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A Note on Approximation Properties of Bernstein-type Operators via Some Summability Methods

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Abstract. In this paper, we focus on two summability methods and investigate some applications of them for the Cheney-Sharma operators. We obtain approximation properties of the Cheney-Sharma operators via power series statistical convergence. We also analyze the convergence rates employing both the modulus of continuity and elements of the Lipschitz class. Additionally, we define r-th order generalization of these operators which is linear but don't satisfy the positivity property and investigate approximation properties of these operators, via A-statistical convergence. We support our results with an example and a graph.

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1. Introduction

Bernstein polynomials are well known lineer positive operators. They have an important place in approximation theory, since their structures gave us first sharp information about the polynomials mentioned in the Weierstrass theorem [2]. After definition of these polynomials is given many generalizations of Bernstein polynomials have been defined. In [6], Cheney-Sharma built generalized Bernstein operators can be described as follows:

$$G_j(f;x) = \left(1 + j\beta_j\right)^{1-j} \sum_{k=0}^{j} f\left(\frac{k}{j}\right)\binom{j}{k} x\left(x + k\beta_j\right)^{k-1} \times (1-x)\left[1 - x + (j-k)\beta_j\right]^{j-k-1}, \tag{1.1}$$

where, $x \in [0,1]$, $j \in \mathbb{N}$, (β_j) is a sequence of non-negative real numbers and $\mathbb{N}$ is the set of all positive integers. Note that, if $\beta_j = 0$ for any $j \in \mathbb{N}$ then the operators (1.1) turn into the Bernstein polynomials [6]. After introducing the operators Cheney and Sharma studied Korovkin type approximation properties of these operators and they proved that the operators (1.1) uniformly converge to the identity operator. In that paper, Cheney and Sharma take the classical condition

$$\lim_{j\to\infty} j\beta_j = 0. \tag{1.2}$$

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Some generalizations of these operators have been defined and investigated approximation to the identity operator via classical convergence [5,16,21,25]. Moreover some preserving properties have been studied for these operators [3,20]. In some different cases the approximation to the identity operator also studied by using some summability methods. If the ordinary convergence of linear positive operators to the function fails, summability methods can be used to obtain convergence more generally sense. For this purpose most of summability methods have been used in approximation theory [8, 10, 17, 23, 26–30]

In the present paper, taking into account power series statistical convergence, investigate convergence of the operators (1.1) and compute rate of the convergence. Furthermore, we define r-th order generalization of these operators and give some approximation properties via A-statistical convergence.

Now, let us recall two summability methods that are considered in this study:

Definition 1.1. Let $A = (a_{nj})$ be a nonnegative regular summability matrix and let K be a subset of positive integer. Then, K is said to have A-density $\delta_A(K)$ if the limit

$$\delta_A(K) := \lim_n \sum_{k\in K} a_{nj}$$

exists. The sequence x is said to be A-statistically convergent to real number α if for any $\varepsilon > 0$

$$\lim_n \sum_{j:\left|x_j-\alpha\right|\geq\varepsilon} a_{nj}.$$

In this case, we write $st_A - \lim x = \alpha$ [9, 18]. Note that x is A-statistically convergent to α if and only if for any $\varepsilon > 0$, $\delta_A(K_\varepsilon) = 0$, where $K_\varepsilon := \left\{k \in \mathbb{N} : \left|x_j - \alpha\right| \geq \varepsilon\right\}$.

Here, we recall the regular power series method [4].

Definition 1.2. Let $\left(s_j\right)$ be a real sequence with $s_0 > 0$ and $s_1, s_2, ... \geq 0$, such that the corresponding power series $s(t) := \sum_{j=0}^{\infty} s_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If

$$\lim_{0<t\to R^-} \frac{1}{s(t)} \sum_{j=0}^{\infty} x_j s_j t^j = L,$$

then we say that $x = \left(x_j\right)$ is convergent in the sense of power series method (P convergent).

In Korovkin type approximation theory some results related to this methods can be found in [26, 27].

Theorem 1.3 ([4]). *A power series method P is regular if and only if for any $j \in \mathbb{N}_0$*

$$\lim_{0<t\to R^-} \frac{s_j t^j}{s(t)} = 0.$$

Now, let us define power series statistical convergence which is introduced in [30]. It is defined with the help of the concept of density with respect to the power series methods.

