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Research Article

Application of the sumudu transform to some equations with fractional derivatives

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ABSTRACT

The aim of the article is to obtain the exact solutions of the linear fractional differential equations by the integral transforms. Exact solutions of the equations with power-law, exponential-decay and Mittag-Leffler kernels have been obtained by the Sumudu transform. We demonstrate some simulations to show the effect of the proposed transforms.

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INTRODUCTION

Physical events occurring in nature are tried to be expressed with mathematical models. Instead of classical derivatives, fractional derivative operators have been used in these models. Riemann-Liouville and Caputo fractional derivatives are the leading fractioanl derivatives. Due to some limitations of these fractional derivatives and having a singular nucleus, there are new fractional derivatives to overcome these disadvantages. Caputo-Fabrizio fractional derivative without a singular kernel and Atangana-Baleanu fractional derivative is given as a result of replacing the kernel of this fractional derivative operator with the more general Mittag-Leffler function [12,13]. The Mittag-Leffler function is an important function because it has many representation properties and types. In this article, linear differential equations containing Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives are solved and their graphs are demonstrated [6].

One of the integral transformations, the Sumudu transform will be used to solve equations involving fractional derivatives. The Sumudu transform is important because it transforms linear equations without changing the property of the function [8,14,24].

Preferably Caputo fractional derivative is chosen because the initial conditions are similar to the classical derivative. After applying the Sumudu transform to the samples, exact solutions will be obtained and comparative graphs will be demonstrated [7,17].

In the second part of the article, fractional derivative definitions will be given. In the third part, the Sumudu transformation and the inverse Sumudu transform will be given. In addition, the relations between the Sumudu

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transform and the Mittag Leffler function will be given. In the fourth chapter, practical examples will be made and graphic drawings will be interpreted [9,10,11].

Preliminaries

In this section, definitions of Caputo, Caputo-Fabrizio and Atangana -Baleanu fractional derivative are given.

Definition 1 The Caputo fractional derivative is given as [11,24]:

$$ {}_{a}^{C}D_{m}^{\varpi} = \frac{1}{\Gamma(n-\varpi)} \int_{a}^{m} (m-\theta)^{n-\varpi-1} f^{(n)}(\theta) d\theta, \ (n-1 \le \varpi \le n). $$

Definition 2 In the sense of Caputo, the fractional derivative of Caputo-Fabrizio is presented by [15,17]:

$$ {}_{0}^{CFC}\boldsymbol{D}_{\boldsymbol{m}}^{\boldsymbol{\varpi}} = \frac{\boldsymbol{M(\varpi)}}{\boldsymbol{1-\varpi}} \int_{0}^{m} \boldsymbol{f'(\theta)}\, e^{-\frac{\varpi}{1-\varpi}(m-\theta)} \boldsymbol{d\theta}, \ (\boldsymbol{0 < \varpi \le 1}) $$

$\boldsymbol{M(\varpi)}$ is the normalization function.

Definition 3 In the sense of Caputo, the fractional derivative of Atangana-Baleanu is given as [12]:

$$ {}_{0}^{ABC}D_{m}^{\varpi} = \frac{B(\varpi)}{1-\varpi} \int_{0}^{m} f'(\theta)\, E_{\varpi}\left(\frac{-\varpi}{1-\varpi}(m-\theta)^{\varpi}\right) d\theta, \ (0 < \varpi \le 1), $$

$B(\varpi)$ is the normalization function.

The reason why we chose the Caputo fractional derivative in our study is that it has similar initial conditions to the initial conditions in classical analysis [16,24]. The Atangana-Baleanu and Caputo-Fabrizio fractional derivatives also were taken in the Caputo sense [20,21].

Sumudu Transform

The Sumudu transform is the dual of the Laplace transform, an important transform defined by Watugala in 1993. The definition and necessary properties of this integral transformation will be given over the set [7, 8]:

$$ A = \left\{ f(m) | \exists K, \tau_1, \tau_2 > 0, |f(m)| < K e^{\frac{|t|}{\tau_j}}, if \ \mathrm{m} \in (-1)^j \times [0,\infty) \right\} $$

the Sumudu transform is defined by

$$ G(\vartheta) = S[f(m)] = \int_{0}^{\infty} f(\vartheta t)\, e^{-m} dm, \vartheta \in (\tau_1, \tau_2). $$

The main features and advantages of the Sumudu transformation

The Sumudu transform, which has unit conservation, does not apply to any frequency domain. In addition, the Sumudu transform is a linear transformation and preserves the properties of linear equations [2,14, 18]. We consider the functions *f*(*m*) and *g*(*m*). The Sumudu transformations of these functions are defined as:

$$ S[f(m)] = G(\vartheta), S[g(m)] = H(\vartheta) $$

The Sumudu transform is linear [14, 23]:

$$ S[\kappa_1 f(m) + \kappa_2 g(m)] = \kappa_1 G(\vartheta) + \kappa_2 H(\vartheta) $$

Sumudu transform of $f(m) = \sum_{n=0}^{\infty} a_n m^n$ power series is obtained as $G(\vartheta) = \sum_{n=0}^{\infty} n!\, a_n \vartheta^n$ [14,22].

The Sumudu transformation converts combinations to permutations [14].

$$ S[(1+m)^{\sigma}] = S\left[\sum_{n=0}^{\sigma} C_n^{\sigma} m\right]^n = \sum_{n=0}^{\sigma} \frac{\sigma!}{n!\,(\sigma-n)!} n!\, \vartheta^n = \sum_{n=0}^{\sigma} \frac{\sigma!}{(\sigma-n)!} \vartheta^n = P_n^{\sigma} \vartheta^n $$

Let $S[f(m)] = G(\vartheta), S[g(m)] = H(\vartheta)$. Sumudu transform of $f * g$ convolution is given as [4, 15]:

$$ S[f(m) * g(m)] = \vartheta G(\vartheta) H(\vartheta). $$

Assume that $S[f(m)] = G(\vartheta)$, $S[g(m)] = H(\vartheta)$. Then, we have [4, 14]:

$$ S[f'(m)] = \frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta} $$

$$ S[f''(m)] = \frac{G(\vartheta)}{\vartheta^2} - \frac{f(0)}{\vartheta^2} - \frac{f'(0)}{\vartheta} $$

…

$$ S\left[f^{(n)}(m)\right] = \frac{G(\vartheta)}{\vartheta^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\vartheta^{n-k}} $$

Duality relation of Laplace and Sumudu transforms

Sumudu transform, which draws attention with its similarity to Laplace transform, provides convenience in solving ordinary, partial and fractional differential equations.

Theorem 1 Let $f(m) \in A$. Then, we have Sumudu transform of $f(t)$ as:

$$ G(\vartheta) = \frac{F\left(\frac{1}{\vartheta}\right)}{\vartheta}, F(s) = \frac{G\left(\frac{1}{s}\right)}{s}. $$

We have

$$ S[cos(m)] = L[sin(m)] = \frac{1}{1+\vartheta^2} $$

$$ S[sin(m)] = L[cos(m)] = \frac{\vartheta}{1+\vartheta^2} $$

The relations of Laplace and Sumudu transformations can be seen in the equations given in the form. While talking about this relationship in the literature, the Sumudu transform is also called the dual of the Laplace transfrom.

An example of the Laplace and Sumudu transform

We consider the following fractional differential equation:

$${}_{0}^{ABC}D_{m}^{\varpi} f(m) = 5m,\ \ \mathrm{m} > 0,\ \ f(0) = 0,\ (0 \leq \varpi < 1)$$

We solve the above equation containing the Atangana-Balenau fractional derivative with the initial conditions by applying the Laplace transform.

