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Research Article

New results on lacunary ideal convergence in fuzzy cone normed spaces

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ABSTRACT

In this paper, some existing theories on convergence of sequences in fuzzy cone normed space (FCNS in short) are extended to lacunary ideal convergence in FCNS. An original concept, named lacunary convergence of sequence in FCNS, is investigated. Also, lacunary I-limit points and lacunary I-cluster points of sequences in FCNS are examined. Furthermore, lacunary Cauchy and lacunary I-Cauchy sequences in FCNS are presented and relationships between them are studied.

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INTRODUCTION

The origins of fuzzy set theory (FS) can be traced back to the initial edition of Zadeh's monograph [1]. However, the theory of FS doesn't always suffice to address the uncertainty surrounding membership degrees. To address this limitation, Atanassov [2] introduced the concept of intuitionistic fuzzy metric spaces (IFS), which serves as an extension of the FS theory. Following the establishment of fuzzy sets, numerous researchers have delved into this concept, exploring novel ideas in topology and analysis. Katsaras [3] introduced the notion of a fuzzy norm in fuzzy topological vector spaces, incorporating fuzzy sets with norm processors. Felbin [4] examined the concept of fuzzy normed spaces. Kramosil and Michalek [5] proposed the idea of a fuzzy metric space (FMS), amalgamating the concepts of fuzzy and probabilistic metric spaces. George and Veeramani [6] established certain qualifications within the realm of FMS. Park [7] extended the scope by generalizing FMS and delving into intuitionistic fuzzy metric spaces (IFMS). The concept of intuitionistic fuzzy normed spaces (IFNS) was introduced by Lael and Nourouzi [8]. Further comprehensive studies regarding fuzziness can be found in references [9-12].

Guang and Xian [13] defined cone metric space as a generalization of metric space by replacing the range of metric with an ordered real Banach space and established some fixed point theorems on contractive mappings on such spaces. Bag [14] generalized the concept of Felbin [4] type fuzzy norm and defined a new concept known as FCNS. Meaningful results on this topic can be examined in [15-17]. In the study [18], some fundamental definitions

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on FCNS were presented and some basic results on finite dimensional FCNS were established.

Güler [19] generalized fuzzy norm by taking ordered Banach space instead of positive real numbers in the definition of fuzzy norm which is given by Lael and Nourouzi [8]. In [19], the notions of I-convergence and I^*- convergence in FCNS were studied. Also, I-limit points and I-cluster points of a sequences in FCNS were examined in the same study.

Statistical convergence was originally introduced by Fast [20]. In the context of IFNS, statistical convergence was first introduced by Karakuş et al. [21]. Several studies on this subject can be found in references [22-29]. Fridy and Orhan [30] explored lacunary statistical convergence through the utilization of lacunary sequences. Kostyrko et al. [31] introduced ideal convergence as a broader type of convergence encompassing statistical convergence, using defined ideals on natural numbers.

In the work [32], existing theories concerning the statistical limit and statistical cluster points of sequences were extended to cover I-limit points and I-cluster points, providing valuable insights and properties.

Nabiev et al. [33] introduced the concept of I-Cauchy and I^*-Cauchy sequences. The exploration of ideal convergence in fuzzy normed spaces was initially undertaken by Kumar and Kumar [34]. Subsequently, I- convergence has been explored within more general abstract spaces, including fuzzy number spaces [35], 2-normed linear spaces [36], and intuitionistic fuzzy normed spaces [37]. Additionally, Yamancı and Gürdal [38] examined lacunary I-convergence and lacunary I-Cauchy sequences in the topology of random n-normed spaces. Debnath [39] focused on lacunary I-convergence in IFNS. Tripathy et al. [40] delved into the concept of I-lacunary convergent sequences. For further background on sequence spaces, classical sets of fuzzy valued sequences, and related topics, readers are advised to consult the monographs [41] and [42], as well as recent papers [43-46].

This paper consists of two sections with the new results in section 2. In the Section 2, the concepts of lacunary ideal convergence, lacunary I-limit points and lacunary I-cluster points of sequences in FCNS are examined. Furthermore, lacunary Cauchy and lacunary I-Cauchy sequences in FCNS are defined and their fundamental properties are studied.

Firstly, we recall some definitions used throughout the paper.

Let $A \subseteq \mathbb{N}$ and $r \in \mathbb{N}$, $\delta_\theta^r(A)$ is named the rth partial lacunary density of A, if

$$\delta_\theta^r(A) = \frac{|A \cap I_r|}{h_r},$$

where $I_r = (k_{r-1}, k_r]$.

