International Electronic Journal of Algebra
Volume 36 (2024) 134-156
DOI: 10.24330/ieja.1411161

STRONGLY J-N-COHERENT RINGS

Zhanmin Zhu

Received: 30 September 2023; Revised: 28 October 2023; Accepted: 28 October 2023
Communicated by Burcu Üngör

Abstract. Let R be a ring and n a fixed positive integer. A right R-module M is called strongly J-n-injective if every R-homomorphism from an n-generated small submodule of a free right R-module F to M extends to a homomorphism of F to M; a right R-module V is said to be strongly J-n-flat, if for every n-generated small submodule T of a free left R-module F, the canonical map $V \otimes T \to V \otimes F$ is monic; a ring R is called left strongly J-n-coherent if every n-generated small submodule of a free left R-module is finitely presented; a ring R is said to be left J-n-semihereditary if every n-generated small left ideal of R is projective. We study strongly J-n-injective modules, strongly J-n-flat modules and left strongly J-n-coherent rings. Using the concepts of strongly J-n-injectivity and strongly J-n-flatness of modules, we also present some characterizations of strongly J-n-coherent rings and J-n-semihereditary rings.

Mathematics Subject Classification (2020): 16D40, 16D50, 16P70, 16E60
Keywords: Strongly J-n-injective module, strongly J-n-flat module, strongly J-n-coherent ring, J-n-semihereditary ring

1. Introduction

Throughout this paper, m and n are positive integers unless otherwise specified, R is an associative ring with identity, I is an ideal of R, $J = J(R)$ is the Jacobson radical, and all modules considered are unitary.

Recall that a ring R is called *left coherent* [2,14] (resp., *left semihereditary* [1]) if every finitely generated left ideal of R is finitely presented (resp., projective). Left coherent rings, left semihereditary rings and their generalizations have been studied by many authors. For example, a ring R is said to be *left J-coherent* [6] (resp., *left J-semihereditary* [6]) if every finitely generated left ideal in $J(R)$ is finitely presented (resp., projective); a ring R is said to be *left n-coherent* [13] (resp., *left n-semihereditary* [18,19]) if every n-generated left ideal of R is finitely presented (resp., projective). By [19, Theorem 1], a ring R is left n-semihereditary if and only

This research was supported by the Natural Science Foundation of Zhejiang Province, China (LY18A010018).

if every n-generated submodule of a projective left R-module is projective. Let I be an ideal of R. Then according to [20], R is called *left I-n-coherent* (resp., *left I-n-semihereditary*) if every n-generated left ideal in I is finitely presented (resp., projective).

In this article, we extend the concept of left J-n-coherent rings to *left strongly J-n-coherent rings*. We call a ring R left strongly J-n-coherent if every n-generated small submodule of a free left R-module is finitely presented, and we call a ring R left J-n-semihereditary if every n-generated small left ideal of R is projective. To characterize left strongly J-n-coherent rings, in Section 2 and Section 3, strongly J-n-injective modules and strongly J-n-flat modules are introduced and studied respectively. In Section 4 and Section 5, left strongly J-n-coherent rings and left J-n-semihereditary rings are investigated respectively.

For any R-module M, M^* denotes $\mathrm{Hom}_R(M,R)$, and M^+ denotes $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Z}$ is the set of integers. In general, for a set S, we write S^n for the set of all formal $1\times n$ matrices whose entries are elements of S, and S_n for the set of all formal $n\times 1$ matrices whose entries are elements of S. Let N be a left R-module, $X\subseteq N_n$ and $A\subseteq R^n$. Then we define $\mathbf{r}_{N_n}(A)=\{u\in N_n : au=0,\forall a\in A\}$ and $\mathbf{l}_{R^n}(X)=\{a\in R^n : ax=0,\forall x\in X\}$.

2. Strongly J-n-injective modules

Recall that a submodule U' of a right R-module U is called a *pure* submodule of U if the canonical map $U'\otimes_R M\to U\otimes_R M$ is a monomorphism for every left R-module M, equivalently, if the canonical map $U'\otimes_R V\to U\otimes_R V$ is a monomorphism for every finitely presented left R-module V. Let I be an ideal of R. Then following [21], a left R-module V is said to be *I-(m,n)-presented*, if there is an exact sequence of left R-modules $0\to K\to R^m\to V\to 0$ with K an n-generated submodule of I^m; a left R-module V is said to be *I-finitely presented*, if it is I-(m,n)-presented for a pair of positive integers m,n. In [21], we extend the concept of pure submodules to *I-(m,n)-pure* submodules, *I-(m,∞)-pure* submodules , *I-(∞,n)-pure* submodules and *I-pure* submodules respectively. Given a right R-module U with submodule U', according to [21], U' is called *I-(m,n)-pure* in U if the canonical map $U'\otimes_R V\to U\otimes_R V$ is a monomorphism for every I-(m,n)-presented left R-module V; U' is said to be *I-(m,∞)-pure* (resp., *I-(∞,n)-pure*) in U in case U' is I-(m,n)-pure in U for all positive integers n (resp., m); U' is said to be *I-pure* in U in case U' is I-(m,n)-pure in U for all positive integers m and n. By [21, Theorem 2.4], we have immediately the following two lemmas.

Lemma 2.1. *Let $U'_R \leq U_R$. Then the following statements are equivalent:*

(1) *U' is J-(n,∞)-pure in U.*

(2) *For every finitely generated free right R-module F and each n-generated small submodule T of F, the canonical map*

$$\mathrm{Hom}_R(F/T, U) \to \mathrm{Hom}_R(F/T, U/U')$$

is surjective.

Lemma 2.2. *Let $U'_R \leq U_R$. Then the following statements are equivalent:*

(1) *U' is J-(∞,n)-pure in U.*

(2) *For every finitely generated small submodule T of R^n_R, the canonical map $\mathrm{Hom}_R(R^n/T, U) \to \mathrm{Hom}_R(R^n/T, U/U')$ is surjective.*

Recall that a right R-module M is called I-(m,n)-*injective* [21], if every R-homomorphism from an n-generated submodule T of I^m to M extends to one from R^m to M. A right R-module M is called *I-n-injective* [20] if it is I-$(1,n)$-injective. Inspired by these concepts, we introduce the concept of strongly J-n-injective modules as follows.

Definition 2.3. A right R-module M is called strongly J-n-injective if every R-homomorphism from an n-generated small submodule of a free right R-module F to M extends to a homomorphism of F to M. A right R-module M is called J-FP-injective if every R-homomorphism from a finitely generated small submodule of a free right R-module F to M extends to a homomorphism of F to M. A ring R is called right strongly J-n-injective (resp., right J-FP-injective) if the right R-module R_R is strongly J-n-injective (resp., J-FP-injective).

It is easy to see that a right R-module M is strongly J-n-injective if and only if it is J-(m,n)-injective for every positive integer m; a right R-module M is J-FP-injective if and only if it is strongly J-n-injective for every positive integer n.

