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Research Article

On non-abelian strongly real Beauville p-groups

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Abstract

We give an infinite family of non-abelian strongly real Beauville p-groups for any odd prime p by considering the lower central quotients of the free product of two cyclic groups of order p.

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1. Introduction

A *Beauville surface* of unmixed type is a compact complex surface isomorphic to $(C_1 \times C_2)/G$, where C_1 and C_2 are algebraic curves of genus at least 2 and G is a finite group acting freely on $C_1 \times C_2$ and faithfully on the factors C_i such that $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$. Then the group G is said to be a *Beauville group*.

The condition for a finite group G to be a Beauville group can be formulated in purely group-theoretical terms.

Definition 1.1. For a couple of elements $x, y \in G$, we define

$$\Sigma(x, y) = \bigcup_{g \in G} \left(\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right),$$

that is, the union of all subgroups of G which are conjugate to $\langle x \rangle$, to $\langle y \rangle$ or to $\langle xy \rangle$. Then G is a Beauville group if and only if the following conditions hold:

(i) G is a 2-generator group.
(ii) There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a *Beauville structure* for G.

Definition 1.2. Let G be a Beauville group. We say that G is *strongly real* if there exists a Beauville structure $\{\{x_1, y_1\}, \{x_2, y_2\}\}$ for G, an automorphism $\theta \in \mathrm{Aut}(G)$, and elements $g_1, g_2 \in G$ such that

$$g_i \theta(x_i) g_i^{-1} = x_i^{-1} \quad \text{and} \quad g_i \theta(y_i) g_i^{-1} = y_i^{-1}$$

for $i = 1, 2$. Then the Beauville structure is called *strongly real Beauville structure.*

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In practice, it is convenient to take $g_1 = g_2 = 1$.

In 2000, Catanese [2] proved that a finite abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, where $n > 1$ and $\gcd(n, 6) = 1$. Since for any abelian group the function $x \longmapsto -x$ is an automorphism, the following result is immediate.

Lemma 1.3. *Every abelian Beauville group is a strongly real Beauville group.*

Thus, there are infinitely many abelian strongly real Beauville p-groups for $p \geq 5$. If the p-group is non-abelian, it is harder to construct a strongly real Beauville structure.

The earliest examples of non-abelian strongly real Beauville p-groups were given by Fairbairn in [4], by constructing the following pair of 2-groups. The groups

$$G = \langle x, y \mid x^8 = y^8 = [x^2, y^2] = (x^i y^j)^4 = 1 \text{ for } i, j = 1, 2, 3\rangle,$$

and

$$G = \langle x, y \mid (x^i y^j)^4 = 1 \text{ for } i, j = 0, 1, 2, 3\rangle$$

are strongly real Beauville groups of order 2^{13} and 2^{14}, respectively.

In [8], the author gave an infinite family of non-abelian strongly real Beauville p-groups for every prime p by considering the quotients of triangle groups. At around the same time, Fairbairn [5] gave another infinite family of non-abelian strongly real Beauville p-groups for odd p, by using wreath products of cyclic p-groups.

In this paper, we give a new infinite family of non-abelian strongly real Beauville p-groups for any odd prime p. To this purpose, we work with the lower central quotients of the free product of two cyclic groups of order p. The main result of this paper is as follows.

Theorem A. *Let $F = \langle x, y \mid x^p, y^p\rangle$ be the free product of two cyclic groups of order p for an odd prime p, and let $i = k(p-1)+1$ for $k \geq 1$. Then the quotient $F/\gamma_{i+1}(F)$ is a strongly real Beauville group.*

Note that in [7], it was shown that all p-central quotients of the free product $F = \langle x, y \mid x^p, y^p\rangle$ are Beauville groups. Observe that since F/F' has exponent p, the lower central series and p-central series of F coincide.

Notation. If p is a prime and G is a finite p-group, then $G^{p^i} = \langle g^{p^i} \mid g \in G\rangle$ and $\Omega_i(G) = \langle g \in G \mid g^{p^i} = 1\rangle$. The exponent of G, denoted by $\exp G$, is the maximum of the orders of all elements of G.

2. Proof of the main theorem

In this section, we give the proof of Theorem A. We begin by recalling the definition of p-central series.