The number $\delta_\theta(A)$ is denoted the lacunary density (θ-density) of A if

$$\delta_\theta(A) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: k \in A\}|, \left(i.e., \delta_\theta(A) = \lim_{r\to\infty} \delta_\theta^r(A)\right)$$

exists. Also, $\Lambda = \{A \subseteq \mathbb{N}: \delta_\theta(A) = 0\}$ is called to be zero density set.

Let $\emptyset \neq S$ be a set, and then a non-empty class $I \subseteq P(S)$ is said to be an ideal on S iff (i) $\emptyset \in I$, (ii) I is additive under union, (iii) for each $A \in I$ and each $B \subseteq A$ we find $B \in I$, An ideal I is called non-trivial if $I \neq \emptyset$ and $S \notin I$. A non-empty family of sets F is called filter on S iff (i) $\emptyset \notin F$, (ii) for each $A, B \in F$ we get $A \cap B \in F$, (iii) for every $A \in F$ and each $B \supseteq A$, we obtain $B \in F$. Relationship between ideal and filter is given as follows:

$$F(I) = \{K \subset S: K^c \in I\},$$

where $K^c = S - K$.

A non-trivial ideal I is (i) an admissible ideal on S iff it contains all singletons.

Throughout the paper, we denote a real Banach space by E and the zero element of E by θ_E.

Let E be a real Banach space and P be a subset of E. Then, P is called a cone if

a) $P \neq \{\theta_E\}$, P is non-empty and closed;
b) $u, v \in R, u, v > 0, t, w \in P \Rightarrow ut + vw \in P$;
c) $t \in P, -t \in P \Rightarrow t = \theta_E$.

For a cone P subset of E, a partial ordering $\preccurlyeq$ in terms of P is determined by $q \preccurlyeq r$ iff $r - q \in P$, $q \prec r$ will stand for $q \preccurlyeq r$ and $q \neq r$ while $q \ll r$ will stand for $r - q \in int(P)$, where $int(P)$ represents the set of the interior points of P.

The sets of the form $[q, r]$ are called order-intervals and are defined as the following:

$$[q, r] = \{z \in E: q \preccurlyeq z \preccurlyeq r\}.$$

It is seen that order-intervals are convex. If $[q,\ r] \subset A$ whenever $q, r \in A$ and $q \preccurlyeq r$, then $A \subset E$ is called order-convex. If ordered topological vector space $(E,\ P)$ has a neighborhoods' base of θ which consist of order- convex sets then, it is order-convex. At this stage, the cone P is called a normal cone. Considering the normed space, this condition comes to mean that the unit ball is order-convex, it is equivalent to the condition that $\exists K$ such that $q, r \in E$ and $\theta \preccurlyeq q \preccurlyeq r \Rightarrow \|q\| \leq K\|r\|$. The smallest constant K is called the normal constant of P. If each of the increasing sequence that is bounded above in P is convergent then, we call P as a regular cone. In other words, if there exists a sequence $\{q_n\}$ such that

$$q_1 \preccurlyeq q_2 \preccurlyeq \cdots \preccurlyeq q_n \preccurlyeq \cdots \preccurlyeq r,$$

for some $r \in E$ then $\exists q \in E$ such that $\lim_{n\to\infty} \|q_n - q\| = 0$. Equivalently the cone P is regular if every decreasing sequence which is bounded from below is convergent. It is well known that if P is regular cone then it is normal cone. Throughout the study, we assume that all cones has nonempty interior.

Triangular norms (t-norms) (TNs) were introduced by Menger [47]. TNs serve as a means to extend the concept of the probability distribution while adhering to the triangle inequality present in metric space terminology. Triangular conorms (t-conorms) (TCs), on the other hand, are dual operations to TNs. Both TNs and TCs play a crucial role in fuzzy operations such as intersections and unions.

Let $*: [0,1] \times [0,1] \to [0,1]$ be an operation. When $*$ satisfies following situations, it is called continuous TN. Take $p, q, r, s \in [0,1]$,

a) $p * 1 = p$,
b) If $p \leq r$ and $q \leq s$, then $p * q \leq r * s$,
c) $*$ is continuous,
d) $*$ is associative and commutative.