Theorem 2.4. *Let M be a right R-module. Then the following statements are equivalent:*

(1) *M is strongly J-n-injective.*

(2) $\mathrm{Ext}^1(F/T, M) = 0$ *for every free right R-module F and every n-generated small submodule T of F.*

(3) $\mathrm{Ext}^1(F/T, M) = 0$ *for every finitely generated free right R-module F and every n-generated small submodule T of F.*

(4) $\mathbf{l}_{M^n}\mathbf{r}_{R_n}\{\alpha_1, \cdots, \alpha_m\} = M\alpha_1 + \cdots + M\alpha_m$ *for every positive integer m and any $\alpha_1, \cdots, \alpha_m \in (J(R))^n$.*

(5) *M is* J*-*(n,∞)*-pure in every module containing M.*
(6) *M is* J*-*(n,∞)*-pure in E(M).*

Proof. It follows from [21, Theorem 3.2]. □

Corollary 2.5. *Every* J*-*(n,∞)*-pure submodule of a strongly* J*-*n*-injective module is strongly* J*-*n*-injective.*

Proof. Let N be a J-(n,∞)-pure submodule of a strongly J-n-injective right R-module M. For any n-generated small submodule T of a finitely generated free right R-module F, we have the exact sequence

$$\mathrm{Hom}(F/T,M)\to\mathrm{Hom}(F/T,M/N)\to\mathrm{Ext}^1(F/T,N)\to\mathrm{Ext}^1(F/T,M)=0.$$

Since N is J-(n,∞)-pure in M, by Lemma 2.1, the sequence

$$\mathrm{Hom}(F/T,M)\to\mathrm{Hom}(F/T,M/N)\to 0$$

is exact. Hence $\mathrm{Ext}^1(F/T,N)=0$, and so N is strongly J-n-injective. □

Theorem 2.6. *The following statements are equivalent for a ring R:*

(1) *R is right strongly J-n-injective.*
(2) *Every finitely generated small submodule T of the left R-module* R^n *is a left annihilator of a subset X of* R_n*.*
(3) *If* $\mathbf{r}_{R_n}(T)\subseteq\mathbf{r}_{R_n}(\alpha)$ *for a finitely generated small submodule T of the left R-module* R^n *and* $\alpha\in R^n$*, then* $\alpha\in T$*.*
(4) R^n/T *is a torsionless left R-module for every finitely generated small submodule T of* R^n*.*
(5) $\mathbf{l}_{R^n}\mathbf{r}_{R_n}(T)=T$ *for every finitely generated small submodule T of the left R-module* R^n*.*

Proof. (1) ⇒ (2) follows from Theorem 2.4(4).

(2) ⇔ (4) ⇔ (5) follows from [22, Lemma 2.3].

(5) ⇒ (3) If $\mathbf{r}_{R_n}(T)\subseteq\mathbf{r}_{R_n}(\alpha)$, then $\alpha\in\mathbf{l}_{R^n}r_{R_n}(\alpha)\subseteq\mathbf{l}_{R^n}r_{R_n}(T)=T$ by (5).

(3) ⇒ (1) Let $F=(R_R)^m, K=\beta_1R+\cdots+\beta_nR$ be an n-generated small submodule of F, and f be a right R-homomorphism from K to R. Write

$$\beta_j=(b_{1j},\cdots,b_{mj}),\ j=1,\cdots,n;$$
$$\alpha_i=(b_{i1},\cdots,b_{in}),\ i=1,\cdots,m;$$
$$\alpha=(f(\beta_1),\cdots,f(\beta_n));$$
$$T=R\alpha_1+\cdots+R\alpha_m.$$

Then T is a small submodule of the left R-module R^n and $\mathbf{r}_{R_n}(T)\subseteq\mathbf{r}_{R_n}(\alpha)$. By (3), $\alpha\in T$, so $\alpha=c_1\alpha_1+\cdots+c_m\alpha_m$ for some $c_1,\cdots,c_m\in R$. Now we define $g : F \to R; (r_1, \cdots, r_m) \mapsto c_1r_1 + \cdots + c_mr_m$, then it is easy to check that $f(\beta_j) = g(\beta_j), j = 1, \cdots, n$, and therefore g extends f. □

Recall that a ring R is called *semiregular* [11] if for any $a \in R$, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. By [11, Theorem B.44], R is semiregular if and only if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. A right R-module M is called *semiregular* if for any $m \in M$, we have $M = P \oplus K$, where P is projective, $P \subseteq mR$, and $mR \cap K$ is small in K. It is easy to see that a ring R is semiregular if and only if the right R-module R_R is semiregular. By [11, Theorem B.51], a module M is semiregular if and only if, for any finitely generated submodule N of M, we have $M = P \oplus K$, where P is projective, $P \subseteq N$, and $N \cap K$ is small in K; and by [11, Theorem B.54], direct sums and direct summands of semiregular modules are semiregular. We recall also that a right R-module M is called *strongly n-injective* [22] if every R-homomorphism from an n-generated submodule of a free right R-module F to M extends to a homomorphism of F to M.

Proposition 2.7. *If R is a semiregular ring, then a right R-module M is strongly n-injective if and only if it is strongly J-n-injective.*

Proof. Necessity is clear. To prove the sufficiency, let N be an n-generated submodule of a finitely generated free right R-module F and $f : N \to M$ be a right R-homomorphism. Since R is semiregular, by [11, Lemma B.54], F is semiregular. So, by [11, Lemma B.51], $F = P \oplus K$, where P is projective, $P \subseteq N$ and $N \cap K$ is small in K. Hence $F = N + K$, $N = P \oplus (N \cap K)$, and so $N \cap K$ is n-generated and small in F. Since M is J-n-injective, there exists a homomorphism $g : F \to M$ such that $g(x) = f(x)$ for all $x \in N \cap K$. Now let $h : F \to M; x \mapsto f(n) + g(k)$, where $x = n + k, n \in N, k \in K$. Then h is a well-defined left R-homomorphism and h extends f. □

3. Strongly J-n-flat modules

Recall that a right R-module V is said to be *n-flat* [13,7], if for every n-generated left ideal T of R, the canonical map $V \otimes T \to V \otimes R$ is monic; a right R-module V is said to be *J-flat* [6], if for every finitely generated left ideal T in $J(R)$, the canonical map $V \otimes T \to V \otimes R$ is monic; a right R-module V is said to be *J-n-flat* [20], if for every n-generated left ideal T in $J(R)$, the canonical map $V \otimes I \to V \otimes R$ is monic. Inspired by these concepts, we introduce the concepts of *strongly J-n-flat modules* and *strongly J-flat modules* as follows.

Definition 3.1. A right R-module V is said to be strongly J-n-flat, if for every n-generated small submodule T of a free left R-module F, the canonical map $V \otimes T \to V \otimes F$ is monic. A right R-module V is said to be strongly J-flat if it is strongly J-n-flat for every positive integer n.

Theorem 3.2. *For a right R-module V, the following statements are equivalent:*

(1) *V is strongly J-n-flat.*

(2) $\mathrm{Tor}_1(V, F/L) = 0$ *for every finitely generated free left R-module F and any n-generated small submodule L of F.*

(3) $\mathrm{Tor}_1(V, F/L) = 0$ *for every free left R-module F and any n-generated small submodule L of F.*

(4) *V^+ is strongly J-n-injective.*

(5) *If the sequence of right R-modules $0 \to U' \to U \to V \to 0$ is exact, then U' is J-(∞, n)-pure in U.*

(6) *For every finitely generated small submodule T of the right R-module R^n and any homomorphism $f : R^n/T \to V$, f factors through a finitely generated free right R-module F, that is, there exist a homomorphism $g : R^n/T \to F$ and a homomorphism $h : F \to V$ such that $f = hg$.*

(7) *For every finitely generated small submodule T of the right R-module R^n and any homomorphism $f : R^n/T \to V$, f factors through a finitely generated projective right R-module P.*

(8) *For every finitely generated small submodule T of the right R-module R^n, if $g : M \to V$ is an epimorphism, then for any homomorphism $f : R^n/T \to V$, there exists a homomorphism $h : R^n/T \to M$ such that $f = gh$.*

Proof. (1) $\Leftrightarrow$ (2) follows from the exact sequence $0 \to \mathrm{Tor}_1(V, F/L) \to V \otimes L \to V \otimes F$.

(2) $\Leftrightarrow$ (3), and (6) $\Leftrightarrow$ (7) are obvious.

(2) $\Leftrightarrow$ (4) follows from the isomorphism $\mathrm{Tor}_1(M, F/L)^+ \cong \mathrm{Ext}^1(F/L, M^+)$.

(2) $\Rightarrow$ (5) Let $0 \to U' \to U \to V \to 0$ be an exact sequence of right R-modules. By (2), the canonical map $U' \otimes F/L \to U \otimes F/L$ is monic for any finitely generated free left R-module F and any n-generated small submodule L of F, and so U' is J-(∞, n)-pure in U.