Definition 2.1. For any group G, put

$$\lambda_n(G) = \gamma_1(G)^{p^{n-1}} \gamma_2(G)^{p^{n-2}} \dots \gamma_n(G)$$

for $n \geq 1$. Thus $\lambda_n(G)$ is a characteristic subgroup of G and

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \dots$$

is a normal series of G, which is called the *p-central series* of G. Then a quotient group $G/\lambda_n(G)$ is said to be a *p-central quotient* of G.

The subgroups $\lambda_n(G)$ have the following properties:

Theorem 2.2 ([10], page 252)**.** *Let G be a group. Then*

(i) $\lambda_n(G) = [\lambda_{n-1}(G), G]\lambda_{n-1}(G)^p$ *for* $n > 1$.

(ii) $[\lambda_n(G), G] = \gamma_2(G)^{p^{n-1}} \gamma_3(G)^{p^{n-2}} \dots \gamma_{n+1}(G)$.

Also observe that if the exponent of $G/\gamma_2(G)$ is p, then by the above theorem we have $\lambda_n(G) = \gamma_n(G)$ for all $n \geq 1$.

We further have the following result regarding the subgroups $\lambda_n(G)$.

Theorem 2.3 ([9], Lemma 9.20)**.** *If $G/\lambda_2(G)$ is generated by the images of $g_1, g_2, \ldots, g_n$, then $\lambda_2(G)/\lambda_3(G)$ is generated by the images of g_i^p for $1 \leq i \leq d$ and $[g_i, g_j]$ for $1 \leq i < j \leq d$. More generally, for $k > 1$, let S be a subset of G which generates G modulo $\lambda_2(G)$, and let T generate $\lambda_k(G)$ modulo $\lambda_{k+1}(G)$. Then $\lambda_{k+1}(G)$ is generated modulo $\lambda_{k+2}(G)$ by $[s,t]$ for $s \in S$, $t \in T$ and t^p for $t \in T$.*

Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order p. By Theorem 2.2(i) and Theorem 2.3, each quotient $\lambda_n(F)/\lambda_{n+1}(F)$ is a finite elementary abelian p-group. Thus, the p-central quotients $F/\lambda_n(F)$ are finite elementary abelian p-groups.

On the other hand, since F/F' has exponent p, the lower central series and p-central series of F coincide, that is,

$$\gamma_n(F) = \lambda_n(F) \tag{2.1}$$

for all $n \geq 1$.

We next continue by stating a lemma regarding the existence of an automorphism of F which sends the generators to their inverses. The proof is left to the reader.

Lemma 2.4. *Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order p. Then the map*

$$\begin{aligned} \theta \colon F &\longrightarrow F \\ x &\longmapsto x^{-1} \\ y &\longmapsto y^{-1}, \end{aligned}$$

is an automorphism of F.

Before we proceed, we will introduce some results regarding the Nottingham group which will help us to determine some properties of F.

The *Nottingham group* $\mathcal{N}$ over the field $\mathbb{F}_p$, for odd p, is the (topological) group of normalized automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series. For any positive integer k, the automorphisms $f \in \mathcal{N}$ such that $f(t) = t + \sum_{i \geq k+1} a_i t^i$ form an open normal subgroup $\mathcal{N}_k$ of $\mathcal{N}$ of index p^{k-1}. Observe that $|\mathcal{N}_k : \mathcal{N}_{k+1}| = p$ for all $k \geq 1$. We have the commutator formula

$$[\mathcal{N}_k, \mathcal{N}_\ell] = \begin{cases} \mathcal{N}_{k+\ell}, & \text{if } k \not\equiv \ell \pmod p, \\ \mathcal{N}_{k+\ell+1}, & \text{if } k \equiv \ell \pmod p \end{cases} \tag{2.2}$$

(see [1], Theorem 2). Thus the lower central series of $\mathcal{N}$ is given by

$$\gamma_i(\mathcal{N}) = \mathcal{N}_{r(i)}, \quad \text{where} \quad r(i) = i + 1 + \left\lfloor \frac{i-2}{p-1} \right\rfloor. \tag{2.3}$$

As a consequence, $|\gamma_i(\mathcal{N}) : \gamma_{i+1}(\mathcal{N})| \leq p^2$. Furthermore, $\gamma_i(\mathcal{N})/\gamma_{i+1}(\mathcal{N}) \cong C_p \times C_p$ if and only if i is of the form $i = k(p-1)+1$ for some $k \geq 0$. In other words, in the lower central series of $\mathcal{N}$, the quotients $\mathcal{N}_{kp+1}/\mathcal{N}_{kp+3}$ are elementary abelian p-groups of order p^2.

