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Investigation of the Spectrum of Nonself-Adjoint Discontinuous Sturm-Liouville Operator

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Abstract
In this paper, we study nonself-adjoint Sturm-Liouville operator containing both the discontinuous coefficient and discontinuity conditions at some point on the positive half-line. The eigenvalues and the spectral singularities of this problem are examined and it is proved that this problem has a finite number of spectral singularities and eigenvalues with finite multiplicities under two different additional conditions. Furthermore, the principal functions corresponding to the eigenvalues and the spectral singularities of this operator are determined.

Keywords: *Discontinuous coefficient, Discontinuity conditions, Eigenvalues and spectral singularities, Nonself-adjoint Sturm-Liouville operator, Principal functions*

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1. Introduction

The development of discontinuous boundary value problems has been great interest recently. It has an important role and making progress in the different field of mathematics and engineering such as mechanics, mathematical physics, geophysics (see [1–4]) and etc. Therefore, discontinuous Sturm-Liouville problems have attracted attention and numerous studies have been done on this subject. The difference between this study from others is that the nonself-adjoint discontinuous Sturm-Liouville problem which includes both a discontinuous coefficient and the discontinuity conditions at the point on the positive half line is investigated. Namely, we take into account the following nonself-adjoint problem created by the Sturm-Liouville equation with discontinuous coefficient

$$\ell(\varphi) = -\varphi'' + q(\xi)\varphi = \mu^2\rho(\xi)\varphi, \quad \xi \in (0,a) \cup (a,\infty), \tag{1.1}$$

with the discontinuity conditions

$$\varphi(a-0) = \alpha\varphi(a+0), \qquad \varphi'(a-0) = \alpha^{-1}\varphi'(a+0) \tag{1.2}$$

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and boundary condition

$$\varphi(0)=0, \tag{1.3}$$

where $0<\alpha\neq 1$, μ is a complex parameter, $\rho(\xi)$ is the piecewise continuous functions

$$\rho(\xi)=\begin{cases}\beta^2, & 0<\xi<a,\\ 1, & a<\xi<\infty\end{cases}$$

with $0<\beta\neq 1$, $q(\xi)$ is a complex-valued function and satisfies the condition

$$\int_0^\infty \xi|q(\xi)|d\xi<\infty. \tag{1.4}$$

The spectral theory of nonself-adjoint operator in the classical case (i.e., $\rho(\xi)\equiv 1$ and $\alpha=1$) was studied by Naimark [5, 6]. He shows that some poles of the resolvent kernel are not the eigenvalues of the operator and belong to the continuous spectrum; moreover, these poles are called spectral singularities and were first introduced by Schwartz [7]. In the self-adjoint case, the operator has a finite number of eigenvalues under the condition (1.4) (see [8]); however, in the nonself-adjoint case, the operator has a finite number of eigenvalues under the additional restriction. For example, the condition

$$\sup_{0\leq\xi<\infty}\{|q(\xi)|\exp(\varepsilon\xi)\}<\infty,\quad \varepsilon>0$$

was introduced by Naimark (see [5]) and it is shown that the number of eigenvalues is finite under this condition. Then, Pavlov weakened this additional condition as follows (see [9]):

$$\sup_{0\leq\xi<\infty}\left\{|q(\xi)|\exp(\varepsilon\sqrt{\xi})\right\}<\infty,\quad \varepsilon>0$$

and demonstrates that the operator has a finite number of eigenvalues. Moreover, Adıvar and Akbulut [10] obtain that the operator has a finite number of the eigenvalues under the following additional condition:

$$\sup_{0\leq\xi<\infty}\left\{|q(\xi)|\exp\left(\varepsilon\xi^\delta\right)\right\}<\infty,\quad \varepsilon>0,\ \frac{1}{2}\leq\delta<1.$$

Note that for any $0<\delta<\frac{1}{2}$, the condition does not provide that the number of eigenvalues is finite (see [11]). The spectral singularities have an essential role in the spectral analysis of the nonself-adjoint operator and Lyantse [12, 13] investigated the influence of the spectral singularities in the spectral expansion with respect to the principal functions of the operator. The investigations on the spectrum, principal functions and the spectral expansion with respect to the principal functions of the nonself-adjoint operator are very attractive and there are many works on the nonself-adjoint operator under different boundary conditions (see [14–22] and the references therein). Moreover, the nonself-adjoint operator with discontinuous coefficient is studied in [10], some spectral properties of the impulsive Sturm-Liouville operator is worked in [23].

To purpose of this study is to investigate the spectrum and the principal functions of the nonself-adjoint discontinuous problem (1.1)-(1.3). In examining this problem, we use new Jost solution of the equation (1.1) with discontinuity condition (1.2). The presence of the discontinuous parameter $\rho(\xi)$ and the discontinuity condition (1.2) strongly influence the structure of the representation of the Jost solution, so the triangular property of the Jost solution representation is lost and the kernel function has a discontinuity along the line $s=\beta(a-\xi)+a$ for $\xi\in(0,a)$ (see [24]).

2. Preliminaries

Assume that the function $e(\xi,\mu)$ satisfies the equation (1.1), discontinuity conditions (1.2) and condition at infinity

$$\lim_{\xi\to\infty}e^{-i\mu\xi}e(\xi,\mu)=1.$$

Note that the function $e(\xi,\mu)$ is defined as a Jost solution of equation (1.1).