(5) $\Rightarrow$ (2) Let $0 \to K \to F_1 \to V \to 0$ be an exact sequence of right R-modules, where F_1 is free. Then by (5), K is J-(∞, n)-pure in F_1. So it follows from the exact sequence

$$0 = \mathrm{Tor}_1^R(F_1, F/L) \to \mathrm{Tor}_1^R(V, F/L) \to K \otimes F/L \to F_1 \otimes F/L$$

that $\mathrm{Tor}_1^R(V, F/L) = 0$ for every finitely generated free left R-module F and any n-generated small submodule L of F.

(5) ⇒ (6) Let $0 \to K \to F_1 \to V \to 0$ be an exact sequence of right R-modules, where F_1 is free. Then by (5), K is J-(∞, n)-pure in F_1. And so, by Lemma 2.2, we have that the canonical map $\mathrm{Hom}(R^n/T, F_1) \to \mathrm{Hom}(R^n/T, V)$ is surjective for any finitely generated small submodule T of R^n_R. This follows that f factors through a finitely generated free right R-module F since R^n/T is finitely generated.

(6) ⇒ (5) Let $0 \to U' \to U \stackrel{\pi}{\to} V \to 0$ be an exact sequence of right R-modules with U J-n-flat. Then for any finitely generated small submodule T of R^n_R and any homomorphism $f : R^n/T \to V$, by (6), there exist a finitely generated free module F, two homomorphisms $g \in \mathrm{Hom}_R(R^n/T, F)$ and $h \in \mathrm{Hom}_R(F, V)$ such that $f = hg$. Since F is projective, there exists a homomorphism $\alpha : F \to U$ such that $h = \pi\alpha$. Thus, αg is a homomorphism from R^n/T to U and $f = \pi(\alpha g)$. So, the canonical map $\mathrm{Hom}_R(R^n/T, U) \to \mathrm{Hom}_R(R^n/T, V)$ is surjective, and then the canonical map $\mathrm{Hom}_R(R^n/T, U) \to \mathrm{Hom}_R(R^n/T, U/U')$ is surjective. By Lemma 2.2, U' is J-(∞, n)-pure in U.

(7) ⇒ (8) Let $g : M \to V$ be an epimorphism and $f : R^n/T \to V$ be any homomorphism, where T is a finitely generated small submodule of R^n. By (7), f factors through a finitely generated projective right R-module P, i.e., there exist $\varphi : R^n/T \to P$ and $\psi : F \to V$ such that $f = \psi\varphi$. Since P is projective, there exists a homomorphism $\theta : P \to M$ such that $\psi = g\theta$. Now write $h = \theta\varphi$, then h is a homomorphism from R^n/T to M, and $f = \psi\varphi = g(\theta\varphi) = gh$. And so (8) follows.

(8) ⇒ (7) Let F_1 be a free module and $\pi : F_1 \to V$ be an epimorphism. By (8), there exists a homomorphism $g : R^n/T \to F_1$ such that $f = \pi g$. Note that $\mathrm{Im}(g)$ is finitely generated, so there is a finitely generated free module F such that $\mathrm{Im}(g) \subseteq F \subseteq F_1$. Let $\iota : F \to F_1$ be the inclusion map and $h = \pi\iota$. Then h is a homomorphism from F to V and $f = hg$. □

Corollary 3.3. *For a right R-module V, the following statements are equivalent:*

(1) *V is strongly J-flat.*
(2) $\mathrm{Tor}_1(V, F/L) = 0$ *for every finitely generated free left R-module F and any finitely generated small submodule L of F.*
(3) $\mathrm{Tor}_1(V, F/L) = 0$ *for every free left R-module F and any finitely generated small submodule L of F.*
(4) *V^+ is J-FP-injective.*
(5) *If the sequence of right R-modules $0 \to U' \to U \to V \to 0$ is exact, then U' is J-pure in U.*

(6) *For every finitely generated small submodule T of a finitely generated free right R-module F, any homomorphism $f : F/T \to V$ factors through a finitely generated free right R-module F_1, that is, there exist a homomorphism $g : F/T \to F_1$ and a homomorphism $h : F_1 \to V$ such that $f = hg$.*

(7) *For every finitely generated small submodule T of a free right R-module F, any homomorphism $f : F/T \to V$ factors through a finitely generated projective right R-module P.*

(8) *For every finitely generated small submodule T of a free right R-module F, if $g : M \to V$ is an epimorphism, then for any homomorphism $f : F/T \to V$, there exists a homomorphism $h : F/T \to M$ such that $f = gh$.*

Proposition 3.4. *Every J-(n, ∞)-pure submodule of a strongly J-n-flat module is strongly J-n-flat.*

Proof. Suppose that V_R is strongly J-n-flat, K is J-(n, ∞)-pure in V. Let $X \in K^n$, $A \in J^{n\times m}$ satisfy $XA = 0$. Then by the strongly J-n-flatness of V, there exist positive integer l, $U \in V^l$ and $C \in R^{l\times n}$ such that $CA = 0$ and $X = UC$. Since K is J-(n, ∞)-pure in V and hence J-(n, l)-pure, by [21, Theorem 2.4(3)], we have $X = YC$ for some $Y \in K^l$. So K is J-(m, n)-flat for any positive integer m by [21, Theorem 4.2(5)], and hence K is strongly J-n-flat. ◻

Corollary 3.5. *Every J-pure submodule of a strongly J-flat module is strongly J-flat.*

Remark 3.6. From Theorem 3.2, the strongly J-n-flatness of V_R can be characterized by the strongly J-n-injectivity of V^+. On the other hand, by [5, Lemma 2.7(1)], the sequence $\mathrm{Tor}_1(V^+, M) \to \mathrm{Ext}^1(M, V)^+ \to 0$ is exact for all finitely presented left R-module M, so if V^+ is strongly J-n-flat, then V is strongly J-n-injective.

Proposition 3.7. *If R is a semiregular ring, then a left R-module M is strongly n-flat if and only if it is strongly J-n-flat.*

Proof. Theorem 3.2(4), Proposition 2.7 and [22, Theorem 3.1(4)] give the desired result. ◻

4. Strongly J-n-coherent rings

Recall that a ring R is called *left (m, n)-coherent* [17] if every n-generated submodule of R^m is finitely presented; a ring R is called *left J-coherent* [6] if every finitely generated left ideal in $J(R)$ is finitely presented; a ring R is called *left J-n-coherent* [20] if every n-generated left ideal in $J(R)$ is finitely presented. Inspired by these concepts, we introduce the concepts of strongly J-n-coherent rings and J-(m,n)-coherent rings as follows.

Definition 4.1. A ring R is called left strongly J-n-coherent if every n-generated small submodule of a free left R-module is finitely presented. A ring R is called left J-(m,n)-coherent if every n-generated small submodule of ${}_RR^m$ is finitely presented.

Recall that a left R-module A is called 2-presented if there exists an exact sequence $F_2 \to F_1 \to F_0 \to A \to 0$ in which every F_i is a finitely generated free module. It is easy to see that a ring R is left strongly J-n-coherent if and only if it is left J-(m,n)-coherent for all positive integers m, if and only if every J-(m,n)-presented left R-module is 2-presented for all positive integers m.

Theorem 4.2. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a left J-coherent ring.*
(2) *Every finitely generated small submodule A of a finitely generated free left R-module F is finitely presented.*
(3) *Every finitely generated small submodule A of a free left R-module F is finitely presented.*
(4) *Every finitely generated small submodule A of a projective left R-module F is finitely presented.*
(5) *For every finitely generated free left R-module F and any finitely generated small submodule A of F, F/A is 2-presented.*

Proof. (1) $\Rightarrow$ (2) Let $F = R^m$. We prove by induction on m. If $m = 1$, then A is a finitely generated left ideal in $J(R)$, by hypothesis, A is finitely presented. Assume that every finitely generated small submodule of the left R-module R^{m-1} is finitely presented. Then for any finitely generated small submodule A of the left R-module R^m, let $B = A \cap (Re_1 \oplus \cdots \oplus Re_{m-1})$, where $e_j \in R^m$ with 1 in the jth position and 0's in all other positions. Then each $a \in A$ has a unique expression $a = b + re_m$, where $b \in (J(R))^{m-1} \oplus 0, r \in J(R)$. If $\varphi : A \to R$ is defined by $a \mapsto r$, then there is an exact sequence $0 \to B \to A \xrightarrow{\varphi} L \to 0$, where $L = \mathrm{Im}(\varphi)$ is a finitely generated left ideal in $J(R)$. By hypothesis, L is finitely presented, and so B is finitely generated. Since B is isomorphic to a small submodule of R^{m-1}, the induction hypothesis gives B is finitely presented. Therefore, A is also finitely presented by [16, 25.1(2)(ii)].

