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Research Article

On products of idempotents and nilpotents

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Abstract

This article studies the ring structure arising from products of idempotents and nilpotents. Thus the argument is concerned essentially with the one-sided IQNN property of rings. We first prove that if the 2 by 2 full matrix ring over a principal ideal domain F of characteristic zero is right IQNN then F contains infinitely many non-integer rational numbers; and that the concepts of right IQNN and right quasi-Abelian are independent of each other. We next introduce a ring property, called *right IAN*, as a generalization of both right IQNN and right quasi-Abelian; and provide several kinds of methods to construct right IAN rings. In the procedure, we also show that the right IQNN and right IAN do not go up to polynomial rings.

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1. Prerequisites

In ring theory we often encounter with products of idempotents and nilpotents. This occurs, for example, in the study of representing an element as a sum of an idempotent and a nilpotent element (or a unit), which is a prominent area of recent ring theory (see [1, 3, 13]). Especially this article concerns the structure of rings which stems from representing a product ea by a product bf, where e, f are idempotents and a, b are nilpotents in a given ring. The results obtained here may provide useful information to the study of representing elements by idempotents and nilpotents.

Throughout this article, every ring is an associative ring with identity unless otherwise stated. Let R be a ring. $I(R)$ is used to denote the set of all idempotents of R, and $I(R)' = I(R)\backslash\{0, 1\}$. We use $U(R)$, $N(R)$, and $N^*(R)$ to denote the group of all units, the set of all nilpotent elements, and upper nilradical (i.e., the sum of all nil ideals) of R, respectively. It is well-known that $N^*(R) \subseteq N(R)$. A nilpotent element is also called a *nilpotent* for simplicity. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix

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ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$, I_n means the identity matrix in $Mat_n(R)$ and use E_{ij} for the matrix with (i,j)-entry 1 and zeros elsewhere. $\mathbb{Z}$ ($\mathbb{Z}_n$) and $\mathbb{Q}$ denote the ring of integers (modulo n) and the field of rational numbers, respectively. For $m, n \in \mathbb{Z}$, $gcd(m,n)$ means the greatest common divisor of m, n. The characteristic of R is denoted by $ch(R)$.

Recall that R is called a *reduced* ring if it has no nonzero nilpotents, i.e., $N(R) = 0$. A ring is called *Abelian* if every idempotent is central. Reduced rings are obviously Abelian. Following [2], a ring R is called *right* (resp., *left*) *quasi-Abelian* provided that either $I(R)'$ is empty, or else for any $(e,a) \in I(R)' \times R$ (resp., $(a,e) \in R \times I(R)'$) there exists $(b,f) \in R \times I(R)'$ (resp., $(f,b) \in I(R)' \times R$) such that $ea = bf$ (resp., $ae = fb$). A ring R is called *quasi-Abelian* if it is both right and left quasi-Abelian. Abelian rings are clearly quasi-Abelian, but not conversely by [2, Example 1.4]. It is easy to check that if R is a ring with $I(R)'$ nonempty then R is right quasi-Abelian if and only if for any $e \in I(R)'$ and $a \in R$, there exists $f \in I(R)'$ such that $ea = eaf$ ([2, Remark 1.3]). Following [11, Definition 1.2], a ring R is said to be *right idempotent-quasi-normalizing on nilpotents* (simply, *right IQNN*) provided that $I(R)'$ is empty, or else for every pair $(e,a) \in I(R)' \times N(R)$ there exists $(b,f) \in N(R) \times I(R)'$ such that $ea = bf$. A *left IQNN* ring is defined symmetrically. A ring is called *IQNN* if it is both right and left IQNN. Abelian rings are clearly IQNN but not conversely as in [11].

In Section 2, we prove that the principal ideal domain F with $ch(F) = 0$ contains infinitely many non-integer rational numbers when $Mat_2(F)$ is a right IQNN ring, and and the domain H such that $K[x] \subseteq H \subseteq K(x)$, where $K(x)$ is the quotient field of the polynomial ring $K[x]$ over a field K, contains infinitely many non-polynomial fractions when $Mat_2(H)$ is a right IQNN ring (Theorem 2.4). In Section 3, we define the concept of a right (left) IAN ring (Definition 3.1) which unifies both the quasi-Abelian ring property and the IQNN ring property, and then study under what conditions these three concepts are interrelated (Proposition 3.9). In addition, it is shown that (i) R is reduced with $I(R) = \{0,1\}$ if and only if $T_n(R)$ for $n \geq 2$ is IAN (Proposition 3.10); (ii) if $D_n(R)$ for $n \geq 2$ is a right IAN ring, then so is R (Proposition 3.14(2)); (iii) some useful examples of right IAN rings are also provided, including the Dorroh extension (Proposition 3.16) and polynomial rings (Remark 3.18).

2. Remarks on right IQNN rings

We first observe the IQNN property of $Mat_2(F)$ over a commutative domain F, showing that the concepts of right IQNN and right quasi-Abelian are independent of each other. We need the following information essentially for our purpose.

Lemma 2.1. *Let F be a commutative domain and $R = Mat_2(F)$. Then the following assertions hold.*

(1) *$I(R)'$ is*

$$\{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$$

where for $t \neq 0, u \neq 0$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}, E_6 = \begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix},$$

and

$$E_7 = \begin{pmatrix} s & t \\ u & 1-s \end{pmatrix} \text{ for } s \notin \{0,1\} \text{ and } s(1-s) = tu;$$

and $N(R)$ is the union of two sets

$$\left\{ B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \mid t \neq 0, u \neq 0 \right\}$$

and

$$\left\{B_4 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a \neq 0, b \neq 0, c \neq 0, \text{ and } a^2 = -bc\right\}.$$

(2) *If $E_7B_4 \neq 0$ then every entry of E_7B_4 is nonzero, and this result also holds for the cases of $B_4E_7, E_4B_4, E_5B_4, B_4E_3$ and B_4E_6.*

(3)(i) *Let $F = \mathbb{Z}$ and p, q be any nonzero integers. If $C = \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} \in R$ is such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$, then there exist $E' \in I(R)'$ and $B \in N(R)$ such that $EA = BE'$, in fact EA is B_4E_2 or B_4E_5.*

(ii) *Let $F = \mathbb{Z}$ and p, q be any nonzero integers such that $p \neq 1$. If $C = \begin{pmatrix} q & 0 \\ p & 0 \end{pmatrix} \in R$ is such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$, then there exist $E' \in I(R)'$ and $B \in N(R)$ such that $EA = BE'$.*

Proof. (1) It follows from [11, Lemma 2.3(2, 3)].