Take V be a linear space over the field K and E be a real Banach space with cone P. Let $*$ be a t-norm. Then, a fuzzy subset $N_C: V \times E \to [0,1]$ is called to be a fuzzy cone norm, if

(FCN1) $\forall m \in E$ with $m \preccurlyeq \theta_E$, $N_C(t,m) = 0$;

(FCN2) $\forall \theta_E \prec m$, $N_C(t,m) = 1 \Leftrightarrow t = \theta_V$, ($\theta_V$ indicates the zero element of V);

(FCN3) $\forall \theta_E \prec m$, $N_C(kt,m) = N_C\left(t, \frac{m}{|k|}\right)$ for all $0 \neq k \in K$;

(FCN4) $\forall t, p \in V$ and $m, n \in E$, $N_C(t,m) * N_C(p,n) \leq N_C(t+p, m+n)$;

(FCN5) $\lim_{\|m\| \to \infty} N_C(t,m) = 1$

Then, $(V, N_C, *)$ is called to be a fuzzy cone normed linear space with regards to E.

Let $(V, N_C, *)$ be a FCNS, $\eta \in V$ and (t_n) be a sequence in V. Then, (t_n) is named to be convergent to η, if for any $m \in E$ with $\theta_E \prec m$ and $\xi \in (0,1)$, $\exists$ a natural number n_0 such that

$$N_C(t_n - \eta, m) > 1 - \xi,$$

$\forall n > n_0$ and $\theta_E \prec m$. We denote this by $N_C - \lim_{n \to \infty} t_n = \eta$.

Let $(V, N_C, *)$ be a FCNS and (t_n) be a sequence in V. Then, (t_n) is named to be a Cauchy sequence, if for any $m \in E$ with $\theta_E \prec m$ and $\xi \in (0,1)$, $\exists$ a natural number n_0 such that

$$N_C(t_{n+p} - t_n, m) > 1 - \xi, \forall n > n_0, p = 1,2, \ldots$$

Lemma 1.1. $N_C(t, .)$ is non-decreasing with regards to E.

Let $(V, N_C, *)$ be a FCNS. For any $m \gg \theta$, $\eta \in V$ and $\xi \in (0,1)$,

$$B_{N_C}(m, \eta, \xi) = \{t \in V: N_C(t - \eta, m) > 1 - \xi\}$$

is called open ball with center η and radius ξ with respect to m.

RESULTS AND DISCUSSION

Currently, we are delving into the examination of certain properties related to the concepts of lacunary ideal convergence, lacunary I-limit points, and lacunary I-cluster points in FCNS. To accomplish our objective, we need to establish a series of definitions. Consistently throughout this article, we will denote FCNS as V.

Definition 2.1. A sequence (t_n) in V is called to be lacunary statistical convergent to $\eta \in V$ with regards to (*w.r.t* in short) fuzzy cone norm on V, if for every $\xi \in (0,1)$ and $m \in E$ with $\theta_E \prec m$ the set

$$A(\xi) = \{n \in \mathbb{N}: N_C(t_n - \eta, m) \leq 1 - \xi\}$$

has lacunary density zero, i.e. $\delta_\theta(A(\xi)) = 0$. We write $S_\theta - lim t_n = \eta(N_C)$.

Definition 2.2. A sequence (t_n) is called to be lacunary convergent to $\eta \in V$ *w.r.t* fuzzy cone norm on V, if for every $\xi \in (0,1)$ and for any $m \in E$ with $\theta_E \prec m$, there exists $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) > 1 - \xi$$

for all $r \geq r_1$. We write $\theta - lim t_n = \eta(N_C)$.

Definition 2.3. Let V be a *FCNS* and I be an ideal on $\mathbb{N}$. A sequence (t_n) in V is called to be lacunary I-convergent to $\eta \in V$ *w.r.t* fuzzy cone norm on V ($I_\theta - FCN$), if for every $\xi \in (0,1)$ and $m \in E$ with $\theta_E \prec m$ the set

$$\left\{r \in \mathbb{N}: \frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) \leq 1 - \xi\right\} \in I.$$

We indicate $I_\theta - lim t_n = \eta(N_C)$.

We start our work with the following result.