Recall that by Remark 3 in [1], $\mathcal{N}$ is topologically generated by the elements $a \in \mathcal{N}_1 \smallsetminus \mathcal{N}_2$ and $b \in \mathcal{N}_2 \smallsetminus \mathcal{N}_3$ given by $a(t) = t(1-t)^{-1}$ and $b(t) = t(1-t^2)^{-1/2}$, which are both of order p.

In the following lemma, we need a result of Klopsch [11, formula (3.4)] regarding the centralizers of elements of order p of $\mathcal{N}$ in some quotients $\mathcal{N}/\mathcal{N}_k$. More specifically, if $f \in \mathcal{N}_k \smallsetminus \mathcal{N}_{k+1}$ is of order p, then for every $\ell = k + 1 + pn$ with $n \in \mathbb{N}$, we have

$$C_{\mathcal{N}/\mathcal{N}_\ell}(f\mathcal{N}_\ell) = C_{\mathcal{N}}(f)\mathcal{N}_{\ell-k}/\mathcal{N}_\ell. \tag{2.4}$$

Lemma 2.5. *Put $G = \mathcal{N}/\mathcal{N}_{kp+3}$ and $N_i = \mathcal{N}_i/\mathcal{N}_{kp+3}$ for $1 \leq i \leq kp+3$. If α is the image of a in G, then the set $\{[\alpha, g] \mid g \in G\}$ does not cover N_{kp+1}.*

Proof. To prove the lemma, we will show that $\{[\alpha, g] \mid g \in G\} \cap N_{kp+2} = 1$. Assume that $[\alpha, g] \in N_{kp+2}$ for some $g \in G$. Since $a \in \mathcal{N}_1 \setminus \mathcal{N}_2$ is of order p, it follows from (2.4) that

$$C_{\mathcal{N}/\mathcal{N}_{kp+2}}(a\mathcal{N}_{kp+2}) = C_{\mathcal{N}}(a)\mathcal{N}_{kp+1}/\mathcal{N}_{kp+2}.$$

Thus we can write $g = ch$, with $[\alpha, c] = 1$ and $h \in N_{kp+1}$. Then $[\alpha, g] = [\alpha, h] \in [G, N_{kp+1}] = 1$, since N_{kp+1} is central in G. □

Lemma 2.6. *Put $H = F/\gamma_{i+1}(F)$, where $i = k(p-1)+1$ for $k \geq 1$ and $H_i = \gamma_i(F)/\gamma_{i+1}(F)$. If u and v are the images of x and y in H, respectively, then the sets $\{[u, h] \mid h \in H\}$ and $\{[v, h] \mid h \in H\}$ do not cover H_i.*

Proof. Let $G = \mathcal{N}/\mathcal{N}_{kp+3}$, and let us call α and β the images of a and b in G, respectively. Since α and β are of order p and $\gamma_{i+1}(G) = 1$, the map

$$\begin{aligned} \psi \colon H &\longrightarrow G \\ u &\longmapsto \alpha \\ v &\longmapsto \beta, \end{aligned}$$

is well-defined and an epimorphism.

By Lemma 2.5, the set of commutators of α does not cover the subgroup $\gamma_i(G) = N_{kp+1}$. It then follows that the set $\{[u, h] \mid h \in H\}$ does not cover H_i. Since the roles of u and v are symmetric, we also conclude that the set $\{[v, h] \mid h \in H\}$ does not cover H_i, as desired. □

To prove the main result, we need the following three lemmas.