Theorem 2.1. *Let a complex-valued function $q(\xi)$ satisfies equation (1.4). Then for all μ from the closed upper half-plane, there exists the Jost solution $e(\xi,\mu)$ of equation (1.1) with discontinuity conditions (1.2), it is unique and representable in the form*

$$e(\xi,\mu)=e_0(\xi,\mu)+\int_{\tau(\xi)}^{\infty}k(\xi,s)e^{i\mu s}ds, \tag{2.1}$$

where

$$e_0(\xi,\mu)=\begin{cases}e^{i\mu\xi}, & \xi>a,\\ \theta^+e^{i\mu(\beta(\xi-a)+a)}+\theta^-e^{i\mu(-\beta(\xi-a)+a)}, & 0<\xi<a,\end{cases}$$

with $\theta^{\pm}=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha\beta}\right)$ and $\theta^++|\theta^-|>1$,

$$\tau(\xi)=\begin{cases}\xi, & \xi>a,\\ \beta(\xi-a)+a, & 0<\xi<a,\end{cases}$$

the kernel function $k(\xi,.)\in L_1(\tau(\xi),\infty)$ for each fixed $\xi\in(0,a)\cup(a,\infty)$ and satisfies the inequality

$$\int_{\tau(\xi)}^{\infty}|k(\xi,s)|ds\le e^{c\sigma_1(\xi)}-1,\quad \sigma_1(\xi)=\int_{\xi}^{\infty}t|q(t)|dt,\quad c=\theta^++|\theta^-|. \tag{2.2}$$

Remark 2.1. The above theorem is proved in [24] when the $q(\xi)$ is real valued function. In case the $q(\xi)$ is complex valued function, the theorem is proved in the same way.

Lemma 2.1. *The following estimate holds:*

$$|k(\xi,s)|\le\frac{c}{2}\sigma\left(\frac{\tau(\xi)+s}{2}\right)e^{(c+1)\sigma_1(\xi)},\quad c=\theta^++|\theta^-|. \tag{2.3}$$

Proof. The function $k(\xi,s)$ is in the form for $0<\xi<a$:

$$\begin{aligned}k(\xi,s) &= k_0(\xi,s)+\frac{1}{2\beta}\int_{\xi}^{a}q(\zeta)\int_{s-\beta(\zeta-\xi)}^{s+\beta(\zeta-\xi)}k(\zeta,u)dud\zeta+\frac{\theta^+}{2}\int_{a}^{\infty}q(\zeta)\int_{s-\zeta+\beta(\xi-a)+a}^{s+\zeta+\beta(a-\xi)-a}k(\zeta,u)dud\zeta\\ &\quad-\frac{\theta^-}{2}\int_{a}^{\beta(a-\xi)+a}q(\zeta)\int_{s+\zeta+\beta(\xi-a)-a}^{s-\zeta+\beta(a-\xi)+a}k(\zeta,u)dud\zeta+\frac{\theta^-}{2}\int_{\beta(a-\xi)+a}^{\infty}q(\zeta)\int_{s-\zeta+\beta(a-\xi)+a}^{s+\zeta+\beta(\xi-a)-a}k(\zeta,u)dud\zeta,\end{aligned}$$

where for $\beta(\xi-a)+a<s<\beta(a-\xi)+a$

$$k_0(\xi,s)=\frac{\theta^+}{2\beta}\int_{\frac{s+\beta(\xi+a)-a}{2\beta}}^{a}q(\zeta)d\zeta+\frac{\theta^-}{2\beta}\int_{\frac{\beta(\xi+a)+a-s}{2\beta}}^{a}q(\zeta)d\zeta+\frac{\theta^+}{2}\int_{a}^{\infty}q(\zeta)d\zeta-\frac{\theta^-}{2}\int_{a}^{\frac{s+\beta(a-\xi)+a}{2}}q(\zeta)d\zeta, \tag{2.4}$$

and for $\beta(a-\xi)+a<s<\infty$

$$k_0(\xi,s)=\frac{\theta^+}{2}\int_{\frac{s+\beta(\xi-a)+a}{2}}^{\infty}q(\zeta)d\zeta+\frac{\theta^-}{2}\int_{\frac{s+\beta(a-\xi)+a}{2}}^{\infty}q(\zeta)d\zeta, \tag{2.5}$$

and for the kernel $k(\xi,s)$ has the form for $\xi>a$

$$k(\xi,s)=k_0(\xi,s)+\frac{1}{2}\int_{\xi}^{\infty}q(\zeta)\int_{s-\zeta+\xi}^{s+\zeta-\xi}k(\zeta,u)dud\zeta,$$