Theorem 2.1. Let V be an *FCNS* and (t_n) be a sequence in V. Then, (t_n) lacunary converges to $\eta \in V$, *w.r.t* fuzzy cone norm on V iff

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) = 1, \forall m \in E \text{ with } \theta_E \prec m.$$

Proof. Let (t_n) be a sequence in V lacunary converges to η. Then, for any $m \in E$ with $\theta_E \prec m$ and $\xi \in (0,1)$, $\exists$ a natural number r_1 such that

$$\frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) > 1 - \xi$$

for all $r \geq r_1$. Since ξ is is arbitrary, it follows that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) = 1, \forall m \in E \text{ with } \theta_E \prec m.$$

Conversely, assume that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) = 1, \forall m \in E \text{ with } \theta_E \prec m.$$

Then, for each $\xi \in (0,1)$ and for any $m \in E$ with $\theta_E \prec m$, there exists $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - \eta, m) > 1 - \xi$$

for all $r \geq r_1$. Thus, (t_n) lacunary converges to η.

Theorem 2.2. Limit of a lacunary convergent sequence in a *FCNS* $(V, N_C, *)$ is unique, provided $*$ is continuous at (1,1).

Proof. Let (t_n) be a lacunary convergent sequence in $(V, N_C, *)$ and $*$ is continuous at (1,1). Suppose that

$\theta - limt_n = \eta_1(N_C)$ and $\theta - limt_n = \eta_2(N_C)$, where $\eta_1 \neq \eta_2$. Then

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta_1, m_1) = 1, \forall m_1 \in E \text{ with } \theta_E \prec m_1$$

and

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta_2, m_2) = 1, \forall m_2 \in E \text{ with } \theta_E \prec m_2.$$

Now,

$$\begin{aligned} N_C(\eta_1 - \eta_2, m_1 + m_2) &= N_C(\eta_1 - t_n + t_n - \eta_2, m_1 + m_2) \\ &\geq N_C(\eta_1 - t_n, m_1) * N_C(t_n - \eta_2, m_2) \\ &= N_C(t_n - \eta_1, m_1) * N_C(t_n - \eta_2, m_2). \end{aligned}$$

Therefore, we write

$$N_C(\eta_1 - \eta_2, m_1 + m_2) \geq \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta_1, m_1) * \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta_2, m_2).$$

Taking limit as $r \to \infty$, we get

$$N_C(\eta_1 - \eta_2, m_1 + m_2) \geq \lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta_1, m_1) * \lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta_2, m_2) = 1 * 1 = 1.$$

Thus, we obtain

$$N_C(\eta_1 - \eta_2, m_1 + m_2) = 1, \forall m_1, m_2 \in E \text{ with } \theta_E \prec m_1, \theta_E \prec m_2.$$

So $\eta_1 - \eta_2 = \theta_V$, by *(FCN2) ($\theta_V$ indicates the zero element of V)*. Hence, $\eta_1 = \eta_2$.

Definition 2.4. A sequence (t_n) in V is called to be lacunary Cauchy *w.r.t* fuzzy cone norm on V, if for every $\xi \in (0,1)$ and for any $m \in E$ with $\theta_E \prec m$, there are $r_0, s \in \mathbb{N}$ providing

$$\frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - t_s, m) > 1 - \xi$$

for all $r > r_0$ and equivalently a sequence $t = (t_n)$ is said to be lacunary Cauchy *w.r.t* fuzzy cone norm if for any $m \in E$ with $\theta_E \prec m$ and $\xi \in (0,1)$, $\exists$ a natural number r_0 such that

$$\frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) > 1 - \xi, \forall r > r_0, p = 1,2, \ldots$$

Theorem 2.3. Take an *FCNS* V. Let (t_n) be a sequence in V. Then, (t_n) is a lacunary Cauchy sequence iff

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) = 1, \forall m \in E\ (\theta_E \prec m), p = 1,2, \ldots$$

Proof. Let V be a FCNS and (t_n) be a lacunary Cauchy sequence in V. Then, for any $m \in E$ with $\theta_E \prec m$ and $\xi \in (0,1)$, $\exists$ a natural number r_0 such that

$$\frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) > 1 - \xi, \forall r > r_0, p = 1,2, \ldots$$

Thus, we get

$$1 - \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) < \xi$$

$\forall r > r_0, p = 1,2, \ldots$ Since ξ is is arbitrary, it follows that

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) = 1, \forall m \in E\ (\theta_E \prec m).$$

Conversely, presume that

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) = 1, \forall m \in E\ (\theta_E \prec m), p = 1,2, \ldots$$

Then, for any $m \in E$ with $\theta_E \prec m$ and $\xi \in (0,1)$, $\exists$ a natural number r_0 such that

$$\frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, m) > 1 - \xi, \forall r > r_0, p = 1,2, \ldots$$

Thus, (t_n) is a lacunary Cauchy sequence in V.