Lemma 2.7. *Let $G = \langle a, b \rangle$ be a finite minimally 2-generated p-group and $o(a) = p$, for some prime p. Then*

$$\Big(\bigcup_{g \in G} \langle a \rangle^g\Big) \bigcap \Big(\bigcup_{g \in G} \langle b \rangle^g\Big) = 1.$$

Proof. We assume that $x = (a^i)^g = (b^j)^h$ for some $i, j \in \mathbb{Z}$ and $g, h \in G$, and prove that $x = 1$. In the quotient $\overline{G} = G/\Phi(G) = \langle \overline{a} \rangle \times \langle \overline{b} \rangle$, we have $\overline{x} \in \langle \overline{a} \rangle \cap \langle \overline{b} \rangle = \overline{1}$ implying that $x \in \Phi(G)$. On the other hand, $x \in \langle a^g \rangle$, where a^g is of order p and $a^g \notin \Phi(G)$. It then follows that $x = 1$. □

Lemma 2.8 ([6], Lemma 3.8)**.** *Let G be a finite p-group and let $x \in G \setminus \Phi(G)$ be an element of order p. If $t \in \Phi(G) \setminus \{[x, g] \mid g \in G\}$ then*

$$\Big(\bigcup_{g \in G} \langle x \rangle^g\Big) \bigcap \Big(\bigcup_{g \in G} \langle xt \rangle^g\Big) = 1.$$

Lemma 2.9 ([7], Lemma 3.1)**.** *Let $\psi \colon G_1 \to G_2$ be a group homomorphism, let $x_1, y_1 \in G_1$ and $x_2 = \psi(x_1)$, $y_2 = \psi(y_1)$. If $o(x_1) = o(x_2)$ then the condition $\langle x_2^{\psi(g)} \rangle \cap \langle y_2^{\psi(h)} \rangle = 1$ implies that $\langle x_1^g \rangle \cap \langle y_1^h \rangle = 1$ for $g, h \in G_1$.*

Let $H = F/\gamma_{i+1}(F)$ and let u and v be the images of x and y in H, respectively. In order to prove the main theorem, we need to know the order of uv. We first recall a result of Easterfield [3] regarding the exponent of $\Omega_j(G)$. More precisely, if G is a finite p-group, then for every $j, k \geq 1$, the condition $\gamma_{k(p-1)+1}(G) = 1$ implies that

$$\exp \Omega_j(G) \leq p^{j+k-1}. \tag{2.5}$$

If we set $k = \left\lceil \frac{i}{p-1} \right\rceil$, we have $\gamma_{k(p-1)+1}(H) \leq \gamma_{i+1}(H) = 1$. Then by (2.5), we get $\exp H \leq p^k$, and hence $o(uv) \leq p^k$. Indeed, we will show that $o(uv) = p^k$. To this purpose, we also need to introduce a result regarding p-groups of maximal class with some specific properties.

Let $G = \langle s \rangle \ltimes A$ where s is of order p and $A \cong \mathbb{Z}_p^{p-1}$. The action of s on A is via θ, where θ is defined by the companion matrix of the pth cyclotomic polynomial $x^{p-1} + \cdots + x + 1$. Then G is the only infinite pro-p group of maximal class. Since $s^p = 1$ and $\theta^{p-1} + \cdots + \theta + 1$ annihilates A, this implies that for every $a \in A$,

$$(sa)^p = s^p a^{s^{p-1}+\cdots+s+1} = 1.$$

Thus all elements in $G \smallsetminus A$ are of order p.

Let P be a finite quotient of G of order p^{i+1} for $i \geq 2$. Let us call P_1 the abelian maximal subgroup of P and $P_j = [P_1, P, \overset{j-1}{\ldots}, P] = \gamma_j(P)$ for $j \geq 2$. Then one can easily check that $\exp P_j = p^{\lceil \frac{i+1-j}{p-1} \rceil}$ and every element in $P_j \smallsetminus P_{j+1}$ is of order $p^{\lceil \frac{i+1-j}{p-1} \rceil}$.

Let $s \in P \smallsetminus P_1$ and $s_1 \in P_1 \smallsetminus \gamma_2(P)$. Since all elements in $P \smallsetminus P_1$ are of order p and $\gamma_{i+1}(P) = 1$, the map

$$\begin{aligned} \psi \colon H &\longrightarrow P \\ u &\longmapsto s^{-1} \\ v &\longmapsto ss_1, \end{aligned}$$

is well-defined and an epimorphism. Then we have $o(uv) \geq o(s_1) = p^k$, and this, together with $\exp H = p^k$, implies that $o(uv) = p^k$.

We are now ready to give the final proof.