where

$$k_0(\xi,s)=\frac{1}{2}\int_{\frac{\xi+s}{2}}^{\infty}q(\zeta)d\zeta.$$

When $\xi>a$, we face the classical case (see [6]) and we have

$$|k(\xi,s)|\le\frac{1}{2}e^{\sigma_1(\xi)}\sigma\left(\frac{\xi+s}{2}\right).$$

Now, let us examine the case $0<\xi<a$. Set

$$\begin{aligned} k_m(\xi,s) &= \frac{1}{2\beta}\int_\xi^a q(\zeta)\int_{s-\beta(\zeta-\xi)}^{s+\beta(\zeta-\xi)} k_{m-1}(\zeta,u)dud\zeta \\ &+\frac{\theta^+}{2}\int_a^\infty q(\zeta)\int_{s-\zeta+\beta(\xi-a)+a}^{s+\zeta+\beta(a-\xi)-a} k_{m-1}(\zeta,u)dud\zeta \\ &-\frac{\theta^-}{2}\int_a^{\beta(a-\xi)+a} q(\zeta)\int_{s+\zeta+\beta(\xi-a)-a}^{s-\zeta+\beta(a-\xi)+a} k_{m-1}(\zeta,u)dud\zeta \\ &+\frac{\theta^-}{2}\int_{\beta(a-\xi)+a}^{\infty} q(\zeta)\int_{s-\zeta+\beta(a-\xi)+a}^{s+\zeta+\beta(\xi-a)-a} k_{m-1}(\zeta,u)dud\zeta, \quad m=1,2... \end{aligned}$$

and $k_0(\xi,s)$ is determined by the formulas (2.4) and (2.5). Then, we obtain for $0<\xi<a$:

$$|k_0(\xi,s)|\le\frac{c}{2}\sigma\left(\frac{s+\beta(\xi-a)+a}{2}\right),$$

$$|k_m(\xi,s)|\le\frac{c}{2}\sigma\left(\frac{s+\beta(\xi-a)+a}{2}\right)\frac{(c+1)^m(\sigma_1(\xi))^m}{m!}.$$

This implies that the series $\sum_{m=0}^{\infty}k_m(\xi,s)$ converges and its sum $k(\xi,s)$ satisfies the inequality

$$|k(\xi,s)|\le\frac{c}{2}\sigma\left(\frac{\beta(\xi-a)+a+s}{2}\right)e^{(c+1)\sigma_1(\xi)},\quad 0<\xi<a.$$

Moreover, since for $\xi>a$

$$|k(\xi,s)|\le\frac{1}{2}e^{\sigma_1(\xi)}\sigma\left(\frac{\xi+s}{2}\right),$$

we obtain that for $\xi\in(0,a)\cup(a,\infty)$ the inequality (2.3) is valid. □

Now, we define $\hat{e}(\xi,\mu)$ as the solution of the equation (1.1) with discontinuity conditions (1.2) and satisfies

$$\lim_{\xi\to\infty}e^{i\mu\xi}\hat{e}(\xi,\mu)=1$$

and when $q(\xi)\equiv 0$ in equation (1.1), the solution has the form:

$$\hat{e}_0(\xi,\mu)=\begin{cases} e^{-i\mu\xi}, & \xi>a,\\ \theta^+e^{-i\mu(-\beta(a-\xi)+a)}+\theta^-e^{-i\mu(\beta(a-\xi)+a)}, & 0<\xi<a.\end{cases} \tag{2.6}$$

The Wronskian of the solutions $e(\xi,\mu)$ and $\hat{e}(\xi,\mu)$ is obtained as

$$w[e(\xi,\mu),\hat{e}(\xi,\mu)]=-2i\mu,\quad Im\mu>0.$$

3. The eigenvalues and spectral singularities

Denote the boundary value problem (1.1)-(1.3) by an operator L operating on the Hilbert space $L_{2,\rho}(0,\infty)$. The values $\lambda=\mu^2$ for which the operator L has a non-zero solution are said eigenvalues and the corresponding nontrivial solutions are defined as eigenfunctions.

Consider $\tilde{e}(\xi,\mu)=e(\xi,-\mu)$ with $Im\mu\le 0$ and the Wronskian of $e(\xi,\mu)$ and $\tilde{e}(\xi,\mu)$ is in the form:

$$w[e(\xi,\mu),\tilde{e}(\xi,\mu)]=-2i\mu,\quad Im\mu=0. \tag{3.1}$$

Let us describe $s(\xi,\mu)$ as the solution of the equation (1.1) under the discontinuity conditions (1.2) and the initial conditions

$$s(0,\mu)=0,\quad s'(0,\mu)=1.$$

It is obtained that

$$s(\xi,\mu)=\frac{\hat{e}(0,\mu)e(\xi,\mu)-e(0,\mu)\hat{e}(\xi,\mu)}{2i\mu},\quad Im\mu>0. \tag{3.2}$$

The following lemma is proved in the same way as in [6]:

Lemma 3.1. *1. The nonself-adjoint operator L does not have positive eigenvalues.*

2. The necessary and sufficient conditions that $\lambda \neq 0$ be an eigenvalue of L are that

$$e(0,\mu)=0,\ \ \lambda=\mu^2,\ \ Im\mu>0.$$

3. The set of eigenvalues of L is bounded, is no more than countable and its limit points can lie only on the half-axis $\lambda \geq 0$.