(2) Let K be the quotient field of F. Consider $E_7 = \begin{pmatrix} s & t \\ u & 1-s \end{pmatrix} \in I(R)'$ and $B_4 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in N(R)$ in (1), where $s \notin \{0, 1\}, s(1-s) = tu$ (then $t, u \neq 0$) and $a \neq 0, a^2 = -bc$ (then $b \neq 0, c \neq 0$). Then we have

$$E_7B_4 = \begin{pmatrix} sa + tc & sb - ta \\ ua + (1-s)c & ub - (1-s)a \end{pmatrix} \text{ and } B_4E_7 = \begin{pmatrix} as + bu & at + b(1-s) \\ cs - au & ct - a(1-s) \end{pmatrix}.$$

From the relations $s(1-s) = tu$ and $a^2 = -bc$, we obtain that $sb - ta = a^{-1}(sab - ta^2) = a^{-1}(sab + tbc) = \frac{b}{a}(sa + tc)$, $ua + (1-s)c = s^{-1}(sua + s(1-s)c) = s^{-1}(sua + tuc) = \frac{u}{s}(sa + tc)$ and $ub - (1-s)a = (as)^{-1}(asub - a^2s(1-s)) = (as)^{-1}(asub + bctu) = \frac{bu}{as}(sa + tc)$; and that $at + b(1-s) = s^{-1}(sat + bs(1-s)) = s^{-1}(sat + btu) = \frac{t}{s}(as + bu)$, $cs - au = a^{-1}(acs - a^2u) = a^{-1}(acs + bcu) = \frac{c}{a}(as + bu)$ and $ct - a(1-s) = (as)^{-1}(asct - a^2s(1-s)) = (as)^{-1}(asct + bctu) = \frac{ct}{as}(as + bu)$, from which we see

$$E_7B_4 = \begin{pmatrix} sa + tc & \frac{b}{a}(sa + tc) \\ \frac{u}{s}(sa + tc) & \frac{bu}{as}(sa + tc) \end{pmatrix} \text{ and } B_4E_7 = \begin{pmatrix} as + bu & \frac{t}{s}(as + bu) \\ \frac{c}{a}(as + bu) & \frac{ct}{as}(as + bu) \end{pmatrix} \text{ in } Mat_2(K).$$

Next consider $E_4 = \begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} \in I(R)'$ and $B_4 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in N(R)$ in (1), where $a \neq 0, a^2 = -bc$ (then $b \neq 0, c \neq 0$). Then

$$E_4B_4 = \begin{pmatrix} a & b \\ ua & ub \end{pmatrix} = \begin{pmatrix} a & -\frac{a}{c}a \\ ua & -\frac{ua}{c}a \end{pmatrix} \text{ and } E_5B_4 = \begin{pmatrix} tc & -ta \\ c & -a \end{pmatrix} = \begin{pmatrix} -\frac{ta}{b}a & -ta \\ -\frac{a}{b}a & -a \end{pmatrix}.$$

In addition, for $E_3 = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}, E_6 = \begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix} \in I(R)'$, we have

$$B_4E_3 = \begin{pmatrix} a & ta \\ c & tc \end{pmatrix} = \begin{pmatrix} a & ta \\ -\frac{a}{b}a & -\frac{ta}{b}a \end{pmatrix} \text{ and } B_4E_6 = \begin{pmatrix} ub & b \\ -ua & -a \end{pmatrix} = \begin{pmatrix} -\frac{ua}{c}a & -\frac{a}{c}a \\ -ua & -a \end{pmatrix},$$

as desired.

(3)(i) By (2), we have the cases that $A = B_2 = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in N(R)$, and E is $E_7 = \begin{pmatrix} p' & m \\ q' & 1-p' \end{pmatrix} \in I(R)'$ (where $p'(1-p') = q'm \neq 0$) or $E_4 = \begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix} \in I(R)'$; that is, EA is $\begin{pmatrix} 0 & p'v \\ 0 & q'v \end{pmatrix}$ with $p = p'v$ and $q = q'v$, or $\begin{pmatrix} 0 & v \\ 0 & uv \end{pmatrix}$ with $p = v$ and $q = uv$.

`Case 1.` When p divides q.

We have

$$BE' = \begin{pmatrix} -q & p \\ -\frac{q^2}{p} & q \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} = EA,$$

noting $B = \begin{pmatrix} -q & p \\ -\frac{q^2}{p} & q \end{pmatrix} \in N(R)$ and $E' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in I(R)'$.

`Case 2.` When q divides p.

We have

$$\begin{pmatrix} p & -\frac{p^2}{q} \\ q & -p \end{pmatrix} \begin{pmatrix} 0 & 1+\frac{p}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} = EA,$$

noting $B = \begin{pmatrix} p & -\frac{p^2}{q} \\ q & -p \end{pmatrix} \in N(R)$ and $E' = \begin{pmatrix} 0 & 1+\frac{p}{q} \\ 0 & 1 \end{pmatrix} \in I(R)'$.

`Case 3.` When p and q do not divide each other, and $gcd(p, q) = 1$.

Note that this case arises from E_7B_2 with $B_2 = E_{12}$ or $B_2 = -E_{12}$. It suffices to consider the case of $B_2 = E_{12}$, i.e., $v = 1$. Then $p = p'$ and $q = q'$ (hence $p(1 - p) = qm$). Evidently $|p|, |q| \geq 2$.

Let $p = p_1^{u_1} \cdots p_f^{u_f}$ and $q = q_1^{v_1} \cdots q_g^{v_g}$ ($u_i, v_j \geq 1$) be the prime number decompositions of p, q respectively, where p_i's and q_j's are distinct. Then, from the hypothesis that $gcd(p, q) = 1$ and $p(1 - p) = qm$, we see that every $q_j^{k_j}$ divides $1 - p$ and, consequently, q divides $1 - p$.

Assume $EA = B_4E_2$. Then

$$\begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & -a \end{pmatrix}$$

and so $b = p$, $a = -q$. But $a^2 = -bc$ implies $q^2 = -pc$, entailing that every p_i divides q, a contradiction. So we must have $EA = B_4E_5$. Then

$$\begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & at+b \\ 0 & ct-a \end{pmatrix}$$

and so $at + b = p$, $ct - a = q$. But $a^2 = -bc$ implies

$$p = at + b = a^{-1}(a^2t + ab) = a^{-1}(-bct + ab) = \frac{-b}{a}(ct - a) = \frac{-b}{a}q$$

and

$$q = ct - a = a^{-1}(act - a^2) = a^{-1}(act + bc) = \frac{c}{a}(at + b) = \frac{c}{a}p,$$

entailing $pa = -qb$ and $qa = pc$. Then $a, c \in q\mathbb{Z}$ and $a, b \in p\mathbb{Z}$ by the distinctness of p_i's and q_j's; hence we can let $a = q^{h_1}p^{h_2}a_1$ with $a_1 \in \mathbb{Z}$ and $h_1, h_2 \geq 1$. It then follows that $b = -\frac{p}{q}a = -q^{h_1-1}p^{h_2+1}a_1$ and $c = \frac{q}{p}a = q^{h_1+1}p^{h_2-1}a_1$.

From the relation $at + b = p$, we have

$$t = \frac{p-b}{a} = \frac{p + q^{h_1-1}p^{h_2+1}a_1}{q^{h_1}p^{h_2}a_1} = \frac{1 + q^{h_1-1}p^{h_2}a_1}{q^{h_1}p^{h_2-1}a_1}.$$

Here if $h_1 \geq 2$ (resp., $h_2 \geq 2$) then q (resp., p) divides 1, a contradiction. Thus $h_1 = h_2 = 1$ and we obtain that $a = qpa_1$, $b = -p^2a_1$, $c = q^2a_1$. Then

$$t = \frac{p-b}{a} = \frac{p + p^2a_1}{qpa_1} = \frac{1 + pa_1}{qa_1},$$

entailing $a_1(qt-p) = 1$ and $a_1 = \frac{1}{qt-p}$. Since $a_1 \in \mathbb{Z}$, we must have $|a_1| = 1$ and $|qt-p| = 1$.

Letting $a_1 = -1$, we obtain that $a = -qp$, $b = p^2$, $c = -q^2$ and $t = -\frac{1-p}{q}$.

