^*) = \frac{1}{\sqrt{2}}\left(T(s) + B(s)\right). \tag{3.6}$$

Here s^* is the arc-length parameter of the Smarandache curve β_2.

We research the Frenet apparatus of the Smarandache β_2 obtained from the curve α. Taking the differential of the equation (3.6) according to s, we get

$$\beta_2' = \frac{d\beta_2}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(T'(s) + B'(s)\right)$$

and

$$T_{\beta_2}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\left(1 - \tau\right)N + \frac{\kappa'}{\kappa}B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\left(1 - \tau\right)^2 + \left(\frac{\kappa'}{\kappa}\right)^2}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\rho_2, \quad \rho_2 = \sqrt{\left(1 - \tau\right)^2 + \left(\frac{\kappa'}{\kappa}\right)^2}. \tag{3.7}$$

So, the tangent vector of the Smarandache curve β_2 is written as follows

$$T_{\beta_2} = \frac{\left(\left(1 - \tau\right)N + \frac{\kappa'}{\kappa}B\right)}{\rho_2}. \tag{3.8}$$

By differentiating the equation (3.8) with respect to s, we obtain

$$\frac{dT_{\beta_2}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_2 T + \eta_2 N + \mu_2 B}{\kappa{\rho_2}^2} \tag{3.9}$$

where

$$\begin{aligned}
\lambda_2 &= \rho_2\kappa^3(-1 + \tau),\\
\eta_2 &= \rho_2(\kappa' - \kappa\tau') + \kappa(-1 + \tau){\rho_2}',\\
\mu_2 &= \kappa'{\rho_2}' - \rho_2(\kappa(-1 + \tau)\tau + \kappa''.
\end{aligned}$$

Substituting the equation (3.7) into the equation (3.9), we get

$$T_{\beta_2}' = \frac{\sqrt{2}}{\kappa{\rho_2}^3}\left(\lambda_2 T + \eta_2 N + \mu_2 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_2 are

$$\kappa_{\beta_2} = \|T'_{\beta_2}\| = \frac{\sqrt{2\left({\lambda_2}^2 + {\eta_2}^2 + {\mu_2}^2\right)}}{\kappa {\rho_2}^3}$$

and

$$N_{\beta_2} = \frac{\lambda_2 T + \eta_2 N + \mu_2 B}{\sqrt{{\lambda_2}^2 + {\eta_2}^2 + {\mu_2}^2}}, \tag{3.10}$$

respectively. From the equations (3.8) and (3.10), the binormal vector of the Smarandache curve β_2 is found as

$$B_{\beta_2} = \frac{1}{\rho_2 q_2}\left(\left(-\eta_2 \frac{\kappa'}{\kappa} + \mu_2 (1-\tau)\right) T + \lambda_2 \frac{\kappa'}{\kappa} N + \lambda_2 (\tau - 1) B\right)$$

where $q_2 = \sqrt{{\lambda_2}^2 + {\eta_2}^2 + {\mu_2}^2}$. To calculate the torsion of the curve, we differentiate the curve β_2' and we get

$$\beta_2'' = \frac{\vartheta_2 T + \sigma_2 N + \omega_2 B}{\sqrt{2}\kappa}$$

where

$$\begin{aligned}
\vartheta_2 &= \kappa^3(-1+\tau), \\
\sigma_2 &= \kappa' - \kappa\tau', \\
\omega_2 &= \kappa(-1+\tau)\tau + \kappa''
\end{aligned}$$

and similarly

$$\beta_2''' = \frac{1}{\sqrt{2}\kappa}\left(\varsigma_2 T + \xi_2 N + \zeta_2 B\right)$$

where

$$\begin{aligned}
\varsigma_2 &= \kappa^2\kappa'(-3+2\tau) + 2\tau', \\
\xi_2 &= \kappa^3(-1+\tau) - \kappa'\tau' + (1+\tau)\kappa'' + \kappa\left((-1+\tau)\tau^2 - \tau''\right), \\
\zeta_2 &= \kappa\tau'(1-3\tau) + (-2+\tau)\tau\kappa' + \kappa''').
\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_2 as follows