Let's apply the Laplace transform to both sides of the equation:

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = L\{5m\}$$

$${}_{0}^{ABC}D_{m}^{\varpi} f(m) = \frac{B(\varpi)}{1-\varpi}\int_0^t f'(\tau)\, E_{\varpi}\left(\frac{-\varpi}{1-\varpi}(m-\tau)^{\varpi}\right) d\tau,\ 0 \leq \varpi<1$$

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = L\left\{\frac{B(\varpi)}{1-\varpi}\int_0^t f'(\tau)\, E_{\varpi}\left(\frac{-\varpi}{1-\varpi}(m-\tau)^{\varpi}\right) d\tau\right\}$$

According to the convolution theorem, we get;

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi} L\left\{f'(m) * E_{\varpi}\left(\frac{-\varpi}{1-\varpi} m^{\varpi}\right)\right\}$$

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi} L\{f'(m)\} L\left\{E_{\varpi}\left(\frac{-\varpi}{1-\varpi} m^{\varpi}\right)\right\}$$

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi}[sF(s) - f(0)]\frac{s^{\varpi}}{s\left(s^{\varpi}+\frac{\varpi}{1-\varpi}\right)}$$

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi} sF(s)\frac{s^{\varpi}}{s\left(s^{\varpi}+\frac{\varpi}{1-\varpi}\right)}$$

$$L\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi}\frac{F(s)s^{\varpi}}{s^{\varpi}+\frac{\varpi}{1-\varpi}}$$

We know

$$L\{5m\} = \frac{5}{s^2}.$$

Then, we get

$$\frac{B(\varpi)}{1-\varpi}\frac{F(s)s^{\varpi}}{s^{\varpi}+\frac{\varpi}{1-\varpi}} = \frac{5}{s^2}$$

$$F(s) = \frac{\frac{5}{s^2}}{\frac{B(\varpi)}{1-\varpi}\frac{s^{\varpi}}{s^{\varpi}+\frac{\varpi}{1-\varpi}}}$$

$$F(s) = \frac{5}{s^2}\left(\frac{1-\varpi}{B(\varpi)} + \frac{\varpi}{B(\varpi)s^{\varpi}}\right)$$

If the inverse Laplace transform is applied, we will reach

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{5}{s^2}\left(\frac{1-\varpi}{B(\varpi)} + \frac{\varpi}{B(\varpi)s^{\varpi}}\right)\right\}$$

$$f(m) = \frac{5}{B(\varpi)}\left[(1-\varpi)m + \frac{\varpi}{\Gamma(\varpi+2)} m^{\varpi+1}\right].$$

We consider:

$${}_{0}^{ABC}D_{t}^{\varpi} f(m) = 5m,\ \mathrm{m} > 0,\ f(0) = 0,\ (0 \leq \varpi < 1)$$

Let's apply the Sumudu transform to both sides of the above equation:

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = S\{5m\}$$

$${}_{0}^{ABC}D_{m}^{\varpi} f(m) = \frac{B(\varpi)}{1-\varpi}\int_0^t f'(\tau)\, E_{\varpi}\left(\frac{-\varpi}{1-\varpi}(m-\tau)^{\varpi}\right) d\tau,\ 0 \leq \varpi<1$$

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = S\left\{\frac{B(\varpi)}{1-\varpi}\int_0^t f'(\tau)\, E_{\varpi}\left(\frac{-\varpi}{1-\varpi}(m-\tau)^{\varpi}\right) d\tau\right\}$$

According to the convolution theorem, we have;

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi} S\left\{f'(m) * E_{\varpi}\left(\frac{-\varpi}{1-\varpi} m^{\varpi}\right)\right\}$$

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi}\vartheta S\{f'(m)\} S\left\{E_{\varpi}\left(\frac{-\varpi}{1-\varpi} m^{\varpi}\right)\right\}$$

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta}\right]\left[\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\varpi}}\right]$$

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi}\vartheta\frac{G(\vartheta)}{\vartheta}\left[\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\varpi}}\right]$$

$$S\{{}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = \frac{B(\varpi)}{1-\varpi+\varpi\vartheta^{\varpi}} G(\vartheta)$$

We know

$$S\{5m\} = 5\vartheta.$$

Then, we get

$$S\{\ {}_{0}^{ABC}D_{m}^{\varpi} f(m)\} = S\{5m\}$$

$$\frac{B(\varpi)}{1-\varpi+\varpi\vartheta^{\varpi}} G(\vartheta) = 5\vartheta$$

$$G(\vartheta) = \frac{5\vartheta}{\frac{B(\varpi)}{1-\varpi+\varpi\vartheta^{\varpi}}}$$

$$G(\vartheta) = \frac{5}{B(\varpi)}\left((1-\varpi)\vartheta + \varpi\vartheta^{\varpi+1}\right)$$

If the inverse Sumudu transform is applied, we will obtain

$$S^{-1}\{G(\vartheta)\} = S^{-1}\left\{\frac{5}{B(\varpi)}\left((1-\varpi)\vartheta + \varpi\vartheta^{\varpi+1}\right)\right\}$$

$$f(m) = \frac{5}{B(\varpi)}\left((1-\varpi)t + \frac{\varpi}{\Gamma(\varpi+2)} m^{\varpi+1}\right)$$

Lemma 1 The Sumudu transform of Caputo derivative for $n-1 < \varpi \leq n, n \in N$, can be obtained as [24]:

$$S[{}_{0}^{C}D_{m}^{\varpi} f(m)] = \vartheta^{n-\varpi}\left[\frac{G(\vartheta)}{\vartheta^n} - \frac{f(0)}{\vartheta^n} - \frac{f'(0)}{\vartheta^{n-1}} - \ldots - \frac{f^{(n-1)}(0)}{\vartheta}\right].$$

Proof.
If the Sumudu transform is applied

$$S[{}_0^C D_m^\varpi f(m)] = S\left[\frac{1}{\Gamma(n-\varpi)}\int_0^m (m-\theta)^{n-\varpi-1} f^{(n)}(\theta)d\theta\right]$$

By the convolution theorem, we will get

$$S[{}_0^C D_m^\varpi f(m)] = \frac{1}{\Gamma(n-\varpi)} S[m^{n-\varpi-1} * f^{(n)}(m)]$$

$$S[{}_0^C D_m^\varpi f(m)] = \frac{1}{\Gamma(n-\varpi)} \vartheta S[m^{n-\varpi-1}] S[f^{(n)}(m)]$$

$$S[{}_0^C D_m^\varpi f(m)] = \frac{1}{\Gamma(n-\varpi)} \vartheta \Gamma(n-\varpi)\vartheta^{n-\varpi-1}\left[\frac{G(\vartheta)}{\vartheta^n} - \frac{f(0)}{\vartheta^n} - \frac{f'(0)}{\vartheta^{n-1}} - \ldots - \frac{f^{(n-1)}(0)}{\vartheta}\right]$$

$$S[{}_0^C D_m^\varpi f(m)] = \vartheta^{n-\varpi}\left[\frac{G(\vartheta)}{\vartheta^n} - \frac{f(0)}{\vartheta^n} - \frac{f'(0)}{\vartheta^{n-1}} - \ldots - \frac{f^{(n-1)}(0)}{\vartheta}\right]$$

Lemma 2 We have the Sumudu transform of the Caputo-Fabrizio fractional derivative for $0 < \varpi \leq 1$ as:

$$S[{}_0^{CFC} D_m^\varpi] = \frac{M(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta}\right]\left[\frac{1-\varpi}{1-\varpi+\varpi\vartheta}\right] \text{ [26].}$$

Proof.
We have

$${}_0^{CFC} D_m^\varpi = \frac{M(\varpi)}{1-\varpi}\int_0^m f'(\theta)\, e^{-\frac{\varpi}{1-\varpi}(m-\theta)} d\theta$$

If the Sumudu transform is applied to the above equation, we will reach

$$S[{}_0^{CFC} D_m^\varpi] = \frac{M(\varpi)}{1-\varpi} S\left[\int_0^m f'(\theta)\, e^{-\frac{\varpi}{1-\varpi}(m-\theta)} d\theta\right]$$