Since q divides $1-p$ as above, we have that $t=-\frac{1-p}{q}\in\mathbb{Z}$ and

$$BE'=\begin{pmatrix}-qp & p^2\\ -q^2 & qp\end{pmatrix}\begin{pmatrix}0 & -\frac{1-p}{q}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & p\\ 0 & q\end{pmatrix}=EA,$$

noting $B=\begin{pmatrix}-qp & p^2\\ -q^2 & qp\end{pmatrix}\in N(R)$ and $E'=\begin{pmatrix}0 & -\frac{1-p}{q}\\ 0 & 1\end{pmatrix}\in I(R)'$.

(ii) Then $0\neq A=\begin{pmatrix}0 & 0\\ s & 0\end{pmatrix}\in N(R)$, and E is $\begin{pmatrix}1-p' & q'\\ m' & p'\end{pmatrix}\in I(R)'$ (where $p'(1-p')=q'm'\neq 0$ (hence $p'\neq 0,1$)) or $\begin{pmatrix}0 & t\\ 0 & 1\end{pmatrix}\in I(R)'$ (where $t\neq 0$); that is, $EA=\begin{pmatrix}q's & 0\\ p's & 0\end{pmatrix}$ with $p=p's$ and $q=q's$, or $EA=\begin{pmatrix}st & 0\\ s & 0\end{pmatrix}$ with $p=s$ and $q=st$. We apply the argument of (i).

`Case 1.` When p divides q.

We have

$$BE'=\begin{pmatrix}q & -\frac{q^2}{p}\\ p & -q\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}q & 0\\ p & 0\end{pmatrix}=EA,$$

noting $B=\begin{pmatrix}q & -\frac{q^2}{p}\\ p & -q\end{pmatrix}\in N(R)$ and $E'=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\in I(R)'$.

`Case 2.` When q divides p.

We have

$$BE'=\begin{pmatrix}-p & q\\ -\frac{p^2}{q} & p\end{pmatrix}\begin{pmatrix}1 & 0\\ 1+\frac{p}{q} & 0\end{pmatrix}=\begin{pmatrix}q & 0\\ p & 0\end{pmatrix}=EA,$$

noting $B=\begin{pmatrix}-p & q\\ -\frac{p^2}{q} & p\end{pmatrix}\in N(R)$ and $E'=\begin{pmatrix}1 & 0\\ 1+\frac{p}{q} & 0\end{pmatrix}\in I(R)'$.

`Case 3.` When p and q do not divide each other, and $gcd(p,q)=1$.

Note $|p|,|q|\geq 2$, and we have

$$BE'=\begin{pmatrix}qp & -q^2\\ p^2 & -qp\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1-p}{q} & 0\end{pmatrix}=\begin{pmatrix}q & 0\\ p & 0\end{pmatrix}=EA,$$

noting $B=\begin{pmatrix}qp & -q^2\\ p^2 & -qp\end{pmatrix}\in N(R)$ and $E'=\begin{pmatrix}1 & 0\\ -\frac{1-p}{q} & 0\end{pmatrix}\in I(R)'$. □

In [11], it is proved that $Mat_2(K)$ is IQNN over a field K, and a question "Is $Mat_2(F)$ also right IQNN when F is a commutative domain?" is raised. The following example answers this question negatively.

Example 2.2. We follow the notation of Lemma 2.1(1).

(1) Let $R=Mat_2(\mathbb{Z})$ and we will show that R is not right IQNN. Note that $\mathbb{Z}$ is a principal ideal domain. Let p be any prime integer. Take $E=\begin{pmatrix}1 & 0\\ p^2 & 0\end{pmatrix}$ and $A=\begin{pmatrix}p & p^2\\ -1 & -p\end{pmatrix}\in R$. Then $E\in I(R)'$ and $A\in N(R)$ such that $EA=\begin{pmatrix}p & p^2\\ p^3 & p^4\end{pmatrix}$. Assume on the contrary that there exist $B\in N(R)$ and $E'\in I(R)'$ such that $EA=BE'$. By Lemma 2.1(2), BE' must be one of the following cases:

$$B_4E_3, B_4E_6, \text{ or } B_4E_7.$$

(Case 1) $EA = B_4E_3 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & ta \\ -\frac{a}{b}a & -\frac{ta}{b}a \end{pmatrix}$ by Lemma 2.1(2), where $a^2 = -bc \neq 0$ and $t \neq 0$. Then $a = p$ and $p^3 = -\frac{a}{b}a = -\frac{p^2}{b}$. So $b = -\frac{1}{p} \in \mathbb{Z}$, a contradiction.

(Case 2) $EA = B_4E_6 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} -\frac{ua}{c}a & -\frac{a}{c}a \\ -ua & -a \end{pmatrix}$ by Lemma 2.1(2), where $a^2 = -bc \neq 0$ and $u \neq 0$. Then $a = -p^4$ and $p^3 = -ua = up^4$, entailing $u = \frac{1}{p} \in \mathbb{Z}$, a contradiction.

(Case 3) $EA = B_4E_7 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} s & t \\ u & 1-s \end{pmatrix} = \begin{pmatrix} as+bu & at+b(1-s) \\ cs-au & ct-a(1-s) \end{pmatrix} = \begin{pmatrix} \alpha & \frac{t}{s}\alpha \\ \frac{c}{a}\alpha & \frac{ct}{as}\alpha \end{pmatrix}$ by Lemma 2.1(2), where $\alpha = as + bu$, $a^2 = -bc \neq 0$ and $s(1-s) = tu \neq 0$. Then $\alpha = p$, $\frac{c}{a}\alpha = p^3$ and $\frac{t}{s}\alpha = p^2$, from which we see $c = p^2a$ and $t = ps$. From $a^2 = -bc$, we obtain $a^2 = -p^2ab$ and $a = -p^2b$.

Now, by the results above, we have

$$p^2 = at + b(1-s) = (-p^2b)(ps) + b(1-s) = -p^3bs + b(1-s) = b(1-(1+p^3)s),$$

from which we see $b = \frac{p^2}{1-(1+p^3)s} \in \mathbb{Z}$. Since $s \neq 0, 1$ and $|1-(1+p^3)s| > p^2$, there cannot exist $b, s \in \mathbb{Z}$ that satisfy the equation $b = \frac{p^2}{1-(1+p^3)s}$.

Consequently $EA \notin N(R) \times I(R)'$, and therefore R is not right IQNN.

(2) Let $R = Mat_2(F)$, where $F = \mathbb{Z}[\sqrt{3}\, i] = \{a + b\sqrt{3}\, i \mid a, b \in \mathbb{Z}\}$ and $i^2 = -1$. Note that $\mathbb{Z}[\sqrt{3}\, i]$ is not a unique factorization domain (hence it is not a principal ideal domain) as can be seen by the two distinct factorizations into irreducible elements: $4 = 2 \cdot 2 = (1+\sqrt{3}\, i)(1-\sqrt{3}\, i)$. We claim that R is not right IQNN. We will use Lemma 2.1(1) freely.

Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in I(R)'$ and $A = \begin{pmatrix} 2 & 1+\sqrt{3}\, i \\ -(1-\sqrt{3}\, i) & -2 \end{pmatrix} \in N(R)$. Then $EA = \begin{pmatrix} 2 & 1+\sqrt{3}\, i \\ 0 & 0 \end{pmatrix}$. Assume on the contrary that there exist $B \in N(R)$ and $E' \in I(R)'$ such that $EA = BE'$. Then BE' must be one of the following:

$$B_2E_6, B_2E_7, B_4E_3, B_4E_6, \text{ and } B_4E_7.$$

(Case 1) $EA = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix} = B_2E_6$ implies $tu = 2$ and $t = 1 + \sqrt{3}\, i$, entailing $u = \frac{1}{2} - \frac{1}{2}\sqrt{3}\, i \notin \mathbb{Z}[\sqrt{3}\, i]$.