Definition 1.4 ([30]). Let P be a regular power series method and let $E \subset \mathbb{N}_0$. If

$$\delta_P(E) := \lim_{0<t\to R^-} \frac{1}{s(t)} \sum_{j\in E}^{\infty} s_j t^j$$

exists, then $\delta_P(E)$ is called the P-density of E.

Definition 1.5 ([30]). Let $x = (x_j)$ be a sequence and let P be a regular power series method. Then, x is said to be P-statistically convergent to L if for any $\varepsilon > 0$

$$\lim_{0<t\to R^-} \frac{1}{s(t)} \sum_{j:\left|x_j-L\right|\geq\varepsilon} s_j t^j = 0,$$

i.e., $\delta_P\left(\{j \in \mathbb{N}_0 : \left|x_j - L\right| \geq \varepsilon\}\right) = 0$. In this case, we write $st_P - \lim x = L$.

Now, we state the rate of power series convergence as in [14].
Let P be a regular power series method and (λ_j) be a positive non-increasing sequence of real numbers. Then, a sequence $x = \{x_j\}$ is said to be P-statistically convergent to L with the rate of

$$o(\lambda(j))\ (j \to \infty)$$

if for any $\varepsilon > 0$

$$\lim_{0<t\to R^-} \frac{1}{s(t)} \sum_{j:|x_j-L|\geq\varepsilon\lambda_j} s_j t^j = 0$$

which is denoted by $x_j - L = st_P - o(\lambda(j))$ as $(j \to \infty)$.

Throughout the paper, we assume that $G_0(f) = 0$ for any $f \in C[0,1]$ and we use the norm of the Banach space $B[0,1]$ is defined for any $f \in C[0,1]$ by

$$\|f\| := \sup_{0\leq x\leq 1} |f(x)|,$$

where $B[0,1]$ is the space of all bounded real functions defined over $[0,1]$.

2. Power Series Statistical Convergence of Cheney-Sharma Operators

In this section, by using power series statistical convergence, we obtain Korovkin type approximation properties of the operators (G_j).

We need the following known lemmas in our proofs:

Lemma 2.1 ([6, 24]). *The followings hold for the operators (1.1) :*

$$G_j(e_0;x) = 1 \tag{2.1}$$

$$G_j(e_1;x) = x$$

$$G_j(e_2;x) - x^2 \leq x\left(x+2\beta_j\right)\left(1+j\beta_j\right) + \frac{x}{j}\left(j\beta_j\right)^2\left(1+j\beta_j\right)$$
$$+\, x\left(x+2\beta_j\right)j\beta_j + \frac{x}{j}\left(j\beta_j\right)^3 + \frac{x}{j} - x^2, \tag{2.2}$$

where $e_i(x) = x^i$, for $i = 0,1,2$.

From the Lemma 2.1, we can give the following result easily.

Lemma 2.2. *The followings hold for the operators (1.1) :*

$$G_j((t-x);x) = 0$$

$$G_j((t-x)^2;x) \leq x^2(2j\beta_j) + (2x\beta_j)(1+2j\beta_j) + \frac{x}{j}\left(j\beta_j\right)^2\left(1+2j\beta_j\right) + \frac{x}{j}.$$

Now, we recall the following Korovkin type P-statistical approximation theorem which is given in [30].

Theorem 2.3. *Let P be a regular power series method and let (L_j) be a sequence of linear positive operators on $C[0,1]$ such that for $i = 0,1,2$*

$$st_p\text{-}\lim \left\|L_j(e_i) - e_i\right\| = 0, \tag{2.3}$$

then for any $f \in C[0,1]$ we have

$$st_p\text{-}\lim \left\|L_j(f) - f\right\| = 0.$$

We are ready to prove the following Korovkin type P-statistical approximation theorem:

Theorem 2.4. *Assume that P is a regular power series method. If (β_j) is a sequence of positive real numbers such that $st_p\text{-}\lim_{j\to\infty} j\beta_j = 0$, then for each $f \in C[0,1]$ we have*

$$st_P\text{-}\lim_{j\to\infty} \left\|G_j(f) - f\right\| = 0.$$

Proof. From Theorem 2.3, it is enough to demonstrate that (2.3) holds for (G_j). Now, considering Lemma 2.1, we get for $i = 0, 1$ that

$$st_P\text{-}\lim_{j\to\infty} \left\| G_j(e_i) - e_i \right\| = 0.$$

Now, using (2.2) one can have

$$\begin{aligned}
\left| G_j(e_2) - e_2 \right| &\le x\left(x + 2\beta_j\right)\left(1 + 2j\beta_j\right) + \frac{x}{j}\left(j\beta_j\right)^2\left(1 + 2j\beta_j\right) + \frac{x}{j} \\
&= x^2\left(\left(1 + 2j\beta_j\right) - 1\right) + 2x\beta_j\left(1 + 2j\beta_j\right) + \frac{x}{j}\left(j\beta_j\right)^2\left(1 + 2j\beta_j\right) + \frac{x}{j} \\
&\le 2j\beta_j + 2\beta_j\left(1 + 2j\beta_j\right) + \frac{1}{j}\left(j\beta_j\right)^2\left(1 + 2j\beta_j\right) + \frac{1}{j}.
\end{aligned}$$

Now, let us define the following sets:

$$\begin{aligned}
N &:= \left\{ j \in \mathbb{N} : \left\| G_j(e_2) - e_2 \right\| \ge \varepsilon \right\}, \\
N_1 &:= \left\{ j \in \mathbb{N} : 2j\beta_j \ge \frac{\varepsilon}{4} \right\}, \\
N_2 &:= \left\{ j \in \mathbb{N} : 2\beta_j\left(1 + 2j\beta_j\right) \ge \frac{\varepsilon}{4} \right\}, \\
N_3 &:= \left\{ j \in \mathbb{N} : \frac{1}{j}\left(j\beta_j\right)^2\left(1 + 2j\beta_j\right) \ge \frac{\varepsilon}{4} \right\}, \\
N_4 &:= \left\{ j \in \mathbb{N} : \frac{1}{j} \ge \frac{\varepsilon}{4} \right\}.
\end{aligned}$$

Then, we see that $N \subseteq N_1 \cup N_2 \cup N_3 \cup N_4$. Therefore, we get

$$\begin{aligned}
0 &\le \delta_P\left(\left\{ j \in \mathbb{N} : \left\| G_j(f) - f \right\| \ge \varepsilon \right\}\right) \\
&\le \delta_P\left\{ j \in \mathbb{N} : 2j\beta_j \ge \frac{\varepsilon}{4} \right\} + \delta_P\left\{ j \in \mathbb{N} : 2\beta_j\left(1 + 2j\beta_j\right) \ge \frac{\varepsilon}{4} \right\} \\
&+ \delta_P\left\{ j \in \mathbb{N} : \frac{1}{j}\left(j\beta_j\right)^2\left(1 + 2j\beta_j\right) \ge \frac{\varepsilon}{4} \right\} + \delta_P\left\{ j \in \mathbb{N} : \frac{1}{j} \ge \frac{\varepsilon}{4} \right\} \\
&= 0,
\end{aligned}$$

$$st_P - \lim_j \left\| G_j(e_2) - e_2 \right\| = 0.$$

This proves the theorem. □

The following example shows that the conditions of Theorem 2.4 are weaker than the approximation Theorem given in [6]:

Example 2.5. We remark that if we take classical conditions (1.2) Theorem 2.4 works. Conversely, we assume that P is the power series method with $\left(s_j\right)$

$$s_j := \begin{cases} 0 & , \quad j = 2k \\ 1 & , \quad j = 2k + 1 \end{cases}$$

and we consider the sequence (β_j)

$$\beta_j := \begin{cases} j & , \quad j = 2k \\ \frac{1}{j^2} & , \quad j = 2k + 1 \end{cases} .$$

Note that, if we take (β_j), approximation theorem given in [6] doesn't work.

Remark 2.6. Figure 1 illustrates the power series statistical convergence of the sequence $G_j(f)$ to f by considering $f(x) = \sin 10t^2$and $\left(\beta_j\right)$ given in Example 2.5.