By the convolution theorem, we obtain

$$S[{}_0^{CFC} D_m^\varpi] = \frac{M(\varpi)}{1-\varpi} S\left[f'(m) * e^{-\frac{\varpi}{1-\varpi}m}\right]$$

$$S[{}_0^{CFC} D_m^\varpi] = \frac{M(\varpi)}{1-\varpi}\vartheta S[f'(m)] S\left[e^{-\frac{\varpi}{1-\varpi}m}\right]$$

$$S[{}_0^{CFC} D_m^\varpi] = \frac{M(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta}\right]\left[\frac{1-\varpi}{1-\varpi+\varpi\vartheta}\right]$$

Lemma 3 We have Sumudu transform of the Atangana-Baleanu fractional derivative for $0 < \varpi \leq 1$ as:

$$S[{}_0^{ABC} D_m^\varpi f(m)] = \frac{B(\varpi)}{1-\varpi}[G(\vartheta) - f(0)]\frac{1-\varpi}{1-\varpi+\varpi\vartheta^\varpi}.$$

Proof.
We have

$${}_0^{ABC} D_m^\varpi f(m) = \frac{B(\varpi)}{1-\varpi}\int_0^m f'(\theta) E_\varpi\left(\frac{-\varpi}{1-\varpi}(m-\theta)^\varpi\right) d\theta$$

If the Sumudu transform is applied, then we will reach

$$S[{}_0^{ABC} D_m^\varpi f(m)] = S\left[\frac{B(\varpi)}{1-\varpi}\int_0^m f'(\theta) E_\varpi\left(\frac{-\varpi}{1-\varpi}(m-\theta)^\varpi\right) d\theta\right]$$

By the convolution theorem, we get

$$S[{}_0^{ABC} D_m^\varpi f(m)] = \frac{B(\varpi)}{1-\varpi} S\left[f'(m) * E_\varpi\left(\frac{-\varpi}{1-\varpi}m^\varpi\right)\right]$$

$$S[{}_0^{ABC} D_m^\varpi f(m)] = \frac{B(\varpi)}{1-\varpi}\vartheta S[f'(m)] S\left[E_\varpi\left(\frac{-\varpi}{1-\varpi}m^\varpi\right)\right]$$

$$S[{}_0^{ABC} D_m^\varpi f(m)] = \frac{B(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi\vartheta^\varpi}$$

$$S[{}_0^{ABC} D_m^\varpi f(m)] = \frac{B(\varpi)}{1-\varpi}[G(\vartheta) - f(0)]\frac{1-\varpi}{1-\varpi+\varpi\vartheta^\varpi}$$

We define the inverse Sumudu transform as:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F\left(\frac{1}{s}\right)\frac{ds}{s}$$

Since it is quite troublesome to calculate the inverse Sumudu transformation, the relations given for the transformation will be used [3,7].

Let's define the Mittag-Leffler function [24,25]. For $\varpi, \sigma > 0$ and $m \in C$ with one parameter, we have:

$$E_\varpi(m) = \sum_{n=0}^{\infty}\frac{m^n}{\Gamma(n\varpi+1)}$$

and with two parameters, we define

$$E_{\varpi,\sigma}(m) = \sum_{n=0}^{\infty}\frac{m^n}{\Gamma(n\varpi+\sigma)}.$$

The relationship of the Mittag-Leffler function with the inverse Sumudu transform is given as [1,15]:

- $S^{-1}\left[\frac{\vartheta^{\sigma-\varpi}}{\vartheta^{1-\varpi}+\vartheta b}\right] = m^{\sigma-1}E_{\varpi,\sigma}(-bm^\alpha),\ \varpi,\sigma>0$ (1)
- $S^{-1}\left[\frac{\vartheta^{-1}}{(\vartheta^{-\varpi}+a\vartheta^{-\sigma})^{n+1}}\right] = m^{\varpi(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\varpi-\sigma)+(n+1)\varpi)}m^{k(\varpi-\sigma)}$

$\varpi \geq \sigma > 0$, b $\in R$. (2)

- $S^{-1}\left[\frac{\vartheta^{-\gamma+1}}{\vartheta^{-\varpi}+a\vartheta^{-\sigma}+b}\right] = m^{\varpi-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\varpi-\sigma)+(n+1)\varpi-\gamma)}m^{k(\varpi-\sigma)+n\varpi}$ (3)
- $S^{-1}\left[\frac{1}{1+b\vartheta^\varpi}\right] = E_\varpi(-bm^\varpi)$ (4)
- $S^{-1}\left[\frac{b\vartheta}{1-b\vartheta}\right] = e^{bm} - 1$ (5)
- $S^{-1}\left[\frac{1}{1-b\vartheta}\right] = e^{bm}$ (6)

Examples
We consider the following examples in this section [23,27].

Example 4.1 We consider [19]

$$D^2 f(m) + D^{\frac{3}{2}} f(m) + f(m) = 1 + m$$

$$f(0) = f'(0) = 1$$

If the Sumudu transform is applied, the following result will be obtained:

$$S\left[D^2 f(m) + D^{\frac{3}{2}} f(m) + f(m)\right] = S[1+m]$$

$$S[D^2 f(m)] + S\left[D^{\frac{3}{2}} f(m)\right] + S[f(m)] = S[1] + S[m]$$

$$\frac{1}{\vartheta^2} G(\vartheta) - \frac{1}{\vartheta^2} f(0) - \frac{1}{\vartheta} f'(0) + \frac{\frac{1}{\vartheta^2} G(\vartheta) - \frac{1}{\vartheta^2} f(0) - \frac{1}{\vartheta} f'(0)}{\frac{1}{\vartheta^{\frac{3}{2}}}} + G(\vartheta) = 1 + \vartheta$$

$$G(\vartheta)\left[\frac{1}{\vartheta^2} + \frac{1}{\vartheta^{\frac{1}{2}}} + 1\right] = \left(\frac{1}{\vartheta^2} + \frac{1}{\vartheta^{\frac{1}{2}}} + 1\right) + \left(1 + \frac{1}{\vartheta} + \vartheta^{\frac{1}{2}}\right)$$

$$G(\vartheta) = \frac{\left(\frac{1}{\vartheta^2} + \frac{1}{\vartheta^{\frac{1}{2}}} + 1\right) + \left(\vartheta + \frac{1}{\vartheta} + v^{\frac{1}{2}}\right)}{\left[\frac{1}{\vartheta^2} + \frac{1}{\vartheta^{\frac{1}{2}}} + 1\right]}$$

$$G(\vartheta) = 1 + \frac{\vartheta + \frac{1}{\vartheta} + \vartheta^{\frac{1}{2}}}{\frac{1}{\vartheta^2} + \frac{1}{\vartheta^{\frac{1}{2}}} + 1} = 1 + \frac{\frac{\vartheta^2 + 1 + \vartheta^{\frac{3}{2}}}{\vartheta}}{\frac{\vartheta^2 + 1 + \vartheta^{\frac{3}{2}}}{\vartheta^2}}$$

$$G(\vartheta) = 1 + \frac{\vartheta^2 + 1 + \vartheta^{\frac{3}{2}}}{\vartheta} \frac{\vartheta^2}{\vartheta^2 + 1 + \vartheta^{\frac{3}{2}}}$$

$$G(\vartheta) = 1 + \vartheta$$

$$S^{-1}[G(\vartheta)] = S^{-1}[1 + \vartheta]$$

$$f(m) = 1 + m$$

Example 4.2

We take into consideration the following problem:

$$\boldsymbol{{}_0^C D_m^{\varpi} f(m) + f(m) = \frac{2m^{2-\varpi}}{\Gamma(3-\varpi)} - \frac{m^{1-\varpi}}{\Gamma(2-\varpi)} + m^2 - m}$$

$$\boldsymbol{f(0) = 0,\ 0 < \varpi \leq 1}$$

If the Sumudu transform is applied, then the following result is obtained

$$S[{}_0^C D_m^{\varpi} f(m)] + S[f(m)] = S\left[\frac{2m^{2-\varpi}}{\Gamma(3-\varpi)}\right] - S\left[\frac{m^{1-\varpi}}{\Gamma(2-\varpi)}\right] + S[m^2] - S[m]$$

If the sumudu transform is applied to the expression $S[{}_0^C D_m^{\varpi} f(m)]$, we will obtain:

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(1-\varpi)} S\left[\int_0^t f'(\theta)(z-\theta)^{-\varpi} d\theta\right]$$

By the convolution theorem, we have

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(1-\varpi)} S[f'(m) * m^{-\varpi}]$$

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(1-\varpi)} \vartheta S[f'(m)] S[m^{-\varpi}]$$

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(1-\varpi)} \vartheta \left(\frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta}\right) \Gamma(1-\varpi) m^{-\varpi}$$

$$S[{}_0^C D_m^{\varpi} f(m)] = \vartheta^{-\varpi}[G(\vartheta) - f(0)] \quad (7)$$

Substituting the results in the equation yields:

$$\vartheta^{-\varpi}[G(\vartheta) - f(0)] + G(\vartheta) = \frac{2}{\Gamma(3-\varpi)} \Gamma(3-\varpi)\vartheta^{2-\varpi} - \frac{1}{\Gamma(2-\varpi)} \Gamma(2-\varpi)\vartheta^{-\varpi} + \Gamma(3)\vartheta^2 - \Gamma(2)\vartheta$$

$$\vartheta^{-\varpi}[G(\vartheta) - f(0)] + G(\vartheta) = 2\vartheta^{2-\varpi} - \vartheta^{-\varpi} + 2\vartheta^2 - \vartheta$$

$$G(\vartheta)[\vartheta^{-\varpi} + 1] = [\vartheta^{-\varpi} + 1][2\vartheta^2 - \vartheta]$$

$$G(\vartheta) = 2\vartheta^2 - \vartheta$$

$$S^{-1}[G(\vartheta)] = S^{-1}[2\vartheta^2 - \vartheta]$$

$$f(m) = m^2 - m$$

Example 4.3

We consider the following problem:

$$\boldsymbol{f''(m) - {}_0^C D_m^{\varpi} f(m) - bf(m) = 8}$$

$$\boldsymbol{f(0) = f'(0) = 0 \quad 1 < \varpi \leq 2}$$

If the Sumudu transform is applied, then the following result is obtained:

$$S[f''(m)] - S[{}_0^C D_m^{\varpi} f(m)] - S[bf(m)] = S[8]$$

If the Sumudu transform is applied to the expression $S[{}_0^C D_m^{\varpi} f(m)]$, then we will get:

$$S[{}_0^C D_m^{\varpi}] = \frac{1}{\Gamma(2-\varpi)} S\left[\int_0^m f''(\theta)(m-\theta)^{1-\varpi} d\theta\right]$$

By the convolution theorem, we get

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(2-\varpi)} S[f''(m) * m^{1-\varpi}]$$

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(2-\varpi)} \vartheta S[f''(m)] S[m^{1-\varpi}]$$

$$S[{}_0^C D_m^{\varpi} f(m)] = \frac{1}{\Gamma(2-\varpi)} \vartheta \left[\frac{1}{\vartheta^2} G(\vartheta) - \frac{1}{\vartheta^2} f(0) - \frac{1}{\vartheta} f'(0)\right] \Gamma(2-\varpi) \vartheta^{1-\varpi}$$

$$S[{}_0^C D_m^{\varpi} f(m)] = \vartheta^{-\varpi} G(\vartheta)$$

If we substitute the result, we will obtain:

$$\frac{1}{\vartheta^2}G(\vartheta)-\frac{1}{\vartheta^2}f(0)-\frac{1}{\vartheta}f'(0)-a\vartheta^{-\varpi}G(\vartheta)-bG(\vartheta)=8$$

$$G(\vartheta)\left[\frac{1}{\vartheta^2}-a\vartheta^{-\varpi}-b\right]=8$$

$$G(\vartheta)=\frac{8}{\frac{1}{\vartheta^2}-a\vartheta^{-\varpi}-b}$$

If the property given in (3) is applied, the we will acquire:

$$S^{-1}[G(\vartheta)]=S^{-1}\left[\frac{8}{\frac{1}{\vartheta^2}-a\vartheta^{-\varpi}-b}\right]$$

$$f(m)=8m^2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^n a^k\binom{n+k}{k}m^{(2-\varpi)k+2n}}{\Gamma\big((2-\varpi)k+2(n+1)+1\big)}$$

Example 4.4

We consider the following problem:

$$\boldsymbol{{}_0^C D_m^{\varpi} f(m) - m\cos m + f(m) - (1+m)\sin m = 0,\quad 0<\varpi\le 1,\ f(0)=1}$$

If the Sumudu transform is applied, we will get

$$\boldsymbol{S[\,{}_0^C D_m^{\varpi} f(m)] - S[m\cos m] + S[f(m)] - S[(1+m)\sin m] = S[0],\ 0<\varpi\le 1}$$

Substituting the result in (7) into the equation yields:

$$[G(\vartheta)-1]\vartheta^{-\varpi}-\frac{\vartheta(1-\vartheta^2)}{(1+\vartheta^2)^2}+G(\vartheta)-\frac{2\vartheta^2}{(1+\vartheta^2)^2}-\frac{\vartheta}{1+\vartheta^2}=0$$

$$[G(\vartheta)-1]\vartheta^{-\varpi}+G(\vartheta)+\frac{-\vartheta+\vartheta^3-2\vartheta^2-\vartheta-\vartheta^3}{(1+\vartheta^2)^2}=0$$

$$G(\vartheta)[\vartheta^{-\varpi}+1]=\frac{2\vartheta^2+2\vartheta}{(1+\vartheta^2)^2}+\vartheta^{-\varpi}$$

$$\boldsymbol{G(\vartheta)=\frac{\frac{2\vartheta^2+2\vartheta}{(1+\vartheta^2)^2}+\vartheta^{-\psi}}{\vartheta^{-\psi}+1}=\frac{2\vartheta^{\varpi}}{1+\vartheta^{\varpi}}\frac{(1+\vartheta)}{(1+\vartheta^2)^2}+\frac{\vartheta^{-\varpi}}{\vartheta^{-\varpi}+1}}$$

If the properties in (1) and (3) are applied, then we will obtain

$$S^{-1}[G(\vartheta)]=S^{-1}\left[\frac{2\vartheta^{\varpi}}{1+\vartheta^{\varpi}}\frac{(1+\vartheta)}{(1+\vartheta^2)^2}\right]+S^{-1}\left[\frac{\vartheta^{-\varpi}}{\vartheta^{-\varpi}+1}\right]$$

$$f(m)=2m^{\varpi+1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}(k+1)m^{2k}\left(\frac{m^{n\varpi+1}}{\Gamma\big((n+1)\varpi+2k+3\big)}+\frac{m^{n\varpi}}{\Gamma\big((n+1)\varpi+2k+2\big)}\right)+E_{\varpi}(-m^{\varpi})$$

Examples with solutions and comparisons over graphs

The derivatives of the examples in this section are taken respectively by Caputo, Atangana-Baleanu and Caputo-Fabrizio derivatives. Obtained solutions are interpreted on graphs by taking $\varpi=0.5, \varpi=0.7, \varpi=0.9$ and $\varpi=1$.