(Case 2) $EA = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} s' & t' \\ u' & 1-s' \end{pmatrix} = B_2E_7$ (where $s' \notin \{0, 1\}$ and $s'(1-s') = t'u'$) implies $tu' = 2$ and $t(1-s') = 1+\sqrt{3}\, i$. Since 2 is irreducible and the only units of $\mathbb{Z}[\sqrt{3}\, i]$ are $1, -1$, we have that

$$(t, u') \in \{(1, 2), (-1, -2), (2, 1), (-2, -1)\}.$$

Letting $(t, u') = (1, 2)$, we have $s' = -\sqrt{3}\, i$ and $2t' = t'u' = s'(1-s') = (-\sqrt{3}\, i)(1+\sqrt{3}\, i) = 3-\sqrt{3}\, i$. From this, we obtain $t' = \frac{3}{2} - \frac{-1}{2}\sqrt{3}\, i \notin \mathbb{Z}[\sqrt{3}\, i]$. When $(t, u') = (-1, -2)$, we have $t' = -\frac{1}{2} + \frac{3}{2}\sqrt{3}\, i \notin \mathbb{Z}[\sqrt{3}\, i]$. When $(t, u') = (2, 1), (-2, -1)$, we have $s', 1-s' \notin \mathbb{Z}[\sqrt{3}\, i]$. In any case, we have a contradiction.

(Case 3) $EA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} = B_4E_3$ implies $c = 0$, a contradiction.

(Case 4) $EA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix} = B_4E_6$ implies $a = 0$, a contradiction.

(Case 5) $EA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} s & t \\ u & 1-s \end{pmatrix} = B_4E_7$ (where $a \neq 0, b \neq 0, c \neq 0, a^2 = -bc$ and $s \notin \{0,1\}$, $s(1-s) = tu$) gives the following equations: $as + bu = 2$ and $cs - au = 0$. Multiplying $cs - au = 0$ by a, we obtain

$$0 = cas - a^2u = cas + bcu = c(as + bu)$$

from the relation $a^2 = -bc$, from which we see $as + bu = 0$ since $c \neq 0$, contrary to $as + bu = 2$.

Summarizing, EA cannot be contained in the set $\{BE' \mid B \in N(R) \text{ and } E' \in I(R)'\}$, and therefore R is not right IQNN.

By help of Example 2.2, we can show that the concepts of right IQNN and right quasi-Abelian are independent of each other.

Example 2.3. (1) Let $R = Mat_2(\mathbb{Z})$. Then R is quasi-Abelian by [8, Theorem 3.4], but R cannot be right IQNN by Example 2.2.

(2) Recall that for a domain R and $n \geq 2$, $T_n(R)$ is right quasi-Abelian if and only if R is a division ring if and only if $T_n(R)$ is left quasi-Abelian, by [2, Theorem 2.1] and its proof. Let F be a domain that is not a division ring. Then $T_2(F)$ is right IQNN by [11, Theorem 3.1], but it is not right quasi-Abelian by the preceding argument.

It is evident that if R is a right quasi-Abelian ring with $N^*(R) = N(R)$ then R is right IQNN. Note that $R = Mat_n(A)$ cannot satisfy the condition $N^*(R) = N(R)$, where $n \geq 2$ and A is any ring. The degree of a polynomial f is denoted by $deg(f)$.

Theorem 2.4. *Let K be a field, $K(x)$ be the quotient field of $K[x]$, and H be a domain with $K[x] \subseteq H \subseteq K(x)$. If $Mat_2(H)$ is right IQNN then H contains infinitely many non-polynomial fractions.*

Proof. Let p be any non-constant polynomial that is prime in $K[x]$, and $q \in K[x]$ be any polynomial satisfying $deg(q) > deg(p)$. Let $R = Mat_2(H)$ and consider $E = \begin{pmatrix} 1 & 0 \\ q & 0 \end{pmatrix} \in I(R)'$ and $A = \begin{pmatrix} p & p^2 \\ -1 & -p \end{pmatrix} \in N(R)$. Then $E \in I(R)'$ and $A \in N(R)$ such that $EA = \begin{pmatrix} p & p^2 \\ pq & p^2q \end{pmatrix}$.

Suppose that R is right IQNN. Then there exist $B \in N(R)$ and $E' \in I(R)'$ such that $EA = BE'$. By Lemma 2.1(2), BE' must be one of the following cases:

$$B_4E_3, B_4E_6, \text{ or } B_4E_7.$$

(Case 1) $EA = B_4E_3 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & ta \\ -\frac{a}{b}a & -\frac{ta}{b}a \end{pmatrix}$ by Lemma 2.1(2), where $a^2 = -bc \neq 0$ and $t \neq 0$. Then $a = p$ and $pq = -\frac{a}{b}a = -\frac{p^2}{b}$, from which H contains $b = -\frac{p}{q} \in K(x) \backslash K[x]$ since $deg(q) > deg(p)$.

(Case 2) $EA = B_4E_6 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} -\frac{ua}{c}a & -\frac{a}{c}a \\ -ua & -a \end{pmatrix}$ by Lemma 2.1(2), where $a^2 = -bc \neq 0$ and $u \neq 0$. Then $a = -p^2q$ and $pq = -ua = up^2q$, entailing that H contains $u = \frac{1}{p} \in K(x) \backslash K[x]$.

(Case 3) $EA = B_4E_7 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} s & t \\ u & 1-s \end{pmatrix} = \begin{pmatrix} as+bu & at+b(1-s) \\ cs-au & ct-a(1-s) \end{pmatrix} = \begin{pmatrix} \alpha & \frac{t}{s}\alpha \\ \frac{c}{a}\alpha & \frac{ct}{as}\alpha \end{pmatrix}$ by Lemma 2.1(2), where $\alpha = as + bu$, $a^2 = -bc \neq 0$ and $s(1-s) = tu \neq 0$. Then $\alpha = p$, $\frac{c}{a}\alpha = pq$ and $\frac{t}{s}\alpha = p^2$, from which we see $c = qa$ and $t = ps$. From $a^2 = -bc$, we obtain $a^2 = -qab$ and $a = -qb$.

Now, by the results above, we have

$$p^2 = at + b(1 - s) = (-qb)(ps) + b(1 - s) = b(1 - (1 + qp)s),$$

and so $b = \frac{p^2}{1-(1+pq)s}$. Since $s \neq 0, 1$ and $|1 - (1 + pq)s| = |(1 + p^2 + pm)s - 1| \geq p^2 + (pm + 1)|s| - 1 \geq p^2 + pm > p^2$, H contains $b = \frac{p^2}{1-(1+pq)s} \in K(x)\backslash K[x]$ since $deg(1 - (1 + pq)s) > deg(p^2)$.

Summarizing, H contains a non-polynomial fraction in any case and, consequently, H contains infinitely many non-polynomial fractions since p and q are taken arbitrarily. □

The Abelian property goes up to polynomials by [10, Lemma 8], but the right quasi-Abelian property does not by [2, Proposition 2.5(1)]. Theorem 2.4 is applicable to show that the right IQNN property is not preserved by polynomial rings. In fact, $Mat_2(\mathbb{Q})$ is IQNN by [11, Theorem 2.4] but $Mat_2(\mathbb{Q})[x]$ is not right IQNN, since $Mat_2(\mathbb{Q})[x] \cong Mat_2(\mathbb{Q}[x])$ and, by Theorem 2.4, $Mat_2(\mathbb{Q}[x])$ is not right IQNN.