$$\tau_{\beta_2} = \frac{\sqrt{2}\left(\varsigma_2\left(\omega_2(1-\tau) - \sigma_2\frac{\kappa'}{\kappa}\right) + \xi_2\vartheta_2\frac{\kappa'}{\kappa} + \zeta_2\vartheta_2(\tau-1)\right)}{\left(\omega_2(1-\tau) - \sigma_2\frac{\kappa'}{\kappa}\right)^2 + \left(\vartheta_2\frac{\kappa'}{\kappa}\right)^2 + \left(\vartheta_2(\tau-1)\right)^2}.$$

Definition 3.3. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors N and B of the curve α can be defined as

$$\beta_3(s^*) = \frac{1}{\sqrt{2}}\left(N(s) + B(s)\right) \tag{3.11}$$

such that s^* is the arc-length parameter of the Smarandache curve β_3.

We investigate the Frenet apparatus of the Smarandache curve β_3 obtained from the curve α. Taking the differential of the equation (3.11) according to s, we get

$$\beta_3' = \frac{d\beta_3}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(N'(s) + B'(s)\right)$$

and

$$T_{\beta_3}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa^2 T + \left(\frac{\kappa'}{\kappa} - \tau\right) N + \left(\frac{\kappa'}{\kappa} + \tau\right) B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\kappa^4 + 2\left(\left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2\right)}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\rho_3, \quad \rho_3 = \sqrt{\kappa^4 + 2\left(\left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2\right)}. \tag{3.12}$$

So, the tangent vector of the Smarandache curve β_3 is written as follows:

$$T_{\beta_3} = \frac{\left(-\kappa^2 T + \left(\frac{\kappa'}{\kappa} - \tau\right) N + \left(\frac{\kappa'}{\kappa} + \tau\right) B\right)}{\rho_3}. \tag{3.13}$$

By differentiating the equation (3.13) with respect to s, we obtain

$$\frac{dT_{\beta_3}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_3 T + \eta_3 N + \mu_3 B}{\kappa {\rho_3}^2} \tag{3.14}$$

where

$$\begin{aligned}
\lambda_3 &= -3\kappa^2\rho_3\kappa' + \kappa^3\left({\rho_3}' + \tau\rho_3\right), \\
\eta_3 &= -2\kappa'\tau\rho_3 - {\rho_3}' + \kappa(-\rho_3(\tau^2 + \tau') + \tau{\rho_3}') - \rho_3\left(\kappa^3 + \kappa''\right), \\
\mu_3 &= \kappa'(2\tau\rho_3 - {\rho_3}') + \kappa(\rho_3(-\tau^2 + \tau') - \tau{\rho_3}') + \rho_3\kappa''.
\end{aligned}$$

Substituting the equation (3.12) into the equation (3.14), we get

$$T'_{\beta_3} = \frac{\sqrt{2}}{\kappa {\rho_3}^3}\left(\lambda_3 T + \eta_3 N + \mu_3 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_3 are

$$\kappa_{\beta_3} = \|{T'}_{\beta_3}\| = \frac{\sqrt{2\left({\lambda_3}^2 + {\eta_3}^2 + {\mu_3}^2\right)}}{\kappa {\rho_3}^3}$$

and

$$N_{\beta_3} = \frac{\lambda_3 T + \eta_3 N + \mu_3 B}{\sqrt{{\lambda_3}^2 + {\eta_3}^2 + {\mu_3}^2}}, \tag{3.15}$$

respectively. From the equations (3.13) and (3.15), the binormal vector of the Smarandache curve β_3 is found as