Example 4.1.1 We consider

$$\boldsymbol{{}_0^C D_m^{\varpi} f(m) - f(m) = 0,\ f(0)=1,\ 0<\varpi\le 1}$$

If the Sumudu transform is applied, we will get

$$S[\,{}_0^C D_m^{\varpi} f(m)] - S[f(m)] = 0$$

Substituting the result in (7) into the equation yields

$$\boldsymbol{\vartheta^{-\varpi}[G(\vartheta)-f(0)] - G(\vartheta) = 0}$$

$$\boldsymbol{G(\vartheta)[\vartheta^{-\varpi}-1]=\vartheta^{-\varpi}}$$

$$\boldsymbol{G(\vartheta)=\frac{\vartheta^{-\varpi}}{\vartheta^{-\varpi}-1}}$$

$$\boldsymbol{S^{-1}[G(\vartheta)]=S^{-1}\left[\frac{\vartheta^{-\varpi}}{\vartheta^{-\varpi}-1}\right]}$$

$$\boldsymbol{f(m)=E_{\varpi}(m^{\varpi})}$$

Example 4.1.2

We consider

$$\boldsymbol{{}_0^{ABC} D_m^{\varpi} f(m) - f(m) = 0,\ f(0)=1,\ 0<\varpi\le 1}$$

If the Sumudu transform is applied, the following result is obtained

$$S[\,{}_0^{ABC} D_m^{\varpi} f(m)] - S[f(m)] = S[0]$$

If the Sumudu transform is applied to the expression $S[\,{}_0^{ABC} D_m^{\varpi} f(m)]$, we will get

$$S[\,{}_0^{ABC} D_m^{\varpi} f(m)]=\frac{B(\varpi)}{1-\varpi}S\left[\int_0^m f'(\theta)E_{\varpi}\left(\frac{-\varpi}{1-\varpi}(m-\theta)^{\varpi}\right)d\theta\right]$$

By the convolution theorem, we have

$$S[\,{}_0^{ABC} D_m^{\varpi} f(m)]=\frac{B(\varpi)}{1-\varpi}S\left[f'(m)*E_{\varpi}\left(\frac{-\varpi}{1-\varpi}m^{\varpi}\right)\right]$$

$$S[\,{}_0^{ABC} D_m^{\varpi} f(m)]=\frac{B(\varpi)}{1-\varpi}\varpi S[f'(m)]S\left[E_{\varpi}\left(\frac{-\varpi}{1-\varpi}m^{\varpi}\right)\right]$$

$$S[\,{}_0^{ABC} D_m^{\varpi} f(m)]=\frac{B(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta}-\frac{f(0)}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\varpi}} \quad (8)$$

If we substitute the result, we will get

$$\frac{B(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta}-\frac{f(0)}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi v^{\psi}}-S[f(m)]=S[0]$$

$$\frac{B(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta}-\frac{1}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi v^{\psi}}-G(\vartheta)=0$$

$$\frac{B(\varpi)}{1-\varpi}\vartheta\frac{G(\vartheta)}{\vartheta}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\psi}}-\frac{B(\varpi)}{1-\varpi}\vartheta\frac{1}{\vartheta}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\varpi}}-G(\vartheta)=0$$

$$\frac{B(\varpi)}{1-\varpi}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\psi}}G(\vartheta)-\frac{B(\varpi)}{1-\varpi}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\psi}}-G(\vartheta)=0$$

$$G(\vartheta)\left[\frac{B(\varpi)}{1-\varpi}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\varpi}}-1\right]=\frac{B(\varpi)}{1-\varpi}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^{\varpi}}$$

$$G(\vartheta)\left[\frac{B(\varpi)-1+\varpi-\varpi\vartheta^{\varpi}}{1-\varpi+\varpi\vartheta^{\varpi}}\right]=\frac{B(\varpi)}{1-\varpi+\varpi\vartheta^{\varpi}}$$

$$G(\vartheta)=\frac{B(\varpi)}{1-\varpi+\varpi\vartheta^{\varpi}}\frac{1-\varpi+\varpi\vartheta^{\varpi}}{B(\varpi)-1+\varpi-\varpi\vartheta^{\varpi}}$$

$$G(\vartheta)=\frac{B(\varpi)}{B(\varpi)-1+\varpi-\varpi\vartheta^{\varpi}}=\frac{\frac{B(\varpi)}{B(\varpi)-1+\varpi}}{\frac{B(\varpi)-1+\varpi-\varpi\vartheta^{\varpi}}{B(\varpi)-1+\varpi}}$$

$$G(\vartheta)=\frac{B(\varpi)}{B(\varpi)-1+\varpi}\left[\frac{1}{1-\frac{\varpi\vartheta^{\varpi}}{B(\varpi)-1+\varpi}}\right]$$

If the property given (4) is applied, then we will obtain

$$S^{-1}[G(\vartheta)]=\frac{B(\varpi)}{B(\varpi)-1+\varpi}S^{-1}\left[\frac{1}{1-\frac{\varpi\vartheta^{\varpi}}{B(\varpi)-1+\varpi}}\right]$$

$$f(m)=\frac{B(\varpi)}{B(\varpi)-1+\varpi}E_{\varpi}\left(\frac{B(\varpi)}{B(\varpi)-1+\varpi}m^{\varpi}\right).$$

Example 4.1.3

We consider

$${}_0^{CFC}D_m^{\varpi}f(m)-f(m)=0,\ f(0)=1,\ 0<\varpi\le 1$$

If the Sumudu transform is applied, then we will get

$$S[{}_0^{CFC}D_m^{\varpi}f(m)]-S[f(m)]=S[0]$$

If the Sumudu transform is applied to the expression $S[{}_0^{CFC}D_m^{\varpi}f(m)]$, we will obtain

$$\boldsymbol{S[{}_0^{CFC}D_m^{\varpi}]=\frac{M(\varpi)}{1-\varpi}S\left[\int_0^m f'(\theta)\,e^{-\frac{\varpi}{1-\varpi}(m-\theta)}d\theta\right]}$$

By the convolution theorem, we have

$$S[{}_0^{CFC}D_m^{\varpi}]=\frac{M(\varpi)}{1-\varpi}S\left[f'(m)*e^{-\frac{\varpi}{1-\varpi}m}\right]$$

$$S[{}_0^{CFC}D_m^{\varpi}]=\frac{M(\varpi)}{1-\varpi}\vartheta S[f'(m)]S\left[e^{-\frac{\varpi}{1-\varpi}m}\right]$$

$$S[{}_0^{CFC}D_m^{\varpi}]=\frac{M(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta}-\frac{f(0)}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi\vartheta}\tag{9}$$

If we substitute the result, we will find

$$\frac{M(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta}-\frac{f(0)}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi\vartheta}-S[f(m)]=S[0]$$

$$\frac{M(\varpi)}{1-\varpi}\vartheta\frac{G(\vartheta)}{\vartheta}\frac{1-\varpi}{1-\varpi+\varpi\vartheta}-\frac{M(\varpi)}{1-\varpi}\vartheta\frac{f(0)}{\vartheta}\frac{1-\varpi}{1-\varpi+\varpi\vartheta}-G(\vartheta)=0$$

$$\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}G(\vartheta)-\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}-G(\vartheta)=0$$

$$G(\vartheta)\left[\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}-1\right]=\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}$$

$$G(\vartheta)\left[\frac{M(\varpi)-(1-\varpi)-\varpi\vartheta}{1-\varpi+\varpi\vartheta}\right]=\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}$$

$$G(\vartheta)=\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}\frac{1-\varpi+\varpi\vartheta}{M(\varpi)-(1-\varpi)-\varpi\vartheta}$$

$$G(\vartheta)=\frac{\frac{M(\varpi)}{M(\varpi)-(1-\varpi)}}{\frac{M(\varpi)-(1-\varpi)-\varpi\vartheta}{M(\varpi)-(1-\varpi)}}=\frac{M(\varpi)}{M(\varpi)-(1-\varpi)}\left(\frac{1}{1-\frac{\varpi\vartheta}{M(\varpi)-(1-\varpi)}}\right)$$

If the property given (6) is applied, then we will reach

$$S^{-1}[G(\vartheta)]=\frac{M(\varpi)}{M(\varpi)-(1-\varpi)}S^{-1}\left(\frac{1}{1-\frac{\varpi\vartheta}{M(\varpi)-(1-\varpi)}}\right)$$

$$f(m)=\frac{M(\varpi)}{M(\varpi)-(1-\varpi)}e^{\frac{\varpi}{M(\varpi)-(1-\varpi)}m}.$$

The graphs of the results obtained in Example 4.1.1, 4.1.2 and 4.1.3 are given as:

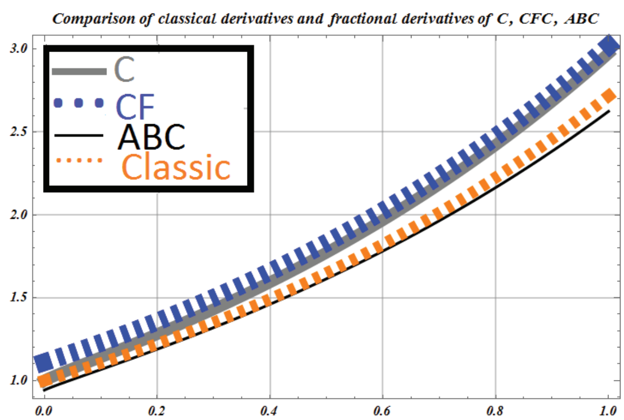

Figure 1. Analysis for ϖ = 0.90.