(2) $\Rightarrow$ (1), and (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4), as well as (2) $\Rightarrow$ (5) are obvious. □

Remark 4.3. By Theorem 4.2, it is easy to see that R is left J-coherent if and only if R is left strongly J-n-coherent for each positive integer n.

Theorem 4.4. *The following statements are equivalent for a ring R:*

(1) *R is left strongly J-n-coherent.*
(2) *If $0 \to K \xrightarrow{f} R^n \xrightarrow{g} T$ is an exact sequence of left R-modules, where T is a finitely generated small submodule of a free left R-module, then K is finitely generated.*
(3) *$\mathbf{l}_{R^n}(X)$ is a finitely generated submodule of ${}_RR^n$ for any finite subset X of J_n.*
(4) *For any finitely generated small submodule S of the right R-module R_n, the dual module $(R_n/S)^*$ is a finitely generated left R-module.*

Proof. (1) ⇒ (2) Since R is left strongly J-n-coherent and $\mathrm{im}(g)$ is an n-generated small submodule of a free left R-module, $\mathrm{im}(g)$ is finitely presented. Noting that the sequence $0 \to \ker(g) \to R^n \to \mathrm{im}(g) \to 0$ is exact, we have that $\ker(g)$ is finitely generated. Thus $K \cong \mathrm{im}(f) = \ker(g)$ is finitely generated.

(2) ⇒ (3) Let $X = \{\alpha_1, \cdots, \alpha_m\}$. Then we have an exact sequence of left R-modules $0 \to \mathbf{l}_{R^n}(X) \to R^n \xrightarrow{g} J^m$, where $g(\beta) = (\beta\alpha_1, \cdots, \beta\alpha_m)$. By (2), $\mathbf{l}_{R^n}(X)$ is a finitely generated left R-module.

(3) ⇒ (1) Let $T = Rt_1 + \cdots + Rt_n$ be an n-generated small submodule of R_m, where $t_j = (a_{1j}, \cdots, a_{mj})', j = 1, \cdots, n$. Write $\alpha_i = (a_{i1}, \cdots, a_{in})', i = 1, \cdots, m$, $X = \{\alpha_1, \cdots, \alpha_m\}$. Then we have an exact sequence of left R-modules $0 \to \mathbf{l}_{R^n}(X) \to R^n \to T \to 0$. By (3), $\mathbf{l}_{R^n}(X)$ is finitely generated, so T is finitely presented.

(3) ⇔ (4) follows from [22, Lemma 4.1]. □

Let $\mathcal{F}$ be a class of right R-modules and M a right R-module. Following [9], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of an $\mathcal{F}$-precover and an $\mathcal{F}$-cover. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Theorem 4.5. *The following statements are equivalent for a ring R:*

(1) *R is left strongly J-n-coherent.*

(2) $\varinjlim \mathrm{Ext}^1(F/T, M_\alpha) \cong \mathrm{Ext}^1(F/T, \varinjlim M_\alpha)$ *for every* n*-generated small submodule* T *of a finitely generated free left* R*-module* F *and direct system* $(M_\alpha)_{\alpha\in A}$ *of left* R*-modules.*

(3) $\mathrm{Tor}_1(\prod N_\alpha, F/T) \cong \prod \mathrm{Tor}_1(N_\alpha, F/T)$ *for any family* $\{N_\alpha\}$ *of right* R*-modules and any* n*-generated small submodule* T *of a finitely generated free left* R*-module* F*.*

(4) *Any direct product of copies of* R_R *is strongly* J*-*n*-flat.*

(5) *Any direct product of strongly* J*-*n*-flat right* R*-modules is strongly* J*-*n*-flat.*

(6) *Any direct limit of strongly* J*-*n*-injective left* R*-modules is strongly* J*-*n*-injective.*

(7) *Any direct limit of injective left* R*-modules is strongly* J*-*n*-injective.*

(8) *A left* R*-module* M *is strongly* J*-*n*-injective if and only if* M^+ *is strongly* J*-*n*-flat.*

(9) *A left* R*-module* M *is strongly* J*-*n*-injective if and only if* M^{++} *is strongly* J*-*n*-injective.*

(10) *A right* R*-module* M *is strongly* J*-*n*-flat if and only if* M^{++} *is strongly* J*-*n*-flat.*

(11) *For any ring* S*,* $\mathrm{Tor}_1(\mathrm{Hom}_S(B, C), F/T) \cong \mathrm{Hom}_S(\mathrm{Ext}^1(F/T, B), C)$ *for the situation* $({}_R(F/T), {}_RB_S, C_S)$ *with* F *a finitely generated free left* R*-module and* T *an* n*-generated small submodule of* F *and* C_S *injective.*

(12) *Every right* R*-module has a strongly* J*-*n*-flat preenvelope.*

(13) *For every finitely generated small submodule* S *of the right* R*-module* R_n*, the right* R*-module* R_n/S *has a finitely generated projective preenvelope.*

Proof. (1) ⇒ (2) follows from [5, Lemma 2.9(2)].

(1) ⇒ (3) follows from [5, Lemma 2.10(2)].

(2) ⇒ (6) ⇒ (7), (3) ⇒ (5) ⇒ (4) are trivial.

(7) ⇒ (1) Let F be a finitely generated free left R-module and T be an n-generated small submodule of F, and let $(E_\alpha)_{\alpha\in A}$ be a direct system of injective left R-modules (with A directed). Then $\varinjlim E_\alpha$ is strongly J-n-injective by (7), and so $\mathrm{Ext}^1(F/T, \varinjlim M_\alpha) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Hom}(F/T, E_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(F, E_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(T, E_\alpha) & \longrightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
\mathrm{Hom}(F/T, \varinjlim E_\alpha) & \longrightarrow & \mathrm{Hom}(F, \varinjlim E_\alpha) & \longrightarrow & \mathrm{Hom}(T, \varinjlim E_\alpha) & \longrightarrow & 0.
\end{array}$$

Since f and g are isomorphism by [16, 25.4(d)], h is an isomorphisms by the Five Lemma. Now, let $(M_\alpha)_{\alpha\in A}$ be any direct system of left R-modules (with A directed). Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varinjlim \mathrm{Hom}(T, M_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(T, E(M_\alpha)) & \longrightarrow & \varinjlim \mathrm{Hom}(T, E(M_\alpha)/M_\alpha) \\
 & & \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 \\
0 & \longrightarrow & \mathrm{Hom}(T, \varinjlim M_\alpha) & \longrightarrow & \mathrm{Hom}(T, \varinjlim E(M_\alpha)) & \longrightarrow & \mathrm{Hom}(T, \varinjlim E(M_\alpha)/M_\alpha),
\end{array}$$

where $E(M_\alpha)$ is the injective hull of M_α. Since T is finitely generated, by [16, 24.9], the maps ϕ_1, ϕ_2 and ϕ_3 are monic. By the above proof, ϕ_2 is an isomorphism. Hence ϕ_1 is also an isomorphism by Five Lemma again, so T is finitely presented by [16, 25.4(d)] again. Therefore R is left strongly J-n-coherent.