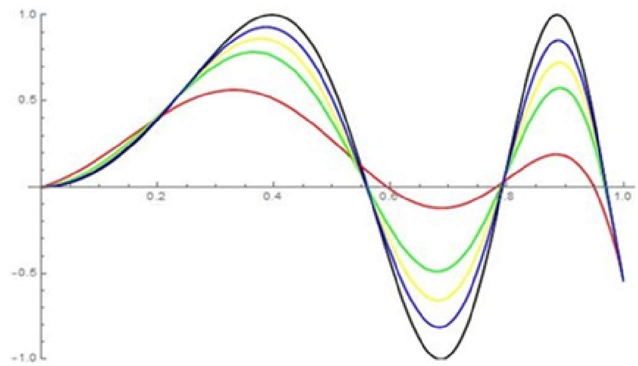

FIGURE 1. Function itself (black), j=12(red), j=30(green), j=50(yellow), j=100(blue)

Now, we compute order of the P-statistical convergence of the operators given with (1.1) by means of the modulus of continuity and elements of Lipschitz class. The modulus of continuity of $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in[0,1]}} |f(x) - f(y)|.$$

It is well known from that for a function $f \in C[0,A]$,

$$\lim_{\delta\to 0^+} \omega(f,\delta) = 0$$

and for any $\delta > 0$

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right).$$

Now, we can give the following result:

Theorem 2.7. *Let P be a regular power series method and (λ_j) be a non-increasing sequence positive real numbers. If (β_j) be a sequence of positive real numbers such that $j\beta_j = st_p - o(\eta_j)$ and $\omega(f,\delta_j) = st_P - o(\lambda_j)$, then*

$$\left\|G_j(f) - f\right\| = st_P - o(\kappa_j)$$

for all $f \in C[0,1]$ and $x \in [0,1]$, $j \in \mathbb{N}$, where

$$\delta_j := \left\{2j\beta_j + 2\beta_j\left(1 + 2j\beta_j\right) + \frac{1}{j}\left(j\beta_j\right)^2 + \frac{1}{j}\right\}^{\frac{1}{2}}$$

and

$$\kappa_j = \max\left\{\eta_j, \lambda_j\right\}.$$

Proof. From Theorem 2 in [22] and equation (2.1), we get

$$\left|G_j(f;x) - f(x)\right| \leq 2\omega\left(f,\delta_j\right).$$

By using Lemma 2.2, we have

$$\delta_j = \sup_{0\leq x\leq 1} G_j\left((t-x)^2;x\right) = \left\{2j\beta_j + 2\beta_j\left(1 + 2j\beta_j\right) + \frac{1}{j}\left(j\beta_j\right)^2 + \frac{1}{j}\right\}^{\frac{1}{2}}$$

and considering $\kappa_j = \max\left\{\eta_j, \lambda_j\right\}$, we reach to

$$\frac{1}{s(t)} \sum_{\left|G_j(f;x)-f(x)\right|\geq\varepsilon\kappa_j} s_j t^j \leq \frac{1}{s(t)} \sum_{\omega\left(f,\delta_j\right)\geq\frac{\varepsilon}{2}\kappa_j} s_j t^j,$$

which implies

$$0 \leq \delta_P\left(\left\{j \in \mathbb{N} : \left\|G_j f - f\right\| \geq \varepsilon\kappa_j\right\}\right) \\ \leq \delta_P\left\{j \in \mathbb{N} : \omega\left(f, \delta_j\right) \geq \frac{\varepsilon}{2}\kappa_j\right\}.$$

By the hypothesis, we obtain

$$\left\|G_j(f) - f\right\| = st_P - o(\kappa_j)\ (j \to \infty).$$

□

Now, we give the rate of convergence of the operators (1.1) via the elements of the Lipschitz class $Lip_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$. Let us recall the following definition.

Definition 2.8. Let f be a real valued continuous function defined on $[0, A]$. Then, f is said to be Lipschitz continuous of order α on $[0, A]$ if

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for $x, y \in [0, A]$ with $M > 0$ and $0 < \alpha \leq 1$. The set of Lipschitz continuous functions is denoted by $Lip_M(\alpha)$.