$$B_{\beta_3} = \frac{1}{\rho_3 q_3}\left(\left(-\eta_3\left(\frac{\kappa'}{\kappa} + \tau\right) + \mu_3\left(\frac{\kappa'}{\kappa} - \tau\right)\right) T + \left(\lambda_3\left(\frac{\kappa'}{\kappa} + \tau\right) + \mu_3\kappa^2\right) N - \left(\lambda_3\left(\frac{\kappa'}{\kappa} - \tau\right) + \eta_3\kappa^2\right) B\right)$$

where $q_3 = \sqrt{{\lambda_3}^2 + {\eta_3}^2 + {\mu_3}^2}$. To calculate the torsion of the curve, we differentiate the equation of the curve β_3'

$$\beta_3'' = \frac{\vartheta_3 T + \sigma_3 N + \omega_3 B}{\sqrt{2}\kappa}$$

where

$$\begin{aligned}
\vartheta_3 &= \kappa^3\tau - 3\kappa^2\kappa' \\
\sigma_3 &= -\kappa^3 - 2\tau\kappa' - \kappa(\tau^2 + \tau') + \kappa'' \\
\omega_3 &= +2\tau\kappa' + \kappa(-\tau^2 + \tau') + \kappa''
\end{aligned}$$

and similarly

$$\beta_3''' = \frac{1}{\sqrt{2}\kappa}\left(\varsigma_3 T + \xi_3 N + \zeta_3 B\right)$$

where

$$\begin{aligned}
\varsigma_3 &= \kappa^5 + \kappa^3(\tau^2 + 2\tau') + 4\kappa^2(\tau\kappa' - \kappa'') - 3\kappa{\kappa'}^2, \\
\xi_3 &= \kappa^3\tau - 6\kappa^2\kappa' + \kappa(\tau^3 - 3\tau\tau' - \tau'') + (-3\kappa'(\tau^2 + \tau') - 3\tau\kappa'' + \kappa'''), \\
\zeta_3 &= -\kappa^3\tau + 3\kappa'(-\tau^2 + \tau') + 3\tau\kappa'' + \kappa(-\tau^3 - 3\tau\tau' + \tau'') + \kappa'''.
\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_3 as follows

$$\tau_{\beta_3} = \frac{\sqrt{2}\left(\left(\omega_3\left(\frac{\kappa'}{\kappa}-\tau\right)+\sigma_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)\varsigma_3+\left(\omega_3\kappa^2-\vartheta_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)\xi_3-\left(\sigma_3\kappa^2+\vartheta_3\left(\frac{\kappa'}{\kappa}-\tau\right)\right)\zeta_3\right)}{\left(\omega_3\left(\frac{\kappa'}{\kappa}-\tau\right)+\sigma_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)^2+\left(\omega_3\kappa^2-\vartheta_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)^2+\left(\sigma_3\kappa^2+\vartheta_3\left(\frac{\kappa'}{\kappa}-\tau\right)\right)^2}.$$

Definition 3.4. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors T, N and B of the curve α can be defined as

$$\beta_4(s^*) = \frac{1}{\sqrt{3}}\left(T(s)+N(s)+B\left(s\right)\right). \tag{3.16}$$

s^* is the arc-length parameter of the Smarandache curve β_4.

We investigate the Frenet apparatus of the Smarandache curve β_4 obtained from the curve α. Taking the differential of the equation (3.16) according to s, we get

$$\beta_4' = \frac{d\beta_4}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(T'(s)+N'(s)+B'\left(s\right)\right)$$

and

$$T_{\beta_4}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(-\kappa^2 T+\left(\frac{\kappa'}{\kappa}-\tau+1\right)N+\left(\frac{\kappa'}{\kappa}+\tau\right)B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\sqrt{\kappa^4+\left(\frac{\kappa'}{\kappa}-\tau+1\right)^2+\left(\frac{\kappa'}{\kappa}+\tau\right)^2}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\rho_4, \quad \rho_4 = \sqrt{\kappa^4+\left(\frac{\kappa'}{\kappa}-\tau+1\right)^2+\left(\frac{\kappa'}{\kappa}+\tau\right)^2}. \tag{3.17}$$