All numbers λ of the form $\lambda=\mu^2$, $Im\mu>0$, $e(0,\mu)\neq 0$ belongs to the resolvent set of L. Assume that $\lambda=\mu^2$ belongs to the resolvent set of L. Then, the resolvent operator $R_{\mu^2}=\left(L-\mu^2 I\right)^{-1}$ exists and has the following representation:

$$R_{\mu^2}(L)=\int_0^\infty r(\xi,s;\mu^2)f(s)ds,$$

where

$$r(\xi,s;\mu^2)=\begin{cases}\frac{\hat{e}(0,\mu)e(\xi,\mu)e(s,\mu)}{2i\mu e(0,\mu)}-\frac{\hat{e}(\xi,\mu)e(s,\mu)}{2i\mu}, & \xi<s<\infty,\\ \\ \frac{\hat{e}(0,\mu)e(\xi,\mu)e(s,\mu)}{2i\mu e(0,\mu)}-\frac{e(\xi,\mu)\hat{e}(s,\mu)}{2i\mu}, & 0<s<\xi.\end{cases}$$

Note that all number $\lambda>0$ belongs to the continuous spectrum of L (see [6]).

The spectral singularities is defined as the poles of the kernel function of the resolvent operator and belong to the continuous spectrum. The operator L which has the compact set of spectral singularities, has zero measure in the sense of Lebesgue. This is provided from the boundary uniqueness theorem of analytic functions [25] (also, see [10]).

Denote the eigenvalues and spectral singularities of the operator L, respectively, as follows:

$$\sigma_d(L)=\left\{\lambda:\lambda=\mu^2,\ Im\mu>0,\ e(0,\mu)=0\right\},$$

$$\sigma_{ss}(L)=\left\{\lambda:\lambda=\mu^2,\ Im\mu=0,\ \mu\neq 0,\ e(0,\mu)=0\right\}.$$

Moreover, the multiplicity of the corresponding eigenvalue and spectral singularity of L is called the multiplicity of the zero of $e(0,\mu)$.

3.1 The finiteness of eigenvalues and spectral singularities

Now, we will demonstrate that the nonself-adjoint operator L has a finite number of eigenvalues and spectral singularities under the two different additional restrictions, respectively.

Additional restriction 1:

$$\int_0^\infty e^{\epsilon\xi}|q(\xi)|d\xi<\infty,\ \ \epsilon>0, \tag{3.3}$$

This condition is introduced by M. A. Naimark (see [6]).

Theorem 3.1. *Assume that the condition (3.3) is valid. Then, the operator L has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. The condition (3.3) implies that

$$\sigma(\xi)=\int_\xi^\infty |q(t)|dt\leq C_\epsilon e^{-\epsilon\xi},$$

$$\sigma_1(\xi)=\int_\xi^\infty t|q(t)|dt\leq C_{\epsilon'} e^{-\epsilon'\xi},$$

where $C_\epsilon>0$, $C_{\epsilon'}>0$ and $0<\epsilon'<\epsilon$ (see [6]). Using these relations and the estimate (2.3), we have

$$|k(\xi,s)|\leq C\exp\left\{-\epsilon\left(\frac{\tau(\xi)+s}{2}\right)\right\}, \tag{3.4}$$

where $C=\frac{c}{2}c_\epsilon e^{(c+1)d_\epsilon}$, $c=\theta^+ +|\theta^-|>1$, $c_\epsilon>0$ and $d_\epsilon>0$. It is obtained from (3.4) that the function $e(0,\mu)$ has an analytic continuation from the real axis to the half plane $Im\mu>-\frac{\epsilon}{2}$. Then, there is no limit points of the sets of the eigenvalues $\sigma_d(L)$ and the spectral singularities $\sigma_{ss}(L)$ on the positive real line. Since $\sigma_d(L)$ and $\sigma_{ss}(L)$ are bounded and $e(0,\mu)$ is holomorphic in the half plane $Im\mu>-\frac{\epsilon}{2}$, the operator L has finite number of eigenvalues and spectral singularities with finite multiplicity. □

Additional restriction 2:

$$\sup_{0\leq\xi<\infty}\left\{\exp(\epsilon\xi^{\delta})|q(\xi)|\right\}<\infty,\ \ \epsilon>0,\ \ \frac{1}{2}\leq\delta<1. \tag{3.5}$$

To prove the finiteness of the eigenvalues and spectral singularities under the condition (3.5), firstly we define the set of zeros of $e(0,\mu)$ in the closed upper half plane $Im\mu\geq 0:$

$$M_1:=\{\mu:\ \mu\in\mathbb{C}_+,\ e(0,\mu)=0\},\quad M_2:=\{\mu:\ \mu\in\mathbb{R},\ \mu\neq 0,\ e(0,\mu)=0\},$$

moreover, define the sets of all limit points of M_1 and M_2 as M_3 and M_4, respectively and the set of all zeros of $e(0,\mu)$ with infinite multiplicity as M_5. We have

$$M_1\cap M_5=\emptyset,\ \ M_3\subset M_2,\ \ M_4\subset M_2,\ \ M_5\subset M_2$$

from the uniqueness theorem of analytic functions (see [26]) and

$$M_3\subset M_5,\quad M_4\subset M_5 \tag{3.6}$$

from the continuity of all derivatives of the function $e(0,\mu)$ up to the real axis.