(4) $\Rightarrow$ (1) Let T be an n-generated small submodule of a finitely generated free left R-module F. By (4), $\mathrm{Tor}_1(\Pi R, F/T) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Pi R)\otimes T & \longrightarrow & (\Pi R)\otimes F & \longrightarrow & (\Pi R)\otimes F/T & \longrightarrow & 0 \\
 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
0 & \longrightarrow & \Pi T & \longrightarrow & \Pi F & \longrightarrow & \Pi(F/T) & \longrightarrow & 0.
\end{array}$$

Since f_2 and f_3 are isomorphisms by [16, 25.4(g)], f_1 is an isomorphism by the Five Lemma. So T is finitely presented by [16, 25.4(g)] again. Hence R is left strongly J-n-coherent.

(5) $\Rightarrow$ (12) Let N be any right R-module. By [9, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ dependent on $\mathrm{Card}(N)$ and $\mathrm{Card}(R)$ such that for any homomorphism $f : N \to F$ with F strongly J-n-flat, there is a pure submodule S of F such that $f(N) \subseteq S$ and $\mathrm{Card}\ S \leq \aleph_\alpha$. Thus f has a factorization $N \to S \to F$ with S strongly J-n-flat by Proposition 3.3. Now let $\{\varphi_\beta\}_{\beta\in B}$ be all such homomorphisms $\varphi_\beta : N \to S_\beta$ with $\mathrm{Card}\ S_\beta \leq \aleph_\alpha$ and S_β strongly J-n-flat. Then any homomorphism $N \to F$ with F strongly J-n-flat has a factorization $N \to S_i \to F$ for some $i \in B$. Thus the homomorphism $N \to \Pi_{\beta\in B} S_\beta$ induced by all φ_β is a strongly J-n-flat preenvelope since $\Pi_{\beta\in B} S_\beta$ is strongly J-n-flat by (5).

(12) $\Rightarrow$ (5) follows from [4, Lemma 1].

(1) $\Rightarrow$ (11) For any finitely generated free left R-module F and each n-generated small submodule T of F, since R is left strongly J-n-coherent, F/T is 2-presented. And so (11) follows from [5, Lemma 2.7(2)].

(11) $\Rightarrow$ (8) Let $S = \mathbb{Z}, C = \mathbb{Q}/\mathbb{Z}$ and $B = M$. Then $\mathrm{Tor}_1(M^+, F/T) \cong \mathrm{Ext}^1(F/T, M)^+$ for any n-generated small submodule T of a finitely generated free left R-module F by (11), and hence (8) holds.

(8) ⇒ (9) Let M be a left R-module. If M is strongly J-n-injective, then M^+ is strongly J-n-flat by (8), and so M^{++} is strongly J-n-injective by Theorem 3.2. Conversely, if M^{++} is strongly J-n-injective, then M, being a pure submodule of M^{++} (see [15, Exercise 41, p.48]), is strongly J-n-injective by Corollary 2.5.

(9) ⇒ (10) If M is a strongly J-n-flat right R-module, then M^+ is a strongly J-n-injective left R-module by Theorem 3.2, and so M^{+++} is strongly J-n-injective by (9). Thus M^{++} is strongly J-n-flat by Theorem 3.2 again. Conversely, if M^{++} is strongly J-n-flat, then M is strongly J-n-flat by Corollary 3.3 as M is a pure submodule of M^{++}.

(10) ⇒ (5) Let $\{N_\alpha\}_{\alpha\in A}$ be a family of strongly J-n-flat right R-modules. Then $\bigoplus_{\alpha\in A} N_\alpha$ is strongly n-flat, and so $(\prod_{\alpha\in A} N_\alpha^+)^+ \cong (\bigoplus_{\alpha\in A} N_\alpha)^{++}$ is strongly n-flat by (10). Since $\bigoplus_{\alpha\in A} N_\alpha^+$ is a pure submodule of $\prod_{\alpha\in A} N_\alpha^+$ by [3, Lemma 1(1)], $(\prod_{\alpha\in A} N_\alpha^+)^+ \to (\bigoplus_{\alpha\in A} N_\alpha^+)^+ \to 0$ splits, and hence $(\bigoplus_{\alpha\in A} N_\alpha^+)^+$ is strongly J-n-flat. Thus $\prod_{\alpha\in A} N_\alpha^{++} \cong (\bigoplus_{\alpha\in A} N_\alpha^+)^+$ is strongly J-n-flat. Since $\prod_{\alpha\in A} N_\alpha$ is a pure submodule of $\prod_{\alpha\in A} N_\alpha^{++}$ by [3, Lemma 1(2)], $\prod_{\alpha\in A} N_\alpha$ is strongly J-n-flat by Corollary 3.3.

(12) ⇒ (13) Let S be a finitely generated small submodule of the right R-module R_n. Then by (12), R_n/S has a strongly n-flat preenvelope $f : R_n/S \to N$. Since N is strongly n-flat, by Theorem 3.2, there exist a free right R-module F, a homomorphism $g : R_n/S \to F$ and a homomorphism $h : F \to N$ such that $f = hg$. Now let $\alpha : R_n/S \to A$ be any right R-homomorphism. Then there exists a homomorphism $\beta : N \to A$ such that $\alpha = \beta f$. So we have a homomorphism βh from F to A such that $\alpha = (\beta h)g$. Hence, $g : R_n/S \to F$ is a finitely generated projective preenvelope of R_n/S.

(13) ⇒ (1) Let S be a finitely generated small submodule of the right R-module R_n. Then R_n/S has a finitely generated projective preenvelope $f : R_n/S \to P$ by (13). So the sequence $\mathrm{Hom}(P, R) \to \mathrm{Hom}(R_n/S, R) \to 0$ is exact, and hence $(R_n/S)^* = \mathrm{Hom}(R_n/S, R)$ is finitely generated since P is finitely generated and projective. Therefore, by Theorem 4.4, R is left strongly J-n-coherent. □

Corollary 4.6. *The following statements are equivalent for a ring R:*

(1) *R is left J-coherent.*

(2) *$\varinjlim \mathrm{Ext}^1(F/T, M_\alpha) \cong \mathrm{Ext}^1(F/T, \varinjlim M_\alpha)$ for every finitely generated small submodule T of a finitely generated free left R-module F and a direct system $(M_\alpha)_{\alpha\in A}$ of left R-modules.*

(3) $\mathrm{Tor}_1(\prod N_\alpha, F/T) \cong \prod \mathrm{Tor}_1(N_\alpha, F/T)$ *for any family* $\{N_\alpha\}$ *of right R-modules and any finitely generated small submodule T of a finitely generated free left R-module F.*

(4) *Any direct product of copies of* R_R *is strongly J-flat.*

(5) *Any direct product of strongly J-flat right R-modules is strongly J-flat.*

(6) *Any direct limit of J-FP-injective left R-modules is J-FP-injective.*

(7) *Any direct limit of injective left R-modules is J-FP-injective.*

(8) *A left R-module M is J-FP-injective if and only if* M^+ *is strongly J-flat.*

(9) *A left R-module M is J-FP-injective if and only if* M^{++} *is J-FP-injective.*

(10) *A right R-module M is strongly J-flat if and only if* M^{++} *is strongly J-flat.*

(11) *For any ring S,* $\mathrm{Tor}_1(\mathrm{Hom}_S(B, C), F/T) \cong \mathrm{Hom}_S(\mathrm{Ext}^1(F/T, B), C)$ *for the situation* $({}_R(F/T), {}_R B_S, C_S)$ *with F a finitely generated free left R-module and T a finitely generated small submodule of F and* C_S *injective.*

(12) *Every right R-module has a strongly J-flat preenvelope.*

(13) *For every finitely generated small submodule S of a finitely generated right R-module F, the right R-module F/S has a finitely generated projective preenvelope.*

Corollary 4.7. *Let R be a semiregular ring. Then it is left strongly n-coherent if and only if it is left strongly J-n-coherent.*

Proof. We need only to proof the sufficiency. Let R be left strongly J-n-coherent. Then by Theorem 4.5(4), any direct product of copies of R_R is strongly J-n-flat. Note that R is a semiregular ring, by Proposition 3.7, any direct product of copies of R_R is strongly n-flat. And so, by [22, Theorem 4.2(4)], R is left strongly n-coherent. □

Corollary 4.8. *Let R be a left strongly J-n-coherent ring. Then every left R-module has a strongly J-n-injective cover.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with B strongly J-n-injective. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ is split. Since R is left strongly J-n-coherent, B^+ is strongly J-n-flat by Theorem 4.5(8), so C^+ is strongly J-n-flat, and hence C is strongly J-n-injective by Remark 3.4. Thus, the class of strongly J-n-injective modules is closed under pure quotients, and so by [10, Theorem 2.5], every left R-module has a strongly J-n-injective cover. □

Proposition 4.9. *Let R be a left J-coherent ring. Then every left R-module has a J-FP-injective cover.*

Proof. It is similar to the proof of Corollary 4.8. □

Corollary 4.10. *The following are equivalent for a left strongly J-n-coherent ring R:*

(1) *Every strongly J-n-flat right R-module is strongly J-flat.*
(2) *Every strongly J-n-injective left R-module is J-FP-injective.*

In this case, R is left J-coherent.