3. Right IAN rings

As noted earlier, if R is a ring with $I(R)'$ nonempty then R is right quasi-Abelian if and only if for any $e \in I(R)'$ and $a \in R$, there exists $f \in I(R)'$ such that $ea = eaf$. Motivated by the arguments of the preceding section, we next consider a class of rings which generalizes both the quasi-Abelian property and the IQNN property, and will show that this new class is quite large.

Definition 3.1. A ring R is said to be *right* (resp., *left*) *idempotent-attaching on nilpotents* (simply, *right* (resp., *left*) IAN) provided that $I(R)'$ is empty, or else for every pair $(e, a) \in I(R)' \times N(R)$ (resp., $(a, e) \in N(R) \times I(R)'$) there exists $f \in I(R)'$ such that $ea = eaf$ (resp., $ae = fae$). A ring is *IAN* if it is both right and left IAN.

The following shows that the IAN property is not right-left symmetric.

Example 3.2. We recall the rings in [11, Example 2.6]. Let $K = \mathbb{Z}_2$ be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K.

Consider the ideal I of A generated by $ba, a^2 - a, b^2$ and set $R_1 = A/I$. Identify the elements in A with their images in R_1 for simplicity. By the argument of [11, Example 2.6(1)], $N(R_1) = \{\alpha ab + \beta b \mid \alpha, \beta \in K\}$ and $I(R_1)' = \{\gamma + a + \delta ab \mid \gamma, \delta \in K\}$, from which we see that

$$N(R_1) \supsetneq I(R_1)'N(R_1) = \{0, ab, ab + b\} = I(R_1)'N(R_1)I(R_1)'$$

and

$$N(R_1)I(R_1)' = \{0, ab, b, ab + b\} = N(R_1) \supsetneq I(R_1)'N(R_1)I(R_1)'.$$

Thus R is right IAN but not left IAN.

We see below a condition under which the IAN property is right-left symmetric. Let R be a ring. Recall that an involution on R is a function $* : R \to R$ which satisfies the properties that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $1^* = 1$, and $(x^*)^* = x$ for all $x, y \in R$. It is well-known that $0^* = 0$, $a^* \in N(R)$ for $a \in N(R)$, and $e^* \in I(R)'$ for $e \in I(R)'$. We use these facts without referring.

Proposition 3.3. *Let R be a ring with an involution $*$. Then R is right IAN if and only if R is left IAN.*

Proof. Let R be right IAN and $a \in N(R)$, $e \in I(R)'$. Then $a^* \in N(R)$ and $e^* \in I(R)'$. So there exists $f \in I(R)'$ such that $e^*a^* = e^*a^*f$. Thus we have

$$ae = ((ae)^*)^* = (e^*a^*)^* = (e^*a^*f)^* = f^*ae.$$

But $f^* \in I(R)'$, so R is left IAN. The converse can be proved analogously. □

Following [7], a ring R is said to be *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. A a ring R is called *unit-regular* [5] if for each $a \in R$ there exists a unit $u \in R$ such that $a = aua$. Every unit-regular ring is clearly regular.

Proposition 3.4. (1) *Every von Neumann regular ring is IAN.*

(2) *Every semisimple Artinian ring is IAN.*

Proof. (1) Let R be a von Neumann regular ring. Suppose $I(R)' \neq \emptyset$, and $e \in I(R)'$, $a \in N(R)$. If $ea = 0$ then $ea = 0 = eae$. So assume $ea \neq 0$. Since R is von Neumann regular, $ea = eabea$ for some $b \in R$. Then $bea \in I(R)$, and $ea \neq 0$ implies $bea \neq 0$. Moreover $bea \in I(R)'$ because $a \in N(R)$. So R is right IAN. The proof of left IAN is done symmetrically.

(2) It directly follows from (1). □

The following, which shows that the class of right IAN rings is not closed under subrings, is an example of practical application of the preceding proposition.

Remark 3.5. (1) The Abelian property passes the extension $D_n(A)$ $(n \geq 2)$ of a ring A, i.e., A is Abelian if and only if so is $D_n(A)$, by [9, Lemma 2]. But this is not valid for the right IAN property as follows. Let $R = D_2(R_0)$ where $R_0 = T_2(\mathbb{Q})$. Then R_0 is both quasi-Abelian and IQNN (hence IAN) by [2, Theorem 2.1] and [11, Theorem 3.1], respectively. Consider $E = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in I(R)'$ and $A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(R)$. Then $EA = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$. Assume that there exists $F = \begin{pmatrix} f & c \\ 0 & f \end{pmatrix} \in I(R)'$ such that $EA = EAF$, where $f = (f_{ij}), c = (c_{ij}) \in R$. Since $F \in I(R)'$, either $f = \begin{pmatrix} 1 & f_{12} \\ 0 & 0 \end{pmatrix}$ or $f = \begin{pmatrix} 0 & f_{12} \\ 0 & 1 \end{pmatrix}$. But $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ implies $f = \begin{pmatrix} 0 & f_{12} \\ 0 & 1 \end{pmatrix}$, hence we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} c + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f = \begin{pmatrix} 0 & c_{22} + f_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which is impossible. Thus R is not right IAN.

(2) $Mat_4(\mathbb{Q})$ is von Neumann regular by [7, Lemma 1.6], hence $Mat_4(\mathbb{Q})$ is IAN by Proposition 3.4(1). The ring R in (1) is a subring of $Mat_4(\mathbb{Q})$. So the class of right IAN rings is not closed under subrings.

Recall that a ring R is called *directly finite* (or *Dedekind finite*) if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Right quasi-Abelian rings are directly finite by [2, Theorem 1.9(1)], hence so are Abelian rings. Thus one may naturally ask whether right IAN rings are directly finite. However the answer is negative by part (1) of the next example, providing another example of an IAN ring which is not that one-sided quasi-Abelian. Furthermore, part (2) shows that the converse of Proposition 3.4(1) need not hold.

Example 3.6. (1) Consider the column finite infinite matrix ring over a field, R say. Then R is von Neumann regular but not directly finite. So R is IAN by Proposition 3.4(1), but neither right nor left quasi-Abelian because one-sided quasi-Abelian rings are directly finite by [2, Theorem 1.9(1)]. This also illuminates that von Neumann regular rings need not right quasi-Abelian, even if unit-regular rings are quasi-Abelian by [2, Theorem 1.9(3)].

(2) There exists an IAN ring that is not von Neumann regular. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let K be the ideal of A generated by $ab, ba, a^2 - a, b^2$; and set $R = A/K$. Identify the elements in A with their images in R for simplicity. Then $a^2 = a$ and $ab = ba = b^2 = 0$ in R.

Every element in R is expressed by $k_0+k_1a+k_2b$ with $k_1,k_2,k_3\in K$. So, we can obtain

$$N(R)=\{kb\mid k\in K\}\text{ and }I(R)'=\{a\}.$$

This yields $I(R)'N(R)=\{0\}=N(R)I(R)'$, so R is IAN.

Now we claim that R is not von Neumann regular. Consider $b\in R$ and assume on the contrary that $b=bcb$ for some $c=k_0+k_1a+k_2b\in R$ with $k_i\in K$. But $bcb=0$ because $ab=ba=b^2=0$, contrary to $b\neq 0$. Thus R is not von Neumann regular.