Lemma 3.2. *Assume that the condition (3.5) is satisfied, then $M_5=\emptyset$.*

Proof. To prove this lemma, we use the following theorem (see [9], also [10, 14]): Suppose that the function ψ is holomorphic function on the upper half plane without real line and all of its derivatives are also continuous on the real axis, and there exists $T>0$ such that

$$|\psi^{(m)}(z)|\leq K_m,\quad m=0,1,...\ z\in\mathbb{C}_+,\ |z|<2T, \tag{3.7}$$

and

$$\left|\int_{-\infty}^{-T}\frac{\ln|\psi(\xi)|}{1+\xi^2}d\xi\right|<\infty,\quad \left|\int_{T}^{\infty}\frac{\ln|\psi(\xi)|}{1+\xi^2}d\xi\right|<\infty. \tag{3.8}$$

If the set Q with linear Lebesgue measure zero is the set of all zeros of the function ψ with infinite multiplicity and if

$$\int_0^h \ln F(s)d\mu(Q_s)=-\infty, \tag{3.9}$$

then $\psi(z)\equiv 0$, where $F(s)=\inf_m\frac{K_m s^m}{m!}$, $m=0,1,...$, $\mu(Q_s)$ is the linear Lebesgue measure of s-neighborhood of Q and h is an arbitrary positive constant.

Now, it is obtained from the relation (2.3) and the condition (3.5) that

$$|k(\xi,s)|\leq\widetilde{C}\exp\left\{-\epsilon\left(\frac{\tau(\xi)+s}{2}\right)^{\delta}\right\},\ \ \widetilde{C}=\frac{c}{2}c_\epsilon e^{(c+1)c_\epsilon},\ \ c=\theta^++|\theta^-|>1.$$

Then, the function $e(0,\mu)$ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis and we have

$$\left|\frac{d^m e(0,\mu)}{d\mu^m}\right|\leq K_m,\ \mu\in\overline{\mathbb{C}}_+,\ m=1,2,..., \tag{3.10}$$

where

$$K_m=\widetilde{C}(\beta a+a)^m\left\{1+\int_0^\infty s^m\exp\left\{-\epsilon\left(\frac{s}{2}\right)^{\delta}\right\}ds\right\},\ \ m=1,2,...$$

Moreover, since the set of zeros of $e(0,\mu)$ is bounded, for sufficiently large T the function $e(0,\mu)$ satisfies the condition (3.8). Thus, it follows from this fact and the relation (3.10) that $e(0,\mu)$ provides the conditions (3.7) and (3.8). Since the function $e(0,\mu)\neq 0$, from (3.9), we have

$$\int_0^h \ln F(s)d\mu(M_{5,s})>-\infty, \tag{3.11}$$

where $F(s) = \inf_m \frac{K_m s^m}{m!}$ and $\mu(M_{5,s})$ is the Lebesgue measure of the s-neighborhood of M_5. The following estimate holds:

$$K_m \leq \left(\widetilde{C}(\beta a + a)^m + Dd^m\right) m^m m!, \tag{3.12}$$

where $D = 4\frac{\widetilde{C}e}{\delta}\epsilon^{-\frac{1}{\delta}}(m+1)$ and $d = 4(\beta a + a)\epsilon^{-\frac{1}{\delta}}$. In fact, we can write

$$\begin{aligned} K_m &= \widetilde{C}(\beta a + a)^m \left\{1 + \int_0^\infty s^m \exp\left\{-\epsilon\left(\frac{s}{2}\right)^\delta\right\} ds\right\} \\ &\leq \widetilde{C}(\beta a + a)^m \left\{1 + \frac{2^{(m+1)}}{\delta}\epsilon^{-\frac{(m+1)}{\delta}}(2m+2)^{m+1} m!\right\} \\ &\leq \widetilde{C}(\beta a + a)^m \left\{1 + \frac{2^{2(m+1)}}{\delta}\epsilon^{-\frac{(m+1)}{\delta}}\left(1 + \frac{1}{m}\right)^m (m+1)m^m m!\right\} \\ &\leq \left(\widetilde{C}(\beta a + a)^m + Dd^m\right) m^m m!. \end{aligned}$$

Putting the estimate (3.12) into $F(s)$, we get

$$\begin{aligned} F(s) &\leq \widetilde{C}\inf_m \{(\beta a + a)^m m^m s^m\} + D\inf_m \{d^m m^m s^m\} \\ &\leq \widetilde{C}\exp\left\{-(\beta a + a)^{-1}s^{-1}e^{-1}\right\} + D\exp\left\{-d^{-1}s^{-1}e^{-1}\right\}. \end{aligned} \tag{3.13}$$

Then, taking into account (3.11) and (3.13), we have

$$\int_0^h \frac{1}{s} d\mu(M_{5,s}) < \infty.$$

This inequality is valid for an arbitrary s if and only if $d\mu(M_{5,s}) = 0$ or $M_5 = \emptyset$. □

Theorem 3.2. *If the condition (3.5) is satisfied, then the operator L has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. It follows from (3.6) and Lemma 3.2 that $M_3 = \emptyset$ and $M_4 = \emptyset$. For this reason, the bounded sets M_1 and M_2 do not have limit points. Thus, the finiteness of the sets of eigenvalues $\sigma_d(L)$ and spectral singularities $\sigma_{ss}(L)$ are found. Moreover, due to $M_5 = \emptyset$, the eigenvalues and spectral singularities have finite multiplicities. □

4. Principal functions

In this section, we examine the principal functions of the nonself-adjoint operator L. Now, assume that the condition (3.5) is provided.