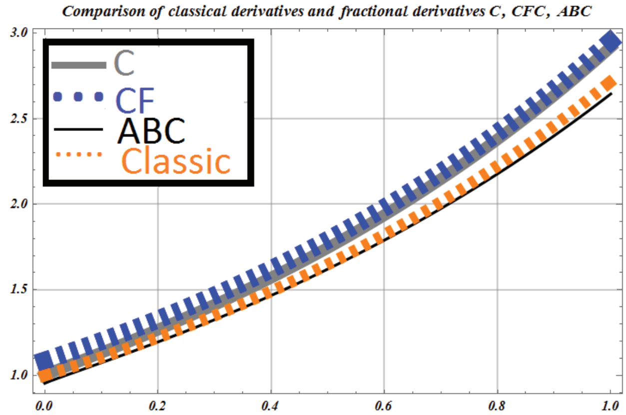

Figure 2. Analysis for ϖ = 0.92.

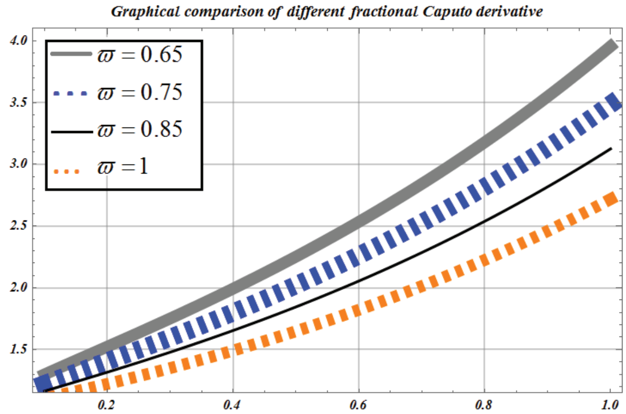

Figure 5. Analysis for Caputo derivative.

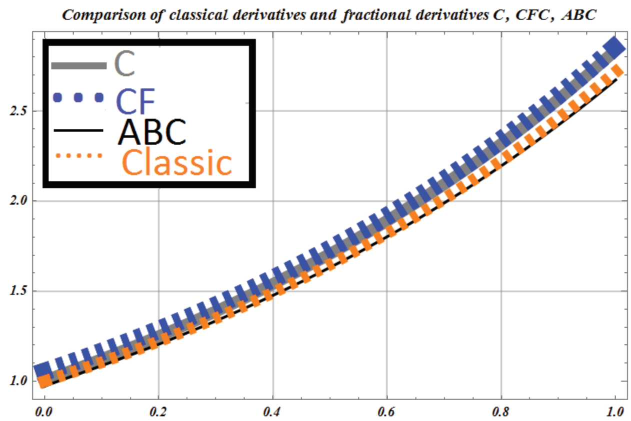

Figure 3. Analysis for ϖ = 0.95.

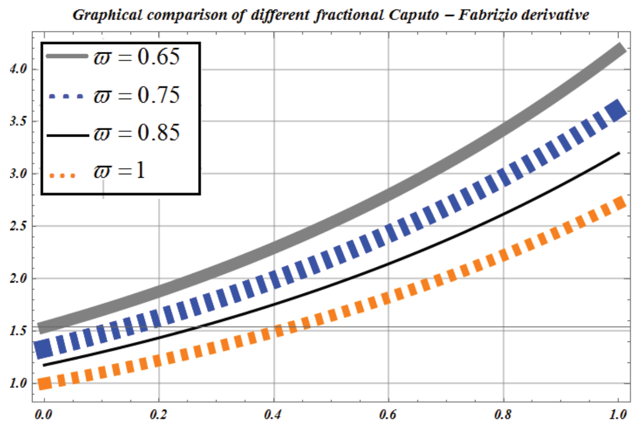

Figure 6. Analysis for Caputo-Fabrizio derivative.

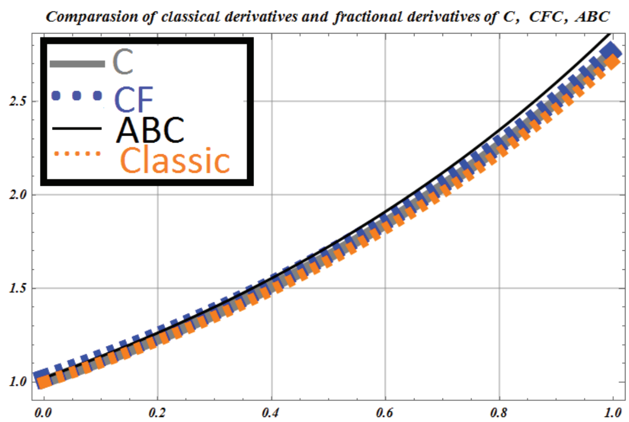

Figure 4. Analysis for ϖ = 0.98.

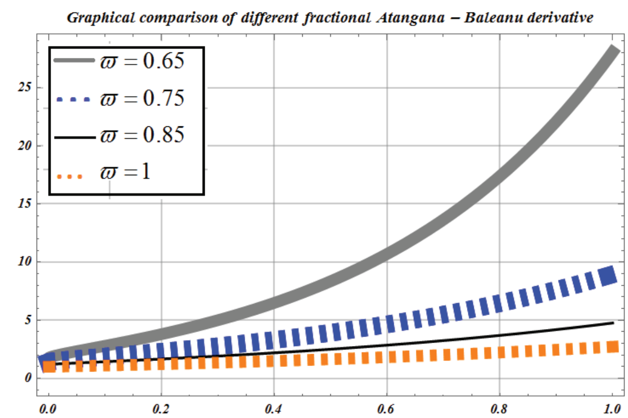

Figure 7. Analysis for Atangana-Baleanu derivative.

- Graphical demonstrations are given for the different ϖ values of the examples containing the fractional derivative operator.
- Figure 1 (ϖ = 0.90) gives a detailed demonstrations of the behavior of the fractional derivative operators. For Figure 2, ϖ = 0.92, for Figure 3 ϖ = 0.95 and for Figure 4 ϖ = 0.98.
- In the graphical demonstrations, it is observed which derivative operator is closer to the classical derivative operator.
- As ϖ gets closer to 1, it is seen that the solution graphs get closer to each other and resemble the classical derivative.
- Figure 5 contains the graphical representations of the solution ϖ = 0.65, ϖ = 0.75, ϖ = 0.85, ϖ = 1 in Example 4.1.1, which includes the Caputo derivative operator. As the value of ϖ gets closer to 1, it is seen that the result gets closer to the classical derivative operator.
- Figure 6 contains the graphical representations of the solution ϖ = 0.65, ϖ = 0.75, ϖ = 0.85, ϖ = 1 in Example 4.1.2, which includes the Caputo-Fabrizio derivative operator. As the value of ϖ gets closer to 1, it is seen that the result gets closer to the classical derivative operator.
- Figure 7 contains the graphical representations of the solution ϖ = 0.65, ϖ = 0.75, ϖ = 0.85, ϖ = 1 in Example 4.1.3, which includes the Atangana-Baleanu derivative operator. As the value of ϖ gets closer to 1, it is seen that the result gets closer to the classical derivative operator.
- When the graphical representations of Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivative operators are examined, it is seen that the Caputo and Caputo Fabrizio derivative operators behave more similarly.