Proof. (1) $\Rightarrow$ (2) Let M be any strongly J-n-injective left R-module. Then M^+ is strongly J-n-flat by Theorem 4.5(8), and so M^+ is strongly J-flat by (1). Thus M^{++} is J-FP-injective. Since M is a pure submodule of M^{++}, and a pure submodule of a J-FP-injective module is J-FP-injective, M is strongly J-FP-injective.

(2) $\Rightarrow$ (1) Let M be a strongly J-n-flat right R-module. Then M^+ is a strongly J-n-injective left R-module by Theorem 3.2, and so M^+ is J-FP-injective by (2). Thus M is strongly J-flat.

In this case, any direct product of strongly J-flat right R-modules is strongly J-flat by Theorem 4.5(5), and so R is left J-coherent by Corollary 4.6. ◻

Proposition 4.11. *The following statements are equivalent for a ring R:*

(1) *R is left strongly J-n-coherent and $_RR$ is strongly J-n-injective.*
(2) *Every right R-module has a monic strongly J-n-flat preenvelope.*
(3) *R is left strongly J-n-coherent and every left R-module has an epic strongly J-n-injective cover.*
(4) *R is left strongly J-n-coherent and every injective right R-module is strongly J-n-flat.*
(5) *R is left strongly J-n-coherent and, for any finitely generated small submodule S of the right R-module R_n, R_n/S embeds in a finitely generated free module.*
(6) *R is left strongly J-n-coherent and, for any finitely generated small submodule S of the right R-module R_n, R_n/S has a monic finitely generated projective preenvelope.*
(7) *R is left strongly J-n-coherent and every flat left R-module is strongly J-n-injective.*

Proof. (1)$\Rightarrow$(2) Let M be any right R-module. Then M has a strongly J-n-flat preenvelope $f : M \to F$ by Theorem 4.5(12). Since $(_RR)^+$ is a cogenerator, there exists an exact sequence $0 \to M \xrightarrow{g} \prod(_RR)^+$. Since $_RR$ is strongly J-n-injective, by Theorem 4.5, $\prod(_RR)^+$ is strongly J-n-flat, and so there exists a right R-homomorphism $h : F \to \prod(_RR)^+$ such that $g = hf$, which shows that f is monic.

(2)⇒(4) Assume (2) holds. Then R is left strongly J-n-coherent by Theorem 4.5(12). Now, let E be an injective right R-module E. Then E has a monic strongly J-n-flat preenvelope F, so E is isomorphic to a direct summand of F, and thus E is strongly J-n-flat.

(4)⇒(1) Since $(_RR)^+$ is injective, by (4), it is strongly J-n-flat. Thus $_RR$ is strongly J-n-injective by Theorem 4.5(8).

(1)⇒(3) Let M be a left R-module. Then M has a strongly J-n-injective cover $\varphi : C \to M$ by Corollary 4.8. On the other hand, there is an exact sequence $F \overset{\alpha}{\to} M \to 0$ with F free. Since $_RR$ is strongly J-n-injective by (1), F is strongly J-n-injective , so there exists a homomorphism $\beta : F \to C$ such that $\alpha = \varphi\beta$. This follows that φ is epic.

(3)⇒(1) Let $f : N \to {}_RR$ be an epic strongly J-n-injective cover. Then the projectivity of $_RR$ implies that $_RR$ is isomorphic to a direct summand of N, and so $_RR$ is strongly J-n-injective.

(1)⇒(5) Let S be any finitely generated small submodule of the right R-module R_n. Since $_RR$ is strongly J-n-injective, by Theorem 2.6(4), M is torsionless. Note that R is left strongly J-n-coherent, by Theorem 4.6(4), R_n/S embeds in a strongly J-n-flat right R-module V. And so R_n/S embeds in a finitely generated free right R-module F by Theorem 3.2(6).

(5)⇒(1) It follows from Theorem 2.6(4).

(2)⇒(6) Clearly, R is left strongly J-n-coherent. Let S be any finitely generated small submodule of the right R-module R_n. By (2), R_n/S has a monic strongly J-n-flat preenvelope $f : R_n/S \to V$. And so, by Theorem 3.2(6), f factors through a finitely generated free right R-module F, that is, there exist a homomorphism $g : R_n/S \to F$ and a homomorphism $h : F \to V$ such that $f = hg$. Now let P be a projective right R-module and φ be a homomorphism from R_n/S to P. Then there exists a homomorphism $\theta : V \to P$ such that $\varphi = \theta f$. Thus, θh is a homomorphism from F to P and $\varphi = (\theta h)g$. Therefore, $g : R_n/S \to F$ is a monic finitely generated projective preenvelope of R_n/S.

(6)⇒(1) It follows from Theorem 2.6(4).

(4)⇒(7) Let M be a flat left R-module. Then M^+ is injective, and so M^+ is strongly J-n-flat by (4). Hence M is strongly J-n-injective by Theorem 4.5(8).

(7)⇒(1) It is obvious. □

Proposition 4.12. *The following statements are equivalent for a ring R:*

(1) *R is left J-coherent and $_RR$ is J-FP-injective.*

(2) *Every right R-module has a monic strongly J-flat preenvelope.*

(3) *R is left J-coherent and every left R-module has an epic J-FP-injective cover.*
(4) *R is left J-coherent and every injective right R-module is strongly J-flat.*
(5) *R is left J-coherent and, for any finitely generated small submodule S of a finitely generated free right R-module F, F/S embeds in a finitely generated free module.*
(6) *R is left J-coherent and, for any finitely generated small submodule S of a finitely generated free right R-module F, F/S has a monic finitely generated projective preenvelope.*
(7) *R is left J-coherent and every flat left R-module is J-FP-injective.*

Proof. It is similar to the proof of Proposition 4.11. □

Theorem 4.13. *The following statements are equivalent for a ring R:*

(1) *R is left strongly J-n-coherent.*
(2) $\mathrm{Ext}^1_R(T,N)=0$ *for any n-generated small submodule T of a free left R-module F and any FP- injective left R-module N.*
(3) $\mathrm{Ext}^2_R(F/T,N)=0$ *for any finitely generated free left R-module F and its n-generated small submodule T and any FP-injective left R-module N.*
(4) *If N is a strongly J-n-injective left R-module,* N_1 *is an FP-injective submodule of N, then* N/N_1 *is strongly J-n-injective.*
(5) *For any FP-injective left R-module N,* $E(N)/N$ *is strongly J-n-injective.*

Proof. (1) ⇒ (2) and (4) ⇒ (5) are obvious.

(2) ⇒ (3) It follows from the isomorphism $\mathrm{Ext}^2_R(F/T,N)\cong\mathrm{Ext}^1_R(T,N)$.