Denote $\mu_1, \mu_2, ..., \mu_\ell$ by the zeros of $e(0,\mu)$ in $\mathbb{C}_+$ with multiplicities $n_1, n_2, ..., n_\ell$, respectively (note that $\mu_1^2, \mu_2^2, ..., \mu_\ell^2$ are the eigenvalues of the operator L). We can write

$$\left\{\frac{d^m}{d\mu^m} W[e(\xi,\mu), s(\xi,\mu)]\right\}_{\mu=\mu_j} = \left\{\frac{d^m}{d\mu^m} e(0,\mu)\right\}_{\mu=\mu_j} = 0 \tag{4.1}$$

for $m = 0, 1, ..., n_j - 1$, $j = \overline{1,\ell}$. In case of $m = 0$, we have

$$e(\xi,\mu_j) = \kappa_0(\mu_j)s(\xi,\mu_j), \ \ \kappa_0(\mu_j) \neq 0, \ \ j = \overline{1,\ell}. \tag{4.2}$$

Lemma 4.1. *The following relation*

$$\left\{\frac{\partial^m}{\partial\mu^m} e(\xi,\mu)\right\}_{\mu=\mu_j} = \sum_{i=0}^m \begin{pmatrix} m \\ i \end{pmatrix} \kappa_{m-i} \left\{\frac{\partial^i}{\partial\mu^i} s(\xi,\mu)\right\}_{\mu=\mu_j} \tag{4.3}$$

is valid for $m = \overline{0, n_j - 1}$, $j = \overline{1,\ell}$ and here $\kappa_0, \kappa_1 ..., \kappa_m$ depend on μ_j.

Proof. To prove of this theorem, we use the mathematical induction. Consider $m=0$. It follows from the relation (4.2) that the proof is trivial. Now, suppose that the formula (4.3) holds for m_0 such that $0<m_0\leq n_j-2$. That is,

$$\left\{\frac{\partial^{m_0}}{\partial\mu^{m_0}}e(\xi,\mu)\right\}_{\mu=\mu_j}=\sum_{i=0}^{m_0}\left(\begin{array}{c}m_0\\ i\end{array}\right)\kappa_{m_0-i}\left\{\frac{\partial^i}{\partial\mu^i}s(\xi,\mu)\right\}_{\mu=\mu_j}. \tag{4.4}$$

Then, we will show that the formula (4.3) is satisfied for m_0+1. If $\varphi(\xi,\mu)$ is the solution of (1.1), then we find

$$\left\{-\frac{d^2}{d\xi^2}+q(\xi)-\mu^2\rho(\xi)\right\}\frac{\partial^m}{\partial\mu^m}\varphi(\xi,\mu)=2\mu m\rho(\xi)\frac{\partial^{m-1}}{\partial\mu^{m-1}}\varphi(\xi,\mu)+m(m-1)\rho(\xi)\frac{\partial^{m-2}}{\partial\mu^{m-2}}\varphi(\xi,\mu). \tag{4.5}$$

Since the functions $e(\xi,\mu)$ and $s(\xi,\mu)$ are solutions of the equation (1.1), using (4.4) and (4.5), we calculate

$$\left\{-\frac{d^2}{d\xi^2}+q(\xi)-\mu_j^2\rho(\xi)\right\}h_{m_0+1}(\xi,\mu_j)=0,$$

where

$$h_{m_0+1}(\xi,\mu_j)=\left\{\frac{\partial^{m_0+1}}{\partial\mu^{m_0+1}}e(\xi,\mu)\right\}_{\mu=\mu_j}-\sum_{i=0}^{m_0+1}\left(\begin{array}{c}m_0+1\\ i\end{array}\right)\kappa_{m_0+1-i}\left\{\frac{\partial^i}{\partial\mu^i}s(\xi,\mu)\right\}_{\mu=\mu_j}.$$

It follows from (4.1) that

$$W[h_{m_0+1}(\xi,\mu_j),s(\xi,\mu_j)]=\left\{\frac{d^{m_0+1}}{d\mu^{m_0+1}}W[e(\xi,\mu),s(\xi,\mu)]\right\}_{\mu=\mu_j}=0. \tag{4.6}$$

Then, this shows that

$$h_{m_0+1}(\xi,\mu_j)=\kappa_{m_0+1}(\mu_j)s(\xi,\mu_j),\ \ j=\overline{1,\ell}.$$

Consequently, we obtain that the formula (4.3) is satisfied for $m=m_0+1$. □

Now, we define the functions

$$\psi_m(\xi,\lambda_j)=\left\{\frac{\partial^m}{\partial\mu^m}e(\xi,\mu)\right\}_{\mu=\mu_j}=\sum_{i=0}^{m}\left(\begin{array}{c}m\\ i\end{array}\right)\kappa_{m-i}\left\{\frac{\partial^i}{\partial\mu^i}s(\xi,\mu)\right\}_{\mu=\mu_j} \tag{4.7}$$

for $m=\overline{0,n_j-1}$, $j=\overline{1,\ell}$ and $\lambda_j=\mu_j^2$.