We will provide methods to construct right IAN rings from given right IAN rings. We use $\prod_{\gamma\in\Gamma}R_\gamma$ (resp., $\bigoplus_{\gamma\in\Gamma}R_\gamma$) to denote the direct product (resp., direct sum) of rings R_γ.

Proposition 3.7. (1) *Let $\{R_i\mid i=1,\dots,n\}$ be a finite family of rings, and $R=\prod_{i=1}^n R_i$. Suppose that R_i is right IAN for all i. Then R is right IAN.*

(2) *Let $\{R_\gamma\mid\gamma\in\Gamma\}$ be an infinite family of rings, and R be the subring of $\prod_{\gamma\in\Gamma}R_\gamma$ generated by $\bigoplus_{\gamma\in\Gamma}R_\gamma$ and $1_{\prod_{\gamma\in\Gamma}R_\gamma}$. Suppose that R_γ is right IAN for all $\gamma\in\Gamma$. Then R is right IAN.*

Proof. (1) Note that $N(R)=\prod_{i=1}^n N(R_i)$ and let $\Gamma=\{1,\dots,n\}$. Let R_i be right IAN for all i. Consider $e=(e_i)\in I(R)'$ and $r=(r_i)\in N(R)$. Then $r_i\in N(R_i)$ for all i. If e is central then $er=ere$. So assume that e is not central. Then there exists $\alpha\in\Gamma$ such that $e_\alpha\in I(R_\alpha)'$. Since R_α is right IAN, there exists $g_\alpha\in I(R_\alpha)'$ such that $e_\alpha r_\alpha=e_\alpha r_\alpha g_\alpha$. Let $f=(f_i)\in I(R)$ be such that $f_\alpha=g_\alpha$ and $f_\beta=1_\beta$ for all $\beta\in\Gamma\backslash\{\alpha\}$. Then $er=erf$ because $e_\alpha r_\alpha=e_\alpha r_\alpha f_\alpha$ and $e_\beta r_\beta=e_\beta r_\beta 1_\beta=e_\beta r_\beta f_\beta$. Clearly $f\in I(R)'$. Therefore R is right IAN. The proof of (2) can be done by adapting the proof of (1) slightly. □

By applying the proof of Proposition 3.7(1), we can also prove that for $\{R_\gamma\mid\gamma\in\Gamma\}$, an infinite family of rings, and the subring R of $\prod_{\gamma\in\Gamma}R_\gamma$ generated by $\bigoplus_{\gamma\in\Gamma}R_\gamma$ and $1_{\prod_{\gamma\in\Gamma}R_\gamma}$, if R_γ is right quasi-Abelian for all $\gamma\in\Gamma$ then R is right quasi-Abelian.

Both right quasi-Abelian rings and right IQNN rings are right IAN clearly, but Example 2.3 shows that each of the preceding implications is proper. The following example shows that there exists a right IAN ring that is neither right IQNN nor right quasi-Abelian.

Example 3.8. Let $R_1=Mat_2(\mathbb{Z})$ and R_2 be the column finite infinite matrix ring over a field. Then R_1 is quasi-Abelian by [8, Theorem 3.4], R_2 is IAN by Propositions 3.4(1), and R_2 is neither right nor left quasi-Abelian by Example 3.6(1). Set next $R=R_1\times R_2$. Then R is right IAN by Proposition 3.7(1). But R is not right quasi-Abelian by [2, Theorem 2.2(2)].

We claim that R is not right IQNN, either. Let $e=(e_1,ba+(1-ba)a),c=(c_1,b(1-ba))\in R$ where $e_1=\begin{pmatrix}1&0\\q&0\end{pmatrix},c_1=\begin{pmatrix}p&p^2\\-1&-p\end{pmatrix}$ (as in the proof of Theorem 2.4) and $a=E_{12}+E_{23}+\cdots+E_{i,i+1}+\cdots$, $b=E_{21}+E_{32}+\cdots+E_{i+1,i}+\cdots$. Then $e\in I(R)'$ and $c\in N(R)$ with $ec=(\begin{pmatrix}p&p^2\\pq&p^2q\end{pmatrix},(1-ba)+b(1-ba))$. Note that $\begin{pmatrix}p&p^2\\pq&p^2q\end{pmatrix}\notin N(R_1)$ and $(1-ba)+b(1-ba)\in I(R_2)'$.

Assume on the contrary that there exist $d=(d_1,d_2)\in N(R)$ and $f=(f_1,f_2)\in I(R)'$ such that $ec=df$. Since $d_1f_1=e_1c_1\notin N(R_1)$ and $d_2f_2=(1-ba)+b(1-ba)\notin N(R_2)$, $(d_1,d_2)\in N(R)$ implies that $f_1\neq 1_{R_1}$ and $f_2\neq 1_{R_2}$. This forces $f_1\in I(R_1)'$ and $f_2\in I(R_2)'$. But there cannot exist $d_1\in N(R_1)$ and $f_1\in I(R_1)'$ such that $e_1c_1=d_1f_1$, by the proof of Theorem 2.4. This contradicts the assumption, and therefore R is not right IQNN.

The arguments above give us the following diagram.

- Abelian ring $\nearrow$ right quasi-Abelian ring $\searrow$ right IAN ring; Abelian ring $\searrow$ right IQNN ring $\nearrow$ right IAN ring

We see some conditions under which the concepts above are equivalent.

Proposition 3.9. *Let R be a ring with $I(R)'$ nonempty.*

(1) *Let R be a ring with $I(R)'N(R) \subseteq N(R)$ (resp., $N(R)I(R)' \subseteq N(R)$). Then R is right (resp., left) IAN if and only if R is right (resp., left) IQNN.*

(2) *Let R be a ring with $N^*(R) = N(R)$. Then we have the following.*

(i) *R is right (resp., left) IQNN if and only if R is right (resp., left) IAN.*

(ii) *If R is right quasi-Abelian, then R is right IQNN.*

Proof. (1) Let $(e, a) \in I(R)' \times N(R)$. If R is right IAN, then $ea = eaf$ for some $f \in I(R)'$. But $ea \in N(R)$ by hypothesis, and so R is right IQNN. The converse is obvious. The proof for the left case is similar.

(2) (i) follows from (1), and (ii) follows from (i). □

The following characterizes the right IAN property of the upper triangular matrix rings under a specific condition.

Proposition 3.10. *Let R be a ring and $n \geq 2$.*

(1) *If R is reduced then $T_n(R)$ is IAN.*

(2) *Let $I(R) = \{0, 1\}$. If $T_n(R)$ is right IAN then R is reduced.*

Proof. Write $T = T_n(R)$. (1) This comes from [11, Theorem 3.1].

(2) The proof is almost the same as one of [11, Theorem 3.4], but we write it for completeness. Let T be right IAN. Assume on the contrary that $a^2 = 0$ for some $0 \neq a \in R$. Consider two matrices $E = \sum_{i=1}^{n-1} E_{ii}$ and $A = a(\sum_{i=1}^{n-1} E_{ii}) + E_{(n-1)n}$ in T. Then $E \in I(T)'$ and $A \in N(T)$ such that $EA = A$. Since T is right IAN, there exists $F = (f_{ij}) \in I(T)'$ such that $EA = EAF = AF$. Then $f_{ii} \in I(R)$ such that $af_{ii} = a$ for all $i = 1, 2, \ldots, n-1$. Since $I(R) = \{0, 1\}$ and $a \neq 0$, $f_{ii} = 1$ for all $i = 1, 2, \ldots, n-1$. But $F \in I(T)'$, and $F^2 = F$ implies that $f_{nn} = 0$ and $f_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n-1\}$ with $i \neq j$; that is, $F = E_{11} + E_{22} + \cdots + E_{(n-1)(n-1)} + f_{1n}E_{1n} + f_{2n}E_{2n} + \cdots + f_{(n-1)n}E_{(n-1)n}$. Thus $1 = af_{(n-1)n}$, and so we get $0 \neq a = a^2 f_{(n-1)n} = 0$, a contradiction. Hence R is reduced. □

$|S|$ denotes the cardinality of a given set S.