So, the tangent vector of the Smarandache curve β_4 is written as follows:

$$T_{\beta_4} = \frac{\left(-\kappa^2 T+\left(\frac{\kappa'}{\kappa}-\tau+1\right)N+\left(\frac{\kappa'}{\kappa}+\tau\right)B\right)}{\rho_4}. \tag{3.18}$$

By differentiating the equation (3.18) with respect to s, we obtain

$$\frac{dT_{\beta_4}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_4 T+\eta_4 N+\mu_4 B}{\kappa{\rho_4}^2} \tag{3.19}$$

where

$$\begin{aligned}
\lambda_4 &= -3\kappa^2\rho_4\kappa'+\kappa^3((-1+\tau)\rho_4+{\rho_4}'),\\
\eta_4 &= \kappa'((1-2\tau)\rho_4-{\rho_4}')+\kappa(-\rho_4(\tau^2+\tau')-(1-\tau){\rho_4}')-\rho_4\left(\kappa^3-\kappa''\right),\\
\mu_4 &= \kappa'(2\tau\rho_4-{\rho_4}')+\kappa(\rho_4(\tau(1-\tau)+\tau')-\tau{\rho_4}')+\rho_4\kappa''.
\end{aligned}$$

Substituting the equation (3.17) into the equation (3.19), we get

$$T_{\beta_4}' = \frac{\sqrt{3}}{\kappa{\rho_4}^3}\left(\lambda_4 T+\eta_4 N+\mu_4 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_4 are

$$\kappa_{\beta_4} = \|{T'}_{\beta_4}\| = \frac{\sqrt{3\left({\lambda_4}^2+{\eta_4}^2+{\mu_4}^2\right)}}{\kappa{\rho_4}^3}$$

and

$$N_{\beta_4} = \frac{\lambda_4 T+\eta_4 N+\mu_4 B}{\sqrt{{\lambda_4}^2+{\eta_4}^2+{\mu_4}^2}}, \tag{3.20}$$

respectively. From the equations (3.18) and (3.20), the binormal vector of the Smarandache curve β_4 is found as

$$B_{\beta_4} = \frac{1}{\rho_4 q_4}\left(\begin{array}{l}\left(\left(-\eta_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)\right)T+\left(\lambda_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\kappa^2\right)N\\ -\left(\lambda_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)+\eta_4\kappa^2\right)B\end{array}\right)$$

where $q_4 = \sqrt{{\lambda_4}^2+{\eta_4}^2+{\mu_4}^2}$. To calculate the torsion of the curve, we differentiate the curve β_4'

$$\beta_4'' = \frac{\vartheta_4 T + \sigma_4 N + \omega_4 B}{\sqrt{3}\kappa}$$

where

$$\begin{aligned}\vartheta_4 &= \kappa^3(\tau-1)-3\kappa^2\kappa',\\ \sigma_4 &= -\kappa^3+\kappa'(1-2\tau)-\kappa(\tau^2+\tau')+\kappa'',\\ \omega_4 &= 2\tau\kappa'+\kappa(\tau(1-\tau)+\tau')+\kappa''\end{aligned}$$

and similarly

$$\beta_4''' = \frac{1}{\sqrt{3}\kappa}\left(\varsigma_4 T + \xi_4 N + \zeta_4 B\right)$$

where

$$\begin{aligned}\varsigma_4 &= \kappa^5+\kappa^3(\tau^2+2\tau')+\kappa^2((-3+4\tau)\kappa'-4\kappa'')-3\kappa{\kappa'}^2,\\ \xi_4 &= \kappa^3(-1+\tau)-6\kappa^2\kappa'+\kappa(\tau((-1+\tau)\tau-3\tau')-\tau'')+(-3\kappa'(\tau^2+\tau')+\kappa''-3\tau\kappa''+\kappa'''),\\ \zeta_4 &= -\kappa^3\tau+(-\tau^3+\tau'-3\tau\tau'+\tau'')+\kappa'((2-3\tau)\tau+3\tau')+3\tau\kappa''+\kappa'''.\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_4 as follows:

$$\tau_{\beta_4} = \frac{\sqrt{3}\left(\left(-\eta_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)\right)\varsigma_4+\left(\lambda_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\kappa^2\right)\xi_4-\left(\lambda_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)+\eta_4\kappa^2\right)\zeta_4\right)}{\left(-\eta_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)\right)^2+\left(\lambda_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\kappa^2\right)^2+\left(\lambda_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)+\eta_4\kappa^2\right)^2}.$$