Example 4.1.4

We consider

$$\boldsymbol{{}_0^C D_m^\varpi f(m) = f(m) + 1,\ f(0) = 0,\ 0<\varpi \leq 1}$$

If the Sumudu transform is applied, we will get

$$\boldsymbol{S\left[{}_0^C D_m^\varpi f(m)\right] = S[f(m)] + S[1]}$$

Substituting the result in (7) into the equation yields

$$\boldsymbol{\vartheta^{-\varpi}[G(\vartheta) - f(0)] = G(\vartheta) + 1}$$

$$\boldsymbol{G(\vartheta)[\vartheta^{-\varpi} - 1] = 1}$$

$$\boldsymbol{G(\vartheta) = \frac{1}{\vartheta^{-\varpi} - 1}}$$

$$\boldsymbol{G(\vartheta) = \frac{\vartheta}{\vartheta}\frac{1}{\vartheta^{-\varpi} - 1} = \frac{\vartheta}{\vartheta^{1-\varpi} - \vartheta}}$$

If the property in (1) is applied, then we will reach:

$$S^{-1}[G(\vartheta)] = S^{-1}\left[\frac{\vartheta}{\vartheta^{1-\varpi} - \varpi}\right]$$

$$f(m) = m^\varpi E_{\varpi,\varpi+1}(m^\varpi)$$

Example 4.1.5

We consider

$$\boldsymbol{{}_0^{ABC} D_m^\varpi f(m) = f(m) + 1,\ f(0)=0,\ 0<\varpi \leq 1}$$

If the Sumudu transform is applied, we will get

$$S\left[{}_0^{ABC} D_m^\varpi f(m)\right] = S[f(m)] + S[1]$$

Substituting the result in (8) yields:

$$\frac{B(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta} - \frac{f(0)}{\vartheta}\right]\left[\frac{1-\varpi}{1-\varpi+\varpi\vartheta^\varpi}\right] = G(\vartheta) + 1$$

$$\frac{B(\varpi)}{1-\varpi}\vartheta\frac{G(\vartheta)}{\vartheta}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^\varpi} - G(\vartheta) = 1$$

$$G(\vartheta)\left[\frac{B(\varpi)}{1-\varpi}\frac{1-\varpi}{1-\varpi+\varpi\vartheta^\varpi} - 1\right] = 1$$

$$G(\vartheta)\left[\frac{B(\varpi) - (1-\varpi) - \varpi\vartheta^\varpi}{1-\varpi+\varpi\vartheta^\varpi}\right] = 1$$

$$G(\vartheta) = \frac{1-\varpi+\varpi\vartheta^\varpi}{B(\varpi) - (1-\varpi) - \varpi\vartheta}$$

$$G(\vartheta) = \frac{1-\varpi}{B(\varpi) - (1-\varpi) - \varpi\vartheta} - \frac{\varpi\vartheta^\varpi}{B(\varpi) - (1-\varpi) - \varpi\vartheta}$$

$$G(\vartheta) = \frac{\frac{1-\varpi}{B(\varpi)-(1-\varpi)}}{\frac{B(\varpi)-(1-\varpi)-\varpi\vartheta}{B(\varpi)-(1-\varpi)}} - \frac{\frac{\varpi\vartheta^\psi}{B(\varpi)-(1-\varpi)}}{\frac{B(\varpi)-(1-\varpi)-\varpi\vartheta}{B(\varpi)-(1-\varpi)}}$$

$$G(\vartheta) = \frac{1-\varpi}{B(\varpi)-(1-\varpi)}\left[\frac{1}{1-\frac{\varpi\vartheta^\varpi}{B(\varpi)-(1-\varpi)}}\right]$$

$$+\frac{\varpi}{B(\varpi)-(1-\varpi)}\left[\frac{\vartheta^\varpi}{1-\frac{\varpi}{B(\varpi)-(1-\varpi)}\vartheta^\varpi}\right]$$

If the properties in (1) and (4) are applied, we will get

$$S^{-1}[G(\vartheta)] = \frac{1-\varpi}{B(\varpi)-(1-\varpi)}S^{-1}\left[\frac{1}{1-\frac{\varpi\vartheta^\varpi}{B(\varpi)-(1-\varpi)}}\right]$$

$$+\frac{\varpi}{B(\varpi)-(1-\varpi)}S^{-1}\left[\frac{\vartheta^\varpi}{1-\frac{\varpi}{B(\varpi)-(1-\varpi)}\vartheta^\varpi}\right]$$

$$f(m) = \frac{1-\varpi}{B(\varpi)-(1-\varpi)}E_\varpi\left(\frac{\varpi}{B(\varpi)-(1-\varpi)}m^\varpi\right) + \frac{\varpi}{B(\varpi)-(1-\varpi)}E_\varpi\left(\frac{\varpi}{B(\varpi)-(1-\varpi)}m^\varpi\right) - 1$$

Example 4.1.6

We consider

$$\boldsymbol{{}_0^{CFC} D_m^\varpi f(m) = f(m) + 1,\ f(0)=0,\ 0<\varpi \leq 1}$$

If the Sumudu transform is applied, we will get

$$S\left[{}_0^{CFC}D_m^{\varpi} f(m)\right] = S[f(m)] + S[1]$$

Substituting the result in (9) into the equation $\frac{M(\varpi)}{1-\varpi}\vartheta\left[\frac{G(\vartheta)}{\vartheta}-\frac{f(0)}{\vartheta}\right]\frac{1-\varpi}{1-\varpi+\varpi\vartheta} = S[f(m)]+S[1]$ yields:

$$\frac{M(\varpi)}{1-\varpi}\vartheta\frac{G(\vartheta)}{\vartheta}\frac{1-\varpi}{1-\varpi+\varpi\vartheta} = G(\vartheta)+1$$

$$\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}G(\vartheta) - G(\vartheta) = 1$$

$$G(\vartheta)\left[\frac{M(\varpi)}{1-\varpi+\varpi\vartheta}-1\right] = 1$$

$$G(\vartheta) = \frac{1-\varpi+\varpi\vartheta}{M(\varpi)-(1-\varpi)-\varpi\vartheta} = \frac{1-\varpi}{M(\varpi)-(1-\varpi)-\varpi\vartheta} + \frac{\varpi\vartheta}{M(\varpi)-(1-\varpi)-\varpi\vartheta}$$

$$G(\vartheta) = \frac{1-\varpi}{M(\varpi)-(1-\varpi)}\left(\frac{1}{1-\frac{\varpi}{M(\varpi)-(1-\varpi)}\vartheta}\right) + \frac{\varpi}{M(\varpi)-(1-\varpi)}\left(\frac{\vartheta}{1-\frac{\varpi}{M(\varpi)-(1-\varpi)}\vartheta}\right)$$

If the property in (5) is applied, we will get

$$S^{-1}[G(\vartheta)] = \frac{1-\varpi}{M(\varpi)-(1-\varpi)}S^{-1}\left(\frac{1}{1-\frac{\varpi}{M(\varpi)-(1-\varpi)}\vartheta}\right) + \frac{\varpi}{M(\varpi)-(1-\varpi)}S^{-1}\left(\frac{\vartheta}{1-\frac{\varpi}{M(\varpi)-(1-\varpi)}\vartheta}\right)$$

$$f(m) = \frac{1-\varpi}{M(\varpi)-(1-\varpi)}e^{\frac{\varpi}{M(\varpi)-(1-\varpi)}m} + e^{\frac{\varpi}{M(\varpi)-(1-\varpi)}m} - 1$$

The graphs of the results obtained in Example 4.1.4, 4.1.5 and 4.1.6 are given as:

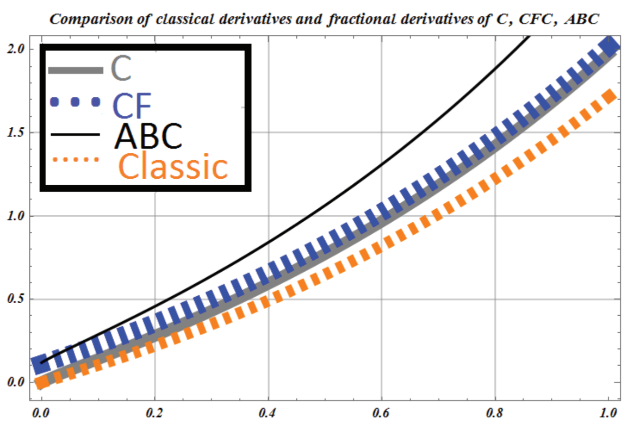

Figure 8. Analysis for ϖ = 0.90

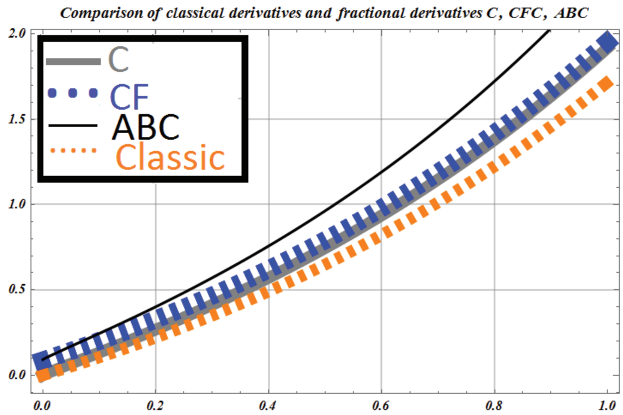

Figure 9. Analysis for ϖ = 0.92

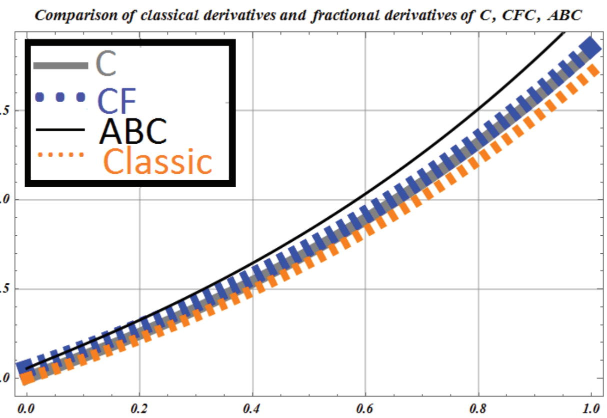

Figure 10. Analysis for ϖ = 0.95

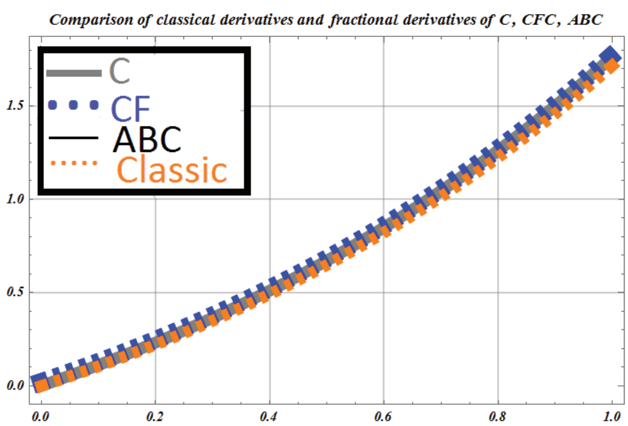

Figure 11. Analysis for ϖ = 0.98

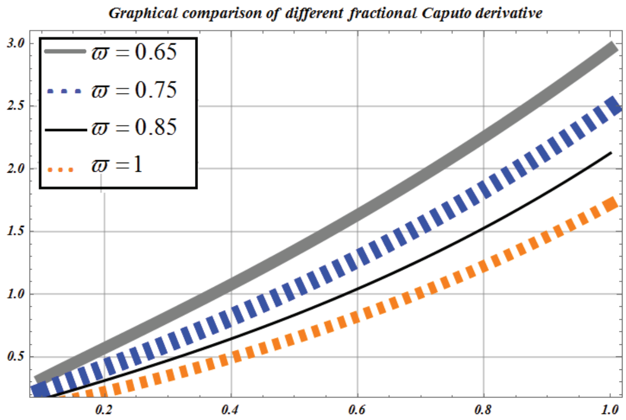

Figure 12. Analysis for Caputo derivative.

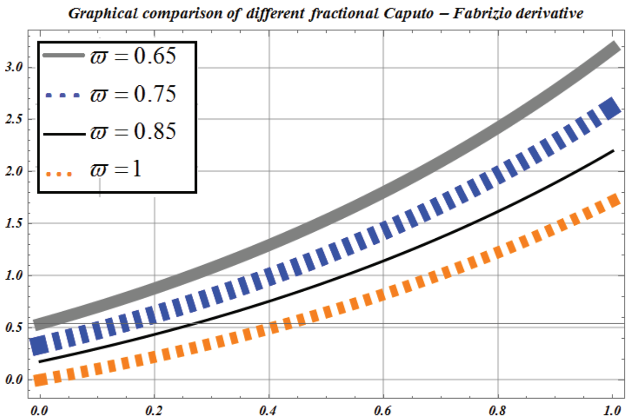

Figure 13. Analysis for Caputo-Fabrizio derivative.

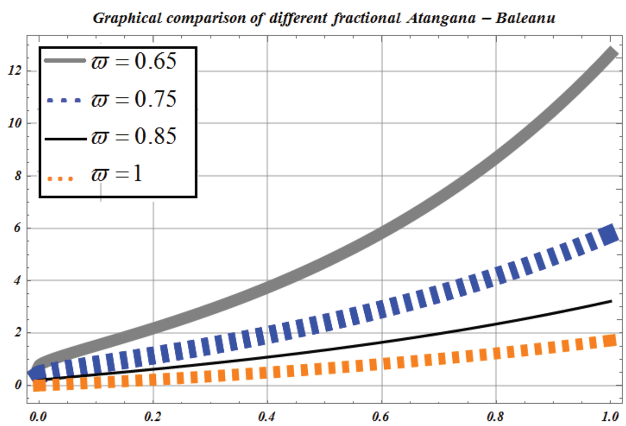

Figure 14. Analysis for Atangana – Baleanu derivative.

- Graphical demonstrations are given for the different ϖ values of the examples containing the fractional derivative operator.
- Figure 8 ($\varpi = 0.90$) gives a detailed demonstrations of the behavior of the fractional derivative operators for Figure 9 $\varpi = 0.92$, for Figure 10 $\varpi = 0.95$ and for Figure 11 $\varpi = 0.98$.
- As ϖ gets closer to 1, it is seen that the solution graphs get closer to each other and resemble the classical derivative.
- Figure 12 contains the graphical representations of the solution $\varpi = 0.65$, $\varpi = 0.75$, $\varpi = 0.85$, $\varpi = 1$ in Example 4.1.4, which includes the Caputo derivative operator. As the value of ϖ gets closer to 1, it is seen that the result gets closer to the classical derivative operator.
- Figure 13 contains the graphical representations of the solution $\varpi = 0.65$, $\varpi = 0.75$, $\varpi = 0.85$, $\varpi = 1$ in Example 4.1.5, which includes the Caputo-Fabrizio derivative operator. As the value of ϖ gets closer to 1, it is seen that the result gets closer to the classical derivative operator.
- Figure 14 contains the graphical representations of the solution $\varpi = 0.65$, $\varpi = 0.75$, $\varpi = 0.85$, $\varpi = 1$ in Example 4.1.6, which includes the Atangana-Baleanu derivative operator. As the value of ϖ gets closer to 1, it is seen that the result gets closer to the classical derivative operator.
- When the graphical representations of Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivative operators are examined, it is seen that the Caputo and Caputo Fabrizio derivative operators behave more similarly.

CONCLUSIONS

Sumudu transform is an effective transform for finding analytical solutions of linear equations. In this article, homogeneous and inhomogeneous linear equations including Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives are investigated. By referring to the relationship between the Sumudu transform and the Mittag-Leffler function and making use of this relationship, the solutions of these equations have been obtained. When the solution graphs are examined, it is seen that Caputo and Caputo-Fabrizio give close solutions. It is also seen that they produce solutions similar to the classical derivative when the order of derivative gets closer to 1.

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