Corollary 3.11. *Suppose that R is a noncommutative right IAN ring of minimal cardinality. Then R is isomorphic to $T_2(\mathbb{Z}_2)$.*

Proof. If $|R|$ has a cube free factorization then R is commutative by [6, Theorem]. So $|R| \geq 2^3$ because R is noncommutative. If $|R| = 2^3$ then R is isomorphic to $T_2(\mathbb{Z}_2)$ by [6, Proposition]. But $T_2(\mathbb{Z}_2)$ is right IAN by Proposition 3.10, from which we see that R is isomorphic to $T_2(\mathbb{Z}_2)$ because R is a right IAN ring of minimal cardinality. □

Notice that, in Proposition 3.10(2), the condition "$I(R) = \{0, 1\}$" is not superfluous by help of [11, Example 3.6], and that the condition "R is reduced" in Proposition 3.10(1) cannot be weakened to the condition "R is Abelian" as follows.

Example 3.12. Consider the non-reduced commutative ring $R = \mathbb{Z}_4$ with $I(R) = \{0, 1\}$. Let $T = T_2(R)$ and use the argument of [11, Example 3.2(2)]. Then

$$I(T)' = \left\{ \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}_4 \right\} \text{ and } N(T) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, z \in \{0, 2\}, y \in \mathbb{Z}_4 \right\}.$$

For $E = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \in I(T)'$ and $A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in N(T)$, assume that there exists $F \in I(T)'$ such that $EA = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = EAF$. Then $F = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$, and it implies $2b = 1$, which is impossible. Thus T is not right IAN.

The following shows an application of Theorem 3.10.

Remark 3.13. A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n and $b \in R$ such that $a^n = a^n b a^n$. Every von Neumann regular ring is clearly π-regular. The ring below shows that the condition "von Neumann regular" in Proposition 3.4(1) cannot be weakened by the condition "π-regular".

Let $R_0 = D_2(F)$ where F is a field. Then $I(R_0) = \{0, 1\}$ clearly. Thus $R = T_2(R_0)$ is not right IAN by Theorem 3.10(2), since R_0 is not reduced. Now we claim that R is π-regular. Let $A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in R$ with $A_1 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $A_2 = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$. We handle each computation case by case.

(Case 1) If $a \neq 0$ and $c \neq 0$ then $A \in U(R)$.

(Case 2) If $a \neq 0$ and $c = 0$ then $A_1 \in U(R_0)$ and $A_2^2 = 0$, and so $A^2 = A^2 \begin{pmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{pmatrix} A^2$.

(Case 3) If $a = 0$ and $c \neq 0$ then $A_1^2 = 0$ and $A_2 \in U(R_0)$, and so $A^2 = A^2 \begin{pmatrix} 0 & 0 \\ 0 & A_2^{-2} \end{pmatrix} A^2$.

(Case 4) If $a = 0$ and $c = 0$ then $A^4 = 0$.

Therefore R is π-regular.

Recall that an ideal I of a ring R is said to be *idempotent-lifting* if idempotents in R/I can be lifted to R. Nil ideals are idempotent-lifting by [12, Proposition 3.6.1].

Proposition 3.14. *For a ring R with $I(R)'$ nonempty, we have the following results.*

(1) *If R is right IAN and I is a nil ideal of R, then so is R/I.*

(2) *If $D_n(R)$ for $n \geq 2$ is a right IAN ring, then so is R.*

Proof. (1) Suppose that R is right IAN and I is a nil ideal of R. Let $\bar{r} = r + I$ with $r \in R$ and $\bar{R} = R/I$. Since I is nil, I is idempotent-lifting by [12, Proposition 3.6.1] and $N(\bar{R}) = \{\bar{a} \mid a \in N(R)\}$. Note $I \cap I(R)' = \emptyset$.

Let $\bar{x} \in I(\bar{R})'$ and $\bar{a} \in N(\bar{R})$. Then $a \in N(R)$, and there exists $e \in I(R)$ such that $\bar{e} = \bar{x}$. Here $\bar{x} \in I(\bar{R})'$ implies $e \in I(R)'$. Since R is right IAN, $ea = eaf$ for some $f \in I(R)'$. Then $f \in I(R)'$ implies $\bar{f} \neq \bar{0}$ because $I \cap I(R)' = \emptyset$. Furthermore, $\bar{f} \neq \bar{1}$ (otherwise, $0 \neq 1 - f \in I$, contrary to $I \cap I(R)' = \emptyset$). Thus $\bar{f} \in I(\bar{R})'$ such that $\bar{x}\bar{a} = \bar{e}\bar{a} = \overline{ea} = \overline{eaf} = \bar{e}\bar{a}\bar{f}$, showing that $\bar{R}$ is right IAN.

(2) It comes from (1) and the fact that $D_n(R)/I \cong R$ for the nilpotent ideal

$$I = \{(a_{ij}) \in D_n(R) \mid a_{ii} = 0 \text{ for all } i\}$$

of $D_n(R)$. □

Example 3.15. (1) The condition "I is nil" in Proposition 3.14(1) cannot be dropped as we see in the following. Let $R = T_2(\mathbb{Z})$. Then R is right IAN by Theorem 3.10(1). Consider the ideal $I = T_2(4\mathbb{Z})$ of R. Then $R/I \cong T_2(\mathbb{Z}_4)$, and $I(\mathbb{Z}_4) = \{0, 1\}$. But $T_2(\mathbb{Z}_4)$ is not right IAN by Theorem 3.10(2) (because $\mathbb{Z}_4$ is not reduced) or Example 3.12. Note that I is not nil. This example also shows that the class of right IAN rings is not closed under homomorphic images.

(2) The converse of Proposition 3.14(1) need not hold. Consider the nil ideal $I = \mathbb{Z}_4 E_{12}$ of $R = T_2(\mathbb{Z}_4)$. Then $R/I \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$ that is Abelian, but R is not right IAN as above.

(3) The converse of Proposition 3.14(2) need not hold by Remark 3.5(1).

Let A be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [4], the *Dorroh extension* of A by S is the Abelian group $A \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. We use $A \times_{dor} S$ to denote the Dorroh extension of A by S.

Proposition 3.16. *Let S be a commutative ring and A be an algebra with identity over S. Write $D = A \times_{dor} S$.*

(1) *Let S be a reduced ring. Then $N(D) = (N(A), 0)$ and D is IAN.*

(2) *Let $ch(A) = 2$. Then we have the following.*

(i) *$I(D) = I(A) \times I(S)$ and $I(A) = \{e + s \mid (e, s) \in I(D)\}$.*

(ii) *$I(D)' = \{(e_1, 0), (e_2, 1) \mid e_1, e_2 \in I(R)\backslash\{0\}\}$ when $I(S)' = \emptyset$.*

(iii) *$I(D)' = \{(e_1, 0), (e_2, 1), (e_3, s) \mid e_1, e_2 \in I(R)\backslash\{0\}, e_3 \in I(R), s \in I(S)'\}$ when $I(S)' \neq \emptyset$.*

(3) *Let $ch(A) = 2$ and suppose that whenever $e \in I(A)'$ and $a \in N(A)$, $ea = ae'$ for some $e' \in I(A)'$. Then D is right IAN.*

Proof. Note that $s \in S$ is identified with $s \cdot 1_A \in A$, and so $A = \{a + s \mid (a, s) \in D\}$.