Example 3.1. Let's plot the graphics of the Smarandache curves based on the modified orthogonal frame of the eight curve which is known as Gerono lemniscate curve [18]. The parametric equation of this curve is given by

$$\alpha(s) = (\sin(s), \sin(s)\cos(s), s).$$

The elements of the Frenet trihedron of the curve $\alpha(s)$ are obtained as

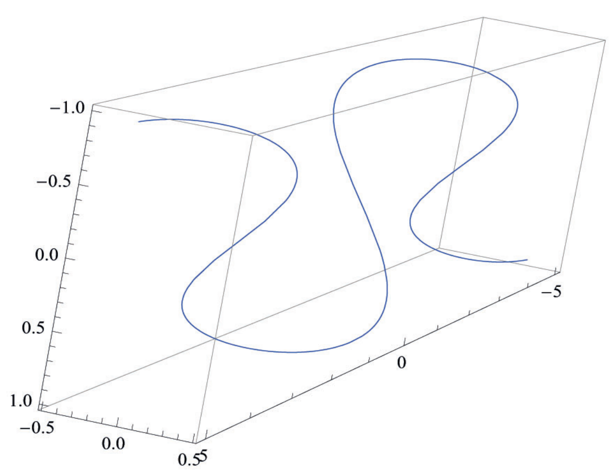


Figure 1. The Gerono lemniscate curve

$$
\begin{aligned}
t(s) &= \frac{\left(\sqrt{2}\cos(s), \sqrt{2}\cos(2s), \sqrt{2}\right)}{\sqrt{4+\cos(2s)+\cos(4s)}},\\
n(s) &= \frac{\left((-1+4\cos(2s)+\cos(4s))\sin(s), -\sin(2s)(6+\cos(2s)), \sin(2s)+2\sin(4s)\right)}{\sqrt{4+\cos(2s)+\cos(4s)}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}},\\
b(s) &= \frac{(4\sin(2s), -2\sin(s), -3\sin(s)-\sin(3s))}{\sqrt{2}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}.
\end{aligned}
$$

The curvature of the curve $\alpha(s)$ is found as

$$
\kappa(s) = \frac{2\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}{(4+\cos(2s)+\cos(4s))^{3/2}}.
$$

Besides the curvature $\kappa(s) = \frac{2\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}{(4+\cos(2s)+\cos(4s))^{3/2}}$ is not differentiable, the principal normal and binormal vectors are discontinuous at $s = 0$ since $n_+ \neq n_-$ and $b_+ \neq b_-$ for $n_+ = \lim\limits_{s\to 0^+} n(s)$, $n_- = \lim\limits_{s\to 0^-} n(s)$ and $b_+ = \lim\limits_{s\to 0^+} b(s)$, $b_- = \lim\limits_{s\to 0^-} b(s)$. Looking for a solution to this problem, Sasai has defined the modified orthogonal frame as an alternative to the Frenet frame. The elements of the modified orthogonal frame of the curve $\alpha(s)$ are obtained as

$$
\begin{aligned}
T(s) &= \frac{\sqrt{2}(\cos(s), \cos(2s), 1)}{\sqrt{4+\cos(2s)+\cos(4s)}},\\
N(s) &= \frac{2\left(\sin(s)(-1+4\cos(2s)+\cos(4s)), -\sin(2s)(6+\cos(2s)), (\sin(2s)+2\sin(4s))\right)}{(4+\cos(2s)+\cos(4s))^2},\\
B(s) &= \frac{\sqrt{2}\left(4\sin(2s), -2\sin(s), -(3\sin(s)+\sin(3s))\right)}{(4+\cos(2s)+\cos(4s))^{3/2}}.
\end{aligned}
$$