Theorem 2.9. *Let P, (δ_j), $\left(\eta_j\right)$, $\left(\lambda_j\right)$, (κ_j) same as in Theorem 2.7 and $(\delta_j) = st_P - o(\lambda_j)$. Then, for all $f \in Lip_M(\alpha)$ such that $0 < \alpha \leq 1$, $M \in \mathbb{R}^+$, we get*

$$\left\|G_j(f) - f\right\| = st_P - o(\kappa_j)\ \ (j \to \infty).$$

Proof. From the Theorem 3 in [22] and by the hypothesis, we have

$$\left|G_j(f; x) - f(x)\right| \leq \left(G_j\left((t - x)^2; x\right)\right)^{\frac{\alpha}{2}} \\ \leq M\delta_j^\alpha.$$

Now, we define the following set

$$K := \left\{j \in \mathbb{N} : \left\|G_j(f) - f\right\| \geq \varepsilon\right\},$$
$$K_1 := \left\{j \in \mathbb{N} : \delta_j^\alpha \geq \frac{\varepsilon}{M}\right\},$$
$$K_1' := \left\{j \in \mathbb{N} : \delta_j \geq \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}\right\},$$

then we can easily see that $K \subseteq K_1 \subseteq K_1'$.

$$\frac{1}{s(t)} \sum_{\left|G_j(f;x)-f(x)\right|\geq\varepsilon\kappa_j} s_j t^j \leq \frac{1}{s(t)} \sum_{\delta_j \geq (\frac{\varepsilon}{M})^{\frac{1}{\alpha}}\kappa_j} s_j t^j,$$

which implies

$$0 \leq \delta_P\left(\left\{j \in \mathbb{N} : \left\|G_j(f) - f\right\| \geq \varepsilon\kappa_j\right\}\right) \\ \leq \delta_P\left\{j \in \mathbb{N} : \delta_j \geq \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}\kappa_j\right\}.$$

Therefore, we have

$$\left\|G_j(f) - f\right\| = st_P - o(\kappa_j)\ (j \to \infty).$$

□

3. An r-th Order Generalization of the Operators (G_j) via A-Statistical Convergence

In this section, we consider r-th order generalization of the positive linear operators (G_j). This generalization was first defined and studied via classical convergence by Kirov and Popova [11]. The authors defined the operators $(L_{j,r})$ as r-th order generalization of the linear positive operators (L_j) replacing the function in the neighbourhood of the point ξ with the r-th degree Taylor series of the function. The operators $(L_{j,r})$ is linear, but don't satify the positivity property. By using A-statistical convergence Agratini [1] investigated r-th order generalization of the linear positive operators and give some applications for the Stancu type operators. Some further results in this direction may be found in [12, 13, 15].

Now, we define the following generalization of the positive linear operators (G_j)

$$G_j^{[r]}(f;x) = \left(1 + j\beta_j\right)^{1-j} \sum_{k=0}^{j} f\left(\frac{k}{j}\right)\binom{j}{k} x\left(x + k\beta_j\right)^{k-1}$$
$$(1-x)\left[1 - x + (j-k)\beta_j\right]^{j-k-1} \sum_{\nu=0}^{r} f^{(\nu)}\left(\frac{t}{j}\right)\frac{\left(x - \frac{t}{j}\right)^{\nu}}{\nu!},$$

where $f \in C^r[0,1]$ $(r \in \mathbb{N}_0)$, $j \in \mathbb{N}$. Here, $C^r[0,1]$ denotes the space of all functions of having continuous r-th order derivative $f^{(r)}$ on the segment $[0,1]$, where as usual, $f^{(0)}(x) = f(x)$.

Note that taking $r = 0$, we obtain the operators $G_j(f;x)$ defined by (1.1)

Theorem 3.1. *Let $A = (a_{n,j})$ be a non-negative regular summability matrix, $r \in \mathbb{N}$, $j \in \mathbb{N}$, $x \in [0,1]$, $\left(\beta_j\right)$ be a sequence of positive real numbers such that $st_A - \lim_{j\to\infty} j\beta_j = 0$, then for any $f \in C^r[0,1]$ with the property $f^{(r)} \in Lip_M(\alpha)$, we have*

$$st_A - \lim_{j\to\infty}\left\|G_j^{[r]}(f;x) - f\right\| = 0.$$

Proof. From (2.1), we have

$$f(x) - G_j^{[r]}(f;x) = \left(1 + j\beta_j\right)^{1-j}\sum_{k=0}^{j} f\left(\frac{k}{j}\right)\binom{j}{k} x\left(x + k\beta_j\right)^{k-1}(1-x)\left[1 - x + (j-k)\beta_j\right]^{j-k-1} \tag{3.1}$$
$$\times\left[f(x) - \sum_{\nu=0}^{r} f^{(\nu)}\left(\frac{k}{j}\right)\frac{\left(x - \frac{k}{j}\right)^{\nu}}{\nu!}\right].$$