(3) ⇒ (4) Let F be a finitely generated free left R-module and T its n-generated small submodule. The exact sequence $0\to N_1\to N\to N/N_1\to 0$ induces the exactness of the sequence

$$0=\mathrm{Ext}^1(F/T,N)\to\mathrm{Ext}^1(F/T,N/N_1)\to\mathrm{Ext}^2(F/T,N_1)=0.$$

Therefore $\mathrm{Ext}^1(F/T,N/N_1)=0$, as desired.

(5) ⇒ (1) Let F be a finitely generated free left R-module and T its n-generated small submodule. Then for any FP-injective module N, $E(N)/N$ is strongly J-n-injective by (5). From the exactness of the two sequences

$$0=\mathrm{Ext}^1(F,N)\to\mathrm{Ext}^1(T,N)\to\mathrm{Ext}^2(F/T,N)\to\mathrm{Ext}^2(F,N)=0$$

and

$$0=\mathrm{Ext}^1(F/T,E(N))\to\mathrm{Ext}^1(F/T,E(N)/N)\to\mathrm{Ext}^2(F/T,N)\to\mathrm{Ext}^2(F/T,E(N))=0,$$

we have

$$\mathrm{Ext}^1(T,N) \cong \mathrm{Ext}^2(F/T,N) \cong \mathrm{Ext}^1(F/T,E(N)/N) = 0,$$

so $\mathrm{Ext}^1(T,N) = 0$. By [8], T is finitely presented. Therefore, R is left strongly J-n-coherent. □

Corollary 4.14. *The following statements are equivalent for a ring R:*

(1) *R is left J-coherent.*

(2) $\mathrm{Ext}^1_R(T,N) = 0$ *for any finitely generated small submodule T of a free left R-module F and any FP-injective left R-module N.*

(3) $\mathrm{Ext}^2_R(F/T,N) = 0$ *for any finitely generated free left R-module F and its finitely generated small submodule T and any FP-injective left R-module N.*

(4) *If N is a J-FP-injective left R-module, N_1 is an FP-injective submodule of N, then N/N_1 is J-FP-injective.*

(5) *For any FP-injective left R-module N, $E(N)/N$ is J-FP-injective.*

5. J-n-semihereditary rings

Definition 5.1. A ring R is said to be left J-n-semihereditary if every n-generated small left ideal of R is projective.

Theorem 5.2. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a left J-n-semihereditary ring.*

(2) *Every n-generated small submodule A of a finitely generated free left R-module F is projective.*

(3) *Every n-generated small submodule A of a free left R-module is projective.*

(4) *If $0 \to K \to P \to V \to 0$ is exact, where V is a J-finitely presented left R-module and P is projective, then K is projective.*

Proof. (1) $\Rightarrow$ (2) Let $F = R^m$. We prove by induction on m. If $m = 1$, then A is an n-generated small left ideal of R, by (1), A is projective. Assume that every n-generated small submodule of the left R-module R^{m-1} is projective. Then for any n-generated small submodule A of the left R-module R^m, let $B = A \cap (Re_1 \oplus \cdots \oplus Re_{m-1})$, where $e_j \in R^m$ with 1 in the jth position and 0's in all other positions. Then each $a \in A$ has a unique expression $a = b + re_m$, where $b \in J(R)^{m-1}, r \in J(R)$. If $\varphi : A \to R$ is defined by $a \mapsto r$, then there is an exact sequence $0 \to B \to A \xrightarrow{\varphi} L \to 0$, where $L = \mathrm{Im}(\varphi)$ is an n-generated small left ideal of R. By (1), L is projective, so $A \cong B \oplus L$ and then B is n-generated. Since B is isomorphic to a small submodule of R^{m-1}, the induction hypothesis gives B, hence A, is projective.

(2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3) are clear.

(2) $\Leftrightarrow$ (4) By the Schanuel's Lemma [12, Theorem 3.62]. □

Corollary 5.3. *If R is a left J-semihereditary ring, then every finitely generated small submodule of a free left R-module is projective.*

Theorem 5.4. *The following statements are equivalent for a ring R:*

(1) *R is a left J-n-semihereditary ring.*

(2) *R is left strongly J-n-coherent and every submodule of a strongly J-n-flat right R-module is strongly J-n-flat.*

(3) *R is left strongly J-n-coherent and every right ideal is strongly J-n-flat.*

(4) *R is left strongly J-n-coherent and every finitely generated right ideal is strongly J-n-flat.*

(5) *Every quotient module of a strongly J-n-injective left R-module is strongly J-n-injective.*

(6) *Every quotient module of an injective left R-module is strongly J-n-injective.*

(7) *Every left R-module has a monic strongly J-n-injective cover.*

(8) *Every right R-module has an epic strongly J-n-flat envelope.*

(9) *For every finitely generated small submodule of the right R-module R^n, R^n/S has an epic finitely generated projective envelope.*

(10) *Every torsionless right R-module is strongly J-n-flat.*

Proof. (2)$\Rightarrow$(3)$\Rightarrow$(4), and (5)$\Rightarrow$(6) are trivial.

(1)$\Rightarrow$(2) Assume (1). Then by Theorem 5.2(2), R is left strongly J-n-coherent. Let A be a submodule of a strongly J-n-flat right R-module B and L an n-generated small submodule of a finitely generated free left R-module F. Then L is projective and hence flat. Thus the exactness of the sequence $0 = \mathrm{Tor}_2(B/A, F) \to \mathrm{Tor}_2(B/A, F/L) \to \mathrm{Tor}_1(B/A, L) = 0$ implies that $\mathrm{Tor}_2(B/A, F/L) = 0$. And so, from the exactness of the sequence $0 = \mathrm{Tor}_2(B/A, F/L) \to \mathrm{Tor}_1(A, F/L) \to \mathrm{Tor}_1(B, F/L) = 0$ we have that $\mathrm{Tor}_1(A, F/L) = 0$, it shows that A is strongly n-flat.

(4)$\Rightarrow$(1) Let L be an n-generated small submodule of a finitely generated free left R-module F. Then L is finitely presented as R is left strongly J-n-coherent. Let I be any finitely generated right ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ implies the exact sequence $0 \to \mathrm{Tor}_2(R/I, F/L) \to \mathrm{Tor}_1(I, F/L) = 0$ since I is strongly J-n-flat. So $\mathrm{Tor}_2(R/I, F/L) = 0$, and hence we obtain an exact sequence $0 = \mathrm{Tor}_2(R/I, F/L) \to \mathrm{Tor}_1(R/I, L) \to 0$. Thus, $\mathrm{Tor}_1(R/I, L) = 0$, and so L is a finitely presented flat left R-module. Therefore, L is projective.

(1)⇒(5) Let M be a strongly J-n-injective left R-module and N be a submodule of M. Then for any n-generated small submodule L of a finitely generated free left R-module F, since L is projective, the exact sequence $0 = \mathrm{Ext}^1(L, N) \to \mathrm{Ext}^2(F/L, N) \to \mathrm{Ext}^2(F, N) = 0$ implies that $\mathrm{Ext}^2(F/L, N) = 0$. Thus the exact sequence $0 = \mathrm{Ext}^1(F/L, M) \to \mathrm{Ext}^1(F/L,\ M/N) \to \mathrm{Ext}^2(F/L, N) = 0$ implies that $\mathrm{Ext}^1(F/L, M/N) = 0$. Therefore, M/N is strongly J-n-injective.

(6)⇒(1) Let L be an n-generated small submodule of a finitely generated free left R-module F. Then for any left R-module M, by (6), $E(M)/M$ is strongly J-n-injective , and so $\mathrm{Ext}^1(F/L, E(M)/M) = 0$. Thus, the exactness of the sequence $0 = \mathrm{Ext}^1(F/L, E(M)/M) \to \mathrm{Ext}^2(F/L, M) \to \mathrm{Ext}^2(F/L, E(M)) = 0$ implies that $\mathrm{Ext}^2(F/L, M) = 0$. Hence, the exactness of the sequence $0 = \mathrm{Ext}^1(F, M) \to \mathrm{Ext}^1(L, M) \to \mathrm{Ext}^2(F/L, M) = 0$ implies that $\mathrm{Ext}^1(L, M) = 0$, this shows that L is projective, as required.