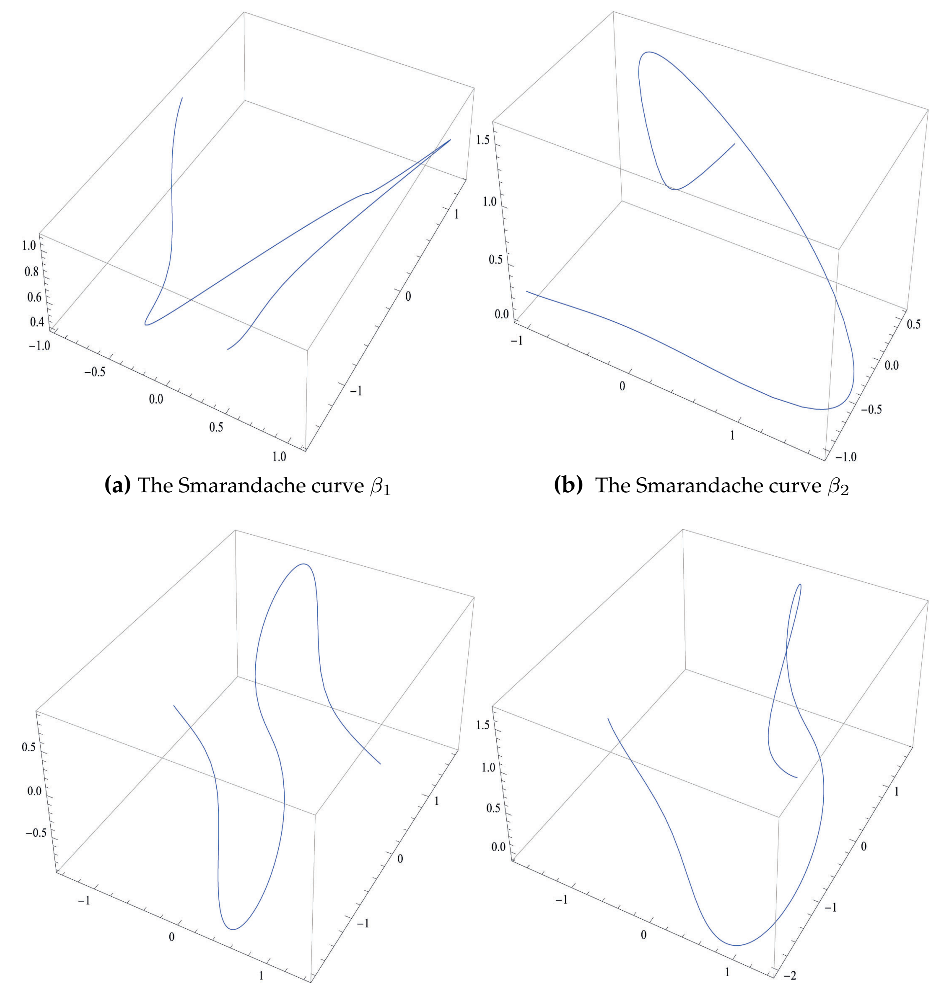
The Smarandache curves $\beta_1, \beta_2, \beta_3$ and β_4 obtained from the curve α are given as

$$
\beta_1 = \begin{pmatrix}
\dfrac{\sqrt{2}\cos(s)}{\sqrt{4+\cos(2s)+\cos(4s)}} + \dfrac{-6\sin(s)+3\sin(3s)+\sin(5s)}{(4+\cos(2s)+\cos(4s))^2}, \dfrac{\sqrt{2}\cos(2s)}{\sqrt{4+\cos(2s)+\cos(4s)}}\\
-\dfrac{2\sin(2s)(6+\cos(2s))}{(4+\cos(2s)+\cos(4s))^2}, \dfrac{\sqrt{2}}{\sqrt{4+\cos(2s)+\cos(4s)}} + \dfrac{2(\sin(2s)+2\sin(4s))}{(4+\cos(2s)+\cos(4s))^2}
\end{pmatrix},
$$