(1) Evidently $N(D) = (N(A), 0)$. Let $E = (e, s) \in I(D)'$ and $B = (b, 0) \in N(D)$. Then $EB = (eb+sb, 0) = (eb+sb, 0)(1, 0) = EB(1, 0)$ and $BE = (be+sb, 0) = (1, 0)(be+sb, 0) = (1, 0)BE$, noting $(1, 0) \in I(D)'$. Thus D is IAN.

(2) If $(a, s) \in I(D)$ then $(a, s)^2 = (r, s)$ implies $a^2 = a, s^2 = s$ because $ch(A) = 2$, and hence $a \in I(A)$ and $s \in I(S)$, showing that $I(D) \subseteq I(A) \times I(S)$. Conversely, if $(e, t) \in I(A) \times I(S)$ then $(e, t)^2 = (e, t)$ and so $I(A) \times I(S) \subseteq I(D)$. Thus $I(D) = I(A) \times I(S)$.

Next, if $e \in I(A)$ then $(e, 0) \in I(D)$, noting $e = e + 0$ with $0 \in S$. Conversely, if $(e, s) \in I(D)$ then $e \in I(A)$ and $s \in I(S)$ as above, from which we see $e + s \in I(A)$. This gives (i), from which we obtain (ii) and (iii) since $1_D = (0, 1)$.

(3) Let $(e, f) \in I(D)'$ and $(a, m) \in N(D)$. Then since $ch(A) = 2$, $e \in I(A)$ by (2). If e is central in A then (e, f) is central in D. So assume that e is not central. Then $e \in I(A)'$. Since $(a, m) \in N(D)$, $a + m \in N(A)$ (in fact, $(a, m)^k = 0$ for some $k \geq 1$ implies that $(a + m)^k = 0$ and $m^k = 0$). By hypothesis, $e(a + m) = (a + m)e'$ for some $e' \in I(A)'$. Note that $(e', f) \in I(D)'$ by (2). Now we have

$$(e, f)(a, m) = (e, f)(a, m) = (e(a + m) + fa, fm) = ((a + m)e' + fa, mf) = (a, m)(e', f)$$

and this yields that

$$(e, f)(a, m) = (e, f)((e, f)(a, m)) = (e, f)((a, m)(e', f)) = ((e, f)(a, m))(e', f).$$

Therefore D is right IAN. □

Let B be a commutative domain with $ch(B) = 2$ and $A = T_2(B)$. Let $S = \mathbb{Z}_2$ and set $D = A \times_{dor} S$. Then D is an example of Proposition 3.16(3). In fact, $I(A)' = \{e = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \mid a, b \in B\}$ and $N(A) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$; and for every $c \in N(A)$, $ec = cE_{22}$ and $fc = 0 = cE_{11}$.

Next we show that the right IAN property does not go up to polynomial rings.

Example 3.17. Let $R_0 = Mat_2(\mathbb{Z})$. Then R_0 is quasi-Abelian (hence IAN) by [8, Theorem 3.4]. Next set $R = R_0[x]$ and note $R \cong Mat_2(\mathbb{Z}[x])$. We will show that $Mat_2(\mathbb{Z}[x])$ is not right IAN.

Consider $E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 2x & -2^2 \\ x^2 & -2x \end{pmatrix}$ in $Mat_2(\mathbb{Z}[x])$. Then $E \in I(Mat_2(\mathbb{Z}[x])'$ and $A \in N(Mat_2(\mathbb{Z}[x]))$ with $EA = \begin{pmatrix} 2x & -2^2 \\ 2x & -2^2 \end{pmatrix}$. Assume on the contrary that $EA =$

EAF for some $F \in I(Mat_2(\mathbb{Z}[x]))'$. Note that EAF is one of EAE_3, EAE_6, EAE_7 by Lemma2.1(2), where we follow the notation of Lemma 2.1(1).

(Case 1) From $EA = EAE_3 = \begin{pmatrix} 2x & -2^2 \\ 2x & -2^2 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2x & 2tx \\ 2x & 2tx \end{pmatrix}$, we obtain $-2^2 = 2tx$ and $t = -\frac{e}{x} \in \mathbb{Z}[x]$, a contradiction.

(Case 2) From $EA = EAE_6 = \begin{pmatrix} 2x & -2^2 \\ 2x & -2^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} -2^2u & -2^2 \\ -2^2u & -2^2 \end{pmatrix}$, we obtain $2x = -2^2u$ and $u = -\frac{x}{2} \in \mathbb{Z}[x]$, a contradiction.

(Case 3) From $EA = EAE_7 = \begin{pmatrix} 2x & -2^2 \\ 2x & -2^2 \end{pmatrix} \begin{pmatrix} s & t \\ u & 1-s \end{pmatrix} = \begin{pmatrix} 2xs - 2^2u & 2xt - 2^2(1-s) \\ 2xs - 2^2u & 2xt - 2^2(1-s) \end{pmatrix}$, we obtain that $2xs - 2^2u = 2x$ and $2xt - 2^2(1-s) = -2^2$, entailing $2x(1-s) = -2^2u$ and $xt = -2s$, and hence $-2^2u = 2x + x^2t$. Letting $u = a_0 + a_1x + \cdots + a_nx^n$ with $a_i \in \mathbb{Z}$, we get $-2^2(a_0 + a_1x + \cdots + a_nx^n) = 2x + x^2t$ and this yields $a_0 = 0$, $-2^2a_1 = 2$. Hence $a_1 = -\frac{1}{2}$, a contradiction.

Summarizing, such F cannot exist in $I(Mat_2(\mathbb{Z}[x]))'$ in any case and, consequently, $Mat_2(\mathbb{Z}[x])$ is not right IAN. Thus R is also not right IAN, as desired.

The following provides simple information for polynomial rings to be right IAN.

Remark 3.18. (1) Let R be a ring and suppose that $R[x]$ is right IAN. We claim that R is also right IAN. Let $e \in I(R)'$ and $a \in N(R)$ be such that $ea \neq 0$. Since $R[x]$ is right IAN, there exists $f(x) = \sum_{i=0}^{m} f_ix^i \in I(R[x])'$ such that $ea = eaf(x)$. Then $0 \neq f_0 \in I(R)$ and $ea = eaf_0$. Here if $f_0 = 1$ then $f(x) = 1$ by a simple computation (see the proof of [10, Lemma 8] for details). Thus $f_0 \in I(R)'$ and R is right IAN.

(2) Let R be a reduced ring with $I(R) = \{0, 1\}$. We claim that $T_n(R)[x]$ is an IAN ring for every $n \geq 2$. Since R is reduced and $I(R) = \{0, 1\}$, we have that $I(R[x]) = \{0, 1\}$ by [10, Lemma 8]. Then $T_n(R[x])$ is IAN by Theorem 3.10(1) since $R[x]$ is reduced. Moreover since $T_n(R)[x] \cong T_n(R[x])$, we have that $T_n(R)[x]$ is IAN.

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