Theorem 4.1. $\psi_m(\xi,\lambda_j)\in L_{2,\rho}(0,\infty),\ m=\overline{0,n_j-1},\ j=\overline{1,\ell}.$

Proof. Since

$$|k(\xi,s)|\leq\widetilde{C}\exp\left\{-\epsilon\left(\frac{\tau(\xi)+s}{2}\right)^\delta\right\},\ \ \widetilde{C}=\frac{c}{2}c_\epsilon e^{(c+1)c_\epsilon},\ \ c=\theta^++|\theta^-|>1,$$

using the integral representation (2.1), we have for $0<\xi<a$

$$\begin{aligned}\left|\left\{\frac{\partial^m}{\partial\mu^m}e(\xi,\mu)\right\}_{\mu=\mu_j}\right| &\leq \xi^m\theta^+e^{-Im\mu_j\xi}+(\beta(a-\xi)+a)^m|\theta^-|e^{-Im\mu_j(\beta(a-\xi)+a)}\\ &\quad+\widetilde{C}\int_{\beta(\xi-a)+a}^{\infty}s^n\exp\left\{-\epsilon\left(\frac{\beta(\xi-a)+a+s}{2}\right)^\delta\right\}e^{-Im\mu_js}ds\end{aligned} \tag{4.8}$$

and for $a<\xi<\infty$

$$\left|\left\{\frac{\partial^m}{\partial\mu^m}e(\xi,\mu)\right\}_{\mu=\mu_j}\right|\leq\xi^me^{-Im\mu_j\xi}+\widetilde{C}\int_\xi^\infty s^m\exp\left\{-\epsilon\left(\frac{\xi+s}{2}\right)^\delta\right\}e^{-Im\mu_js}ds. \tag{4.9}$$

Since $\lambda_j=\mu_j^2$, $j=\overline{1,\ell}$ are eigenvalues of the operator L, it is obtained from (4.8) and (4.9) for $Im\mu_j>0$ that

$$\left\{\frac{\partial^m}{\partial\mu^m}e(\xi,\mu)\right\}_{\mu=\mu_j}\in L_{2,\rho}(0,\infty),\ \ m=\overline{0,n_j-1},\ \ j=\overline{1,\ell}.$$

Consequently, from (4.7) we have $\psi_m(\xi,\lambda_j)\in L_{2,\rho}(0,\infty)$, $m=\overline{0,n_j-1}$, $j=\overline{1,\ell}$. □

Definition 4.1. The functions $\psi_0(\xi,\lambda_j)$, $\psi_1(\xi,\lambda_j)$,...,$\psi_{n_j-1}(\xi,\lambda_j)$ are called the principal functions associated with eigenvalues $\lambda_j=\mu_j^2$, $j=\overline{1,\ell}$ of the operator L. The function $\psi_0(\xi,\lambda_j)$ is the eigenfunction, $\psi_1(\xi,\lambda_j)$, $\psi_2(\xi,\lambda_j)$,..., $\psi_{n_j-1}(\xi,\lambda_j)$ are the associated functions of $\psi_0(\xi,\lambda_j)$ corresponding to eigenvalue λ_j.

Now, we define the spectral singularities of L: $\mu_{\ell+1},\mu_{\ell+2},...,\mu_p$ are the zeros of the function $e(0,\mu)$ in $\mathbb{R}-\{0\}$ with multiplicities $n_{\ell+1},n_{\ell+2},...,n_p$, respectively. Then, using the similar way in Lemma 4.1, we obtain

$$\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r}=\sum_{j=0}^{\eta}\begin{pmatrix}\eta\\ j\end{pmatrix}\tau_{\eta-j}(\mu_r)\left\{\frac{\partial^j}{\partial\mu^j}s(\xi,\mu)\right\}_{\mu=\mu_r}\tag{4.10}$$

for $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$. Denote the functions

$$\psi_\eta(\xi,\lambda_r)=\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r}=\sum_{j=0}^{\eta}\begin{pmatrix}\eta\\ j\end{pmatrix}\tau_{\eta-j}(\mu_r)\left\{\frac{\partial^j}{\partial\mu^j}s(\xi,\mu)\right\}_{\mu=\mu_r}\tag{4.11}$$

for $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$ and $\lambda_j=\mu_j^2$.

Theorem 4.2. *The functions $\psi_\eta(\xi,\lambda_r)$ do not belong to $L_{2,\rho}(0,\infty)$, $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$.*

Proof. Take into account the relations (4.8) and (4.9) for $\mu=\mu_r$, $r=\ell+1,\ell+2,...,p$ and since $Im\mu_r=0$ for the spectral singularities, we have

$$\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r}\notin L_{2,\rho}(0,\infty),\ \eta=\overline{0,n_r-1},\ r=\ell+1,\ell+2,...,p.$$

As a result, from the definition of the functions (4.11), we find $\psi_\eta(\xi,\lambda_r)\notin L_{2,\rho}(0,\infty)$, $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$. □

Now, we introduce the Hilbert spaces

$$H_{\zeta,\rho}=\left\{f:\|f\|_{\zeta,\rho}<\infty\right\},\ \ H_{-\zeta,\rho}=\left\{f:\|f\|_{-\zeta,\rho}<\infty\right\},\ \ \zeta=1,2,...$$

with the norms

$$\|f\|^2_{\zeta,\rho}=\int_0^\infty(1+\tau(s))^{2\zeta}|f(s)|^2\rho(s)ds,\quad \|f\|^2_{-\zeta,\rho}=\int_0^\infty(1+\tau(s))^{-2\zeta}|f(s)|^2\rho(s)ds,$$

respectively and evidently, $H_{0,\rho}=L_{2,\rho}(0,\infty)$.