Proof of the Main Theorem. Let H and H_i be as defined in Lemma 2.6. Let u and v be the images of x and y in H, respectively. By Lemma 2.6, there exist $w, z \in H_i$ such that $w \notin \{[u,h] \mid h \in H\}$ and $z \notin \{[v,h] \mid h \in H\}$. Observe that w and z are central elements in H, and since H_i is elementary abelian, they have order p in H. We claim that $\{u, v\}$ and $\{(uw)^{-1}, vz\}$ form a Beauville structure in H. Let $X = \{u, v, uv\}$ and $Y = \{(uw)^{-1}, vz, u^{-1}vw^{-1}z\}$.

Assume first that $\tilde{x} \in X$ is of order p, and let $\tilde{y} \in Y$. If $\langle \tilde{x}\Phi(H) \rangle \neq \langle \tilde{y}\Phi(H) \rangle$ in $H/\Phi(H)$, then by Lemma 2.7, $\langle \tilde{x} \rangle^g \cap \langle \tilde{y} \rangle^h = 1$ for every $g, h \in H$. Otherwise, we are in one of the following two cases: $\tilde{x} = u$ and $\tilde{y} = (uw)^{-1}$, or $\tilde{x} = v$ and $\tilde{y} = vz$. Then the condition $\langle \tilde{x} \rangle^g \cap \langle \tilde{y} \rangle^h = 1$ follows by Lemma 2.8.

We now assume that $\tilde{x} = uv$. Again applying Lemma 2.7, we get $\langle \tilde{x} \rangle^g \cap \langle \tilde{y} \rangle^h = 1$ where $\tilde{y} = (uw)^{-1}$ or $\tilde{y} = vz$, which is of order p. Thus we are only left with the case when $\tilde{x} = uv$ and $\tilde{y} = u^{-1}vw^{-1}z$. Recall that the map $\psi \colon H \longrightarrow P$ is an epimorphism such that $\psi(u) = s^{-1}$ and $\psi(v) = ss_1$. Then $\psi(u^{-1}vw^{-1}z)$ is an element outside P_1, which is of order p . Thus $\langle \psi(u^{-1}vw^{-1}z)^{\psi(g)} \rangle \cap \langle s_1^{\psi(h)} \rangle = 1$ for all $g, h \in H$. Since $o(uv) = o(s_1)$, the condition $\langle \tilde{x} \rangle^g \cap \langle \tilde{y} \rangle^h = 1$ for all $g, h \in H$ follows by Lemma 2.9. This completes the proof that H is a Beauville group.

We next show that the Beauville structure $\{\{u, v\}, \{(uw)^{-1}, vz\}\}$ is strongly real. By Lemma 2.4, we know that the map θ is an automorphism of F. Since $\theta(\gamma_n(F)) = \gamma_n(\theta(F)) = \gamma_n(F)$ for all $n \geq 1$, the map θ induces an automorphism $\overline{\theta} \colon H \longrightarrow H$ such that $\overline{\theta}(u) = u^{-1}$ and $\overline{\theta}(v) = v^{-1}$. Now we only need to check if $\overline{\theta}((uw)^{-1}) = uw$ and $\overline{\theta}(vz) = (vz)^{-1}$. Note that

$$\overline{\theta}((uw)^{-1}) = \overline{\theta}(w^{-1})u = u\overline{\theta}(w^{-1}),$$

and

$$\overline{\theta}(vz) = v^{-1}\overline{\theta}(z) = \overline{\theta}(z)v^{-1}$$

where the last equalities follow from the fact that both w and z are central in H. Thus it suffices to see that $\overline{\theta}(w^{-1}) = w$ and $\overline{\theta}(z) = z^{-1}$.

Note that H_i is generated by the commutators of length i in u and v. Now by using commutator identities (see Proposition 1.1.6, [12]) and taking into account that i is odd and $H_i \leq Z(H)$, one can show that

$$\overline{\theta}([x_{j_1}, x_{j_2}, \ldots, x_{j_i}]) = [x_{j_1}^{-1}, x_{j_2}^{-1}, \ldots, x_{j_i}^{-1}] = [x_{j_1}, x_{j_2}, \ldots, x_{j_i}]^{-1},$$

where each x_{j_k} is either u or v. Hence the automorphism $\overline{\theta}$ sends the generators of H_i to their inverses. Since H_i is abelian, this implies that for every $t \in H_i$ we have $\overline{\theta}(t) = t^{-1}$.

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