Applying the Taylor's formula, we may write that

$$f(x) - \sum_{\nu=0}^{r} f^{(\nu)}\left(\frac{k}{j}\right)\frac{\left(x - \frac{k}{j}\right)^{\nu}}{\nu!} = \frac{\left(x - \frac{k}{j}\right)^{r}}{(r-1)!}\int_0^1 (1-s)^{r-1}\left[f^{(r)}\left(\frac{k}{j} + s\left(x - \frac{k}{j}\right)\right) - f^{(r)}\left(\frac{k}{j}\right)\right]ds. \tag{3.2}$$

Because of $f^{(r)} \in Lip_M(\alpha)$, we obtain

$$\left|f^{(r)}\left(\frac{k}{j} + s\left(x - \frac{k}{j}\right)\right) - f^{(r)}\left(\frac{k}{j}\right)\right| \le M s^{\alpha}\left|x - \frac{k}{j}\right|^{\alpha}. \tag{3.3}$$

On the other hand, from the the well-known expression of the Beta function, we get

$$\int_0^1 (1-s)^{r-1} s^{\alpha} ds = B(1+\alpha, r) = \frac{\alpha}{\alpha + r}B(\alpha, r). \tag{3.4}$$

By considering (3.3) and (3.4) in (3.2), we have

$$\left|f(x) - \sum_{\nu=0}^{r} f^{(\nu)}\left(\frac{k}{j}\right)\frac{\left(x - \frac{k}{j}\right)^{\nu}}{\nu!}\right| \le \frac{M}{(r-1)!}\frac{\alpha}{\alpha + r}B(\alpha, r)\left|x - \frac{k}{j}\right|^{r+\alpha}. \tag{3.5}$$

Taking (3.5) in (3.1), we arrive that

$$\left|f(x)-G_j^{[r]}(f;x)\right| \leq \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\,G_j\left(|x-t|^{r+\alpha};x\right).$$

Thus, we have

$$\left\|G_j^{[r]}(f;x)-f\right\| \leq K(\alpha,r)\left\|G_j\left(g_x^{r+\alpha};x\right)\right\|,$$

where $K(\alpha,r)=\frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)$.

Now, we take a function $g_x\in C[0,1]$ which is defined by $g_x(t)=|x-t|$. Since $g_x(x)=0$, From Lemma 3.4 in [7], we can write

$$st_A-\lim_j\left\|G_j\left(g_x^{r+\alpha};x\right)\right\|=0. \tag{3.6}$$

On the other hand, for an arbitrary $\varepsilon>0$ let us establish the following sets

$$R:=\left\{j\in\mathbb{N}:\left\|G_j^{[r]}(f;x)-f\right\|\geq\varepsilon\right\},$$

$$S:=\left\{j\in\mathbb{N}:\left\|G_j\left(|x-t|^{r+\alpha};x\right)\right\|\geq\frac{\varepsilon}{K(\alpha,r)}\right\},$$

Then, we see that $R\subseteq S$. Therefore, we have $\sum_{j\in R}a_{nj}\leq\sum_{j\in S}a_{nj}$. (3.6) yields that for all $f\in C^r[0,1]$ such that $f^{(r)}\in Lip_M(\alpha)$, we obtain

$$st_A-\lim_{j\to\infty}\left\|G_j^{[r]}(f;x)-f\right\|=0.$$

□

Note that, if we take $r=0$, we obtain Theorem 2 in [19]. By using inequality (3.5), we can obtain the following result.

Corollary 3.2. *Let $x\in[0,1]$, $r\in\mathbb{N}$. Then for all $f\in C^r[0,1]$ such that $f^{(r)}\in Lip_M(\alpha)$ and $j\in\mathbb{N}$, we have*

$$\left|G_j^{[r]}(f;x)-f\right|\leq\frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\,\omega\Big(\left\|G_j\left(g_x^{r+\alpha},\delta_j\right)\right\|\Big),$$

where δ_j same as in Theorem 2.7.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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