Let n_0 denotes the greatest of the multiplicities of the spectral singularities of L:

$$n_0=\max\{n_{\ell+1},n_{\ell+2},...,n_p\}.$$

We put

$$H_{+,\rho}=H_{(n_0+1),\rho},\quad H_-=H_{-(n_0+1),\rho}$$

Then, we have

$$H_{+,\rho}\subset L_{2,\rho}(0,\infty)\subset H_{-,\rho}$$

and for all $f\in H_{+,\rho}$, $\|f\|_{+,\rho}\geq\|f\|_\rho\geq\|f\|_{-,\rho}$, where $\|.\|_{\pm,\rho}=\|.\|_{\pm(n_0+1),\rho}$, $\|.\|_\rho=\|.\|_{0,\rho}$ (see [6]). We are particularly interested in the space $H_{\pm,\rho}$ because the space $H_{-,\rho}$ contains the principal functions for the spectral singularities. Now, we will prove this claim by using following lemma.

Lemma 4.2. *The following relation holds:*

$$\sup_{0\leq\xi<\infty}\frac{|e^{(n)}(\xi,\mu)|}{(1+\tau(\xi))^n}<\infty,\ \ e^{(n)}=\left(\frac{d}{d\mu}\right)^n e,\ \ Im\mu=0,\ \ n=0,1,2,...\tag{4.12}$$

Proof. Using the integral representation (2.1), we obtain for $Im\mu = 0$

$$|e^{(n)}(\xi,\mu)| \leq \xi^n\theta^+ + (\beta(a-\xi)+a)^n|\theta^-| + \widetilde{C}\int_{\beta(\xi-a)+a}^{\infty} s^n \exp\left\{-\epsilon\left(\frac{\beta(\xi-a)+a+s}{2}\right)^{\delta}\right\}ds, \quad 0<\xi<a \tag{4.13}$$

and

$$|e^{(n)}(\xi,\mu)| \leq \xi^n + \widetilde{C}\int_{\xi}^{\infty} s^n \exp\left\{-\epsilon\left(\frac{\xi+s}{2}\right)^{\delta}\right\}ds, \quad a<\xi<\infty. \tag{4.14}$$

Then, taking into account (4.13) and (4.14), we find $\sup_{0\leq\xi<\infty}\frac{|e^{(n)}(\xi,\mu)|}{(1+\tau(\xi))^n} < \infty$ for $Im\mu = 0$. □

Theorem 4.3. $\psi_\eta(\xi,\lambda_r) \in H_{-(\eta+1),\rho}$, $\eta = \overline{0, n_r - 1}$, $r = \ell+1, \ell+2, ..., p$.

Proof. Using the relation (4.12), we have

$$\left\|e^{(\eta)}(\xi,\mu)\right\|^2_{-(\eta+1),\rho} = \int_0^{\infty}|\frac{e^{(\eta)}(\xi,\mu)}{(1+\tau(\xi))^{\eta+1}}|^2\rho(\xi)d\xi < \infty.$$

That is, the functions $e^{(\eta)}(\xi,\mu) = \frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu) \in H_{-(\eta+1)}$ for $Im\mu = 0$ and $\eta = 0, 1, 2, ...$. Then, we get

$$\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r} \in H_{-(\eta+1),\rho}$$

for $Im\mu_r = 0$, $\eta = \overline{0, n_r - 1}$ and $r = \ell+1, \ell+2, ..., p$. Consequently, it follows from the formula (4.11) that $\psi_\eta(\xi,\lambda_r) \in H_{-(\eta+1),\rho}$, $\eta = \overline{0, n_r - 1}$, $r = \ell+1, \ell+2, ..., p$. □

Definition 4.2. The functions $\psi_0(\xi,\lambda_r), \psi_1(\xi,\lambda_r), ..., \psi_{n_r-1}(\xi,\lambda_r)$ are defined as the principal functions associated with the spectral singularities $\lambda_r = \mu_r^2$, $r = \ell+1, \ell+2, ..., p$ of operator L. The function $\psi_0(\xi,\lambda_r)$ is the generalized eigenfunction, $\psi_1(\xi,\lambda_r), ..., \psi_{n_r-1}(\xi,\lambda_r)$ are the generalized associated functions of $\psi_0(\xi,\lambda_r)$ corresponding to spectral singularity λ_r.

5. Conclusion

In this paper, we examine the spectrum and the principal functions of the nonself-adjoint discontinuous Sturm-Liouville operator which contains the discontinuous coefficient and the discontinuity conditions at the point on the positive half line. When examining the spectrum of the considered problem (1.1)-(1.3), we use the newly constructed Jost solution of the equation (1.1) with discontinuity condition (1.3). This solution is completely different from the classical Jost solution because of the presence of the discontinuous coefficient $\rho(\xi)$ and discontinuity condition (1.2). We point out that the triangular property of the Jost solution representation is lost and the kernel function has a discontinuity along the line $s = \beta(\xi - a) + a$ for $\xi \in (0, a)$. Under two different additional conditions, it is proved that the problem (1.1)-(1.3) has finite number of eigenvalues and spectral singularities with finite multiplicity. Finally using the additional restriction (3.5) which is weaker than the restriction (3.3), we determine the principal functions corresponding to the eigenvalues and the spectral singularities of the problem (1.1)-(1.3).

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