(5)⇒(7) Since R is left strongly J-n-coherent by (2), for any left R-module M, there is a strongly J-n-injective cover $f : E \to M$ by Corollary 4.8. Note that $im(f)$ is strongly J-n-injective by (5), and $f : E \to M$ is a strongly J-n-injective precover, so for the inclusion map $i : im(f) \to M$, there is a homomorphism $g : im(f) \to E$ such that $i = fg$. Hence $f = f(gf)$. Observing that $f : E \to M$ is a strongly J-n-injective cover and gf is an endomorphism of E, gf is an automorphism of E, and thus $f : E \to M$ is a monic strongly J-n-injective cover.

(7)⇒(5) Let M be a strongly J-n-injective left R-module and N be a submodule of M. By (7), M/N has a monic strongly J-n-injective cover $f : E \to M/N$. Let $\pi : M \to M/N$ be the natural epimorphism. Then there exists a homomorphism $g : M \to E$ such that $\pi = fg$. Thus f is an isomorphism, and therefore $M/N \cong E$ is strongly J-n-injective.

(2)⇔(8) Since the class of strongly J-n-flat right R-modules is closed under direct summands and isomorphisms, so by [4, Theorem 2], the class of strongly J-n-flat right R-modules is closed under direct product and submodules if and only if every right R-module has an epic strongly J-n-flat envelope.

(8)⇒(9) Let $M = R^n/S$, where S is a finitely generated small submodule of R^n. Then by (8), M has an epic strongly J-n-flat envelope $f : M \to N$. By Theorem 3.2(6), f factors through a finitely generated free right R-module F, that is, there exist $g : M \to F$ and $h : F \to N$ such that $f = hg$. Since F is strongly J-n-flat, there exists $\varphi : N \to F$ such that $g = \varphi f$. So $f = (h\varphi)f$, and hence $h\varphi = 1$ since f is epic. Hence, N is isomorphic to a direct summand of F, and thus N is finitely generated projective. Therefore, $f : M \to N$ is a finitely generated projective envelope of M.

(9)⇒(2) Clearly R is left strongly J-n-coherent by Theorem 4.5(13). Now suppose that A is a submodule of a strongly J-n-flat right R-module B and $\iota : A \to B$ is the inclusion. Let L be a finitely generated small submodule of R_R^n. Then for any homomorphism $f : R^n/L \to A$, ιf factors through a finitely generated free right R-module F by Theorem 3.2(6). So there exists homomorphism $g : R^n/L \to F$ and $h : F \to B$ such that $\iota f = hg$. By (9), R^n/L has an epic finitely generated projective envelope $\alpha : R^n/L \to P$. So there exists a homomorphism $\beta : P \to F$ such that $g = \beta\alpha$. It is easy to see that $Ker(\alpha) \subseteq Ker(f)$. Now we define $\gamma : P \to A$ by $\gamma(x) = f(y)$, where $x = \alpha(y)$, then γ is a right R-homomorphism and $f = \gamma\alpha$. Therefore, A is strongly J-n-flat by Theorem 3.2(6).

(2)⇒(10) Let M be a torsionless right R-module. Then there exists an exact sequence $0 \to M \to \prod R_R$. By (2), R is left strongly J-n-coherent and every submodule of a strongly J-n-flat right R-module is strongly J-n-flat, so M is strongly J-n-flat by Theorem 4.6(4).

(10)⇒(3) Assume (10). Then $\prod R_R$ is strongly J-n-flat, and hence R is left strongly J-n-coherent by Theorem 4.5(4). Moreover, every right ideal of R is torsionless and so is strongly J-n-flat. □

Corollary 5.5. *The following statements are equivalent for a ring R:*

(1) *R is a left J-semihereditary ring.*
(2) *R is left J-coherent and submodules of strongly J-flat right R-modules are strongly J-flat.*
(3) *R is left J-coherent and every right ideal is strongly J-flat.*
(4) *R is left strongly J-coherent and every finitely generated right ideal is strongly J-flat.*
(5) *Every quotient module of a J-FP-injective left R-module is J-FP-injective.*
(6) *Every quotient module of an injective left R-module is J-FP-injective.*
(7) *Every left R-module has a monic J-FP-injective cover.*
(8) *Every right R-module has an epic strongly J-flat envelope.*
(9) *For every finitely generated small submodule S of a free right R-module F, F/S has an epic finitely generated projective envelope.*
(10) *Every torsionless right R-module is strongly J-flat.*

Corollary 5.6. *Let R be a semiregular ring. Then it is left n-semihereditary if and only if it is left J-n-semihereditary.*

Proof. We need only to prove the sufficiency. Suppose R is left J-n-semihereditary, then by Theorem 5.4(6), every quotient module of an injective left R-module is strongly J-n-injective. Since R is semiregular, every strongly J-n-injective left R-module is strongly n-injective by Proposition 2.7. So every quotient module of an injective left R-module is strongly n-injective and hence n-injective. Hence, by [17, Theorem 3(2)], R is left n-semihereditary. □

Acknowledgement. The author would like to thank the referee for the valuable suggestions and comments.

References

[1] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, NJ, 1956.

[2] S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc., 97 (1960), 457-473.

[3] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc., 81 (1981), 175-177.

[4] J. L. Chen and N. Q. Ding, *A note on existence of envelopes and covers*, Bull. Austral. Math. Soc., 54 (1996), 383-390.

[5] J. L. Chen and N. Q. Ding, *On n-coherent rings*, Comm. Algebra, 24 (1996), 3211-3216.

[6] N. Q. Ding, Y. L. Li and L. X. Mao, *J-coherent rings*, J. Algebra Appl., 8 (2009), 139-155.

[7] D. D. Dobbs, *On n-flat modules over a commutative ring*, Bull. Austral. Math. Soc., 43 (1991), 491-498.

[8] E. E. Enochs, *A note on absolutely pure modules*, Canad. Math. Bull., 19 (1976), 361-362.

[9] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, Walter de Gruyter & Co., Berlin, 2000.

[10] H. Holm and P. Jørgensen, *Covers, precovers, and purity*, Illinois. J. Math., 52 (2008), 691-703.

[11] W. K. Nicholson and M. F. Yousif, Quasi-Frobenius Rings, Cambridge University Press, Cambridge, 2003.

[12] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, New York-London, 1979.

[13] A. Shamsuddin, *n-injective and n-flat modules*, Comm. Algebra, 29 (2001), 2039-2050.

[14] B. Stenström, *Coherent rings and FP-injective modules*, J. London. Math. Soc., 2 (1970), 323-329.

[15] B. Stenström, Rings of Quotients, Springer-Verlag, New York-Heidelberg, 1975.

[16] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[17] X. X. Zhang, J. L. Chen and J. Zhang, *On (m, n)-injective modules and (m, n)-coherent rings*, Algebra Colloq., 12 (2005), 149-160.

[18] X. X. Zhang and J. L. Chen, *On n-semihereditary and n-coherent rings*, Int. Electron. J. Algebra, 1 (2007), 1-10.

[19] Z. M. Zhu and Z. S. Tan, *On n-semihereditary rings*, Sci. Math. Jpn., 62 (2005), 455-459.

[20] Z. M. Zhu, *I-n-coherent rings, I-n-semihereditary rings, and I-regular rings*, Ukrainian Math. J., 66 (2014), 857-883.

[21] Z. M. Zhu, *I-pure submodules, I-FP-injective modules and I-flat modules*, Br. J. Math. Comput. Sci., 8 (2015), 170-188.

[22] Z. M. Zhu, *Strongly n-coherent rings*, Chinese Ann. Math. Ser. A, 38 (2017), 313-326.

Zhanmin Zhu
Department of Mathematics
College of Data Science
Jiaxing University
Jiaxing, Zhejiang Province, 314001, P. R. China
e-mail: zhuzhanminzjxu@hotmail.com