$$
\beta_2 = \begin{pmatrix}
\dfrac{9\cos(s)+2\cos(3s)+\cos(5s)+8\sin(2s)}{\sqrt{2}(4+\cos(2s)+\cos(4s))^{3/2}}, \dfrac{1+9\cos(2s)+\cos(4s)+\cos(6s)-4\sin(s)}{\sqrt{2}(4+\cos(2s)+\cos(4s))^{3/2}},\\
\dfrac{2(4+\cos(2s)+\cos(4s)-3\sin(s)-\sin(3s))}{\sqrt{2}(4+\cos(2s)+\cos(4s))^{3/2}}
\end{pmatrix},
$$

$$
\beta_3 = \begin{pmatrix}
\dfrac{2\sin(s)(4\cos(2s)+\cos(4s)-1)}{(4+\cos(2s)+\cos(4s))^2} + \dfrac{4\sqrt{2}\sin(2s)}{(4+\cos(2s)+\cos(4s))^{3/2}},\\
\dfrac{-2\sin(s)(13\cos(s)+\cos(3s))}{(4+\cos(2s)+\cos(4s))^2} + \dfrac{2\sqrt{2}\sin(s)}{(4+\cos(2s)+\cos(4s))^{3/2}},\\
\dfrac{2(\sin(2s)+2\sin(4s))}{(4+\cos(2s)+\cos(4s))^2} - \dfrac{\sqrt{2}(3\sin(s)+\sin(3s))}{(4+\cos(2s)+\cos(4s))^{3/2}}
\end{pmatrix},
$$

$$\beta_4=\begin{pmatrix}\dfrac{\sqrt{2}\cos(s)}{\sqrt{4+\cos(2s)+\cos(4s)}}+\dfrac{2\sin(s)\left(4\cos(2s)+\cos(4s)-1\right)}{\left(4+\cos(2s)+\cos(4s)\right)^2}+\dfrac{4\sqrt{2}\sin(2s)}{\left(4+\cos(2s)+\cos(4s)\right)^{3/2}},\\ \dfrac{\sqrt{2}\cos(2s)}{\sqrt{4+\cos(2s)+\cos(4s)}}-\dfrac{2\sin(s)\left(13\cos(s)+\cos(3s)\right)}{\left(4+\cos(2s)+\cos(4s)\right)^2}-\dfrac{2\sqrt{2}\sin(s)}{\left(4+\cos(2s)+\cos(4s)\right)^{3/2}},\\ \dfrac{\sqrt{2}}{\sqrt{4+\cos(2s)+\cos(4s)}}+\dfrac{2\left(\sin(2s)+2\sin(4s)\right)}{\left(4+\cos(2s)+\cos(4s)\right)^2}-\dfrac{\sqrt{2}\left(3\sin(s)+\sin(3s)\right)}{\left(4+\cos(2s)+\cos(4s)\right)^{3/2}}\end{pmatrix}.$$



(a) The Smarandache curve β_1 **(b)** The Smarandache curve β_2

(c) The Smarandache curve β_3 **(d)** The Smarandache curve β_4

Figure 2. The Smarandache curves for $s \in [-2, 2]$

4. Conclusion

In this paper, we investigate the geometric properties of the Smarandache curves with respect to the modified orthogonal frame. Sasai presented the modified orthogonal frame as an alternative to the Frenet frame. Because the principal normal and binormal vectors of the Frenet frame of a space curve become discontinuous at the points where the curvature is zero, However, the Smarandache curves have not been examined under these conditions yet. For this reason, this research is a new study in the geometry field.

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