Theorem 2.4. In an *FCNS*, with $*$ continuous at (1.1), every lacunary convergent sequence is also a lacunary Cauchy sequence *w.r.t* fuzzy cone norm.

Proof. Let (t_n) be a lacunary convergent sequence in $(V, N_C, *)$ and converges to η. Then

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta, m) = 1, \forall m \in E\ (\theta_E \prec m).$$

For $\theta_E \prec m$, $\theta_E \prec n$ and $p = 1,2, \ldots$, we get

$$\begin{aligned} N_C(t_{n+p} - t_n, n + m) &= N_C(t_{n+p} - \eta + \eta - t_n, n + m) \\ &\geq N_C(t_{n+p} - \eta, n) * N_C(\eta - t_n, m) \\ &= N_C(t_{n+p} - \eta, n) * N_C(t_n - \eta, m). \end{aligned}$$

Therefore, we write

$$\frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, n + m) \geq \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - \eta, n) * \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta, m).$$

Taking limit as $r \to \infty$, we get

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, n + m) \geq \lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - \eta, n) * \lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_n - \eta, m) = 1 * 1 = 1.$$

Thus, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{n\in I_r} N_C(t_{n+p} - t_n, n + m) = 1, \forall m, n \in E \text{ with } \theta_E \prec m, \theta_E \prec n, p = 1,2, \ldots$$

So, (t_n) is a lacunary Cauchy sequence *w.r.t* fuzzy cone norm on V.

Theorem 2.5. If $N_C - limt_n = \eta$, then $S_\theta - limt_n = \eta(N_C)$.

The converse of Theorem 2.5 is not true in general which follows from the following example.

Example 2.1. *Let $E = \mathbb{R}$. Then, $P = [0, \infty) \subset E$ is a* normal cone. Let $V = \mathbb{R}^2$, $u * v = uv$ and $N_C: V \times E \to [0,1]$ contemplated by $N_C(t, m) = \frac{m}{m + \|t\|}$ for all $t \in V$ and $m \in E$ with $\theta \prec m$. We take a sequence (t_k) by

$$t_k = \begin{cases} k, & if\ n - [\![\sqrt{h_r}]\!] + 1 \leq k \leq n, r \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Consider

$$K_r(\xi, m) := \{k \in I_r : N_C(t_k, m) \leq 1 - \xi\}$$

for every $\xi \in (0,1)$ and $m \in E$ with $\theta_E \prec m$. Then, we get

$$K_r(\xi, m) := \left\{k \in I_r : \frac{m}{m + \|t_k\|} \leq 1 - \xi\right\}$$
$$= \left\{k \in I_r : \|t_k\| \geq \frac{m\xi}{1 - \xi} > 0\right\} \subseteq \{k \in I_r : t_k = k\}.$$

Thus

$$\frac{1}{h_r}|\{k \in I_r : k \in K_r(\xi, m)\}| \leq \frac{[\![\sqrt{h_r}]\!]}{h_r} \to 0$$

as $r \to \infty$. Therefore, we obtain $S_\theta - limt_n = 0(N_C)$. But the sequence (t_k) is not convergent *w.r.t* fuzzy cone norm.

Theorem 2.6. Take V as an *FCNS*. Then, $S_\theta - limt_n = \eta(N_C)$ iff there exists an increasing index sequence $K = \{n_i\}$ of natural numbers such that $\delta_\theta(K) = 1$ and $N_C - limt_{ni} = \eta$.

Theorem 2.7. Take V as an FCNS. Then, $S_\theta - limt_n = \eta(N_C)$ iff there exists a sequence $s = (s_n)$ such that $N_C - lims_n = \eta$ and $\delta_\theta(\{n \in \mathbb{N} : t_n = s_n\}) = 1$.

Proof. Let $S_\theta - limt_n = \eta(N_C)$. By Theorem 2.6, we have an increasing index sequence $K = \{n_i\}$ of natural numbers such that $\delta_\theta(K) = 1$ and $N_C - limt_{ni} = \eta$. Think the sequence $s = (s_n)$ given by

$$s_n = \begin{cases} t_n, & if\ n \in K \\ \eta, & \text{otherwise.} \end{cases}$$

Then, s serves our purpose.

Conversely presume that t and s are be sequences such that

$$N_C - lims_n = \eta \text{ and } \delta_\theta(\{n \in \mathbb{N} : t_n = s_n\}) = 1.$$

Then, for every $\xi \in (0,1)$ and $m \in E$ with $\theta_E \prec m$, we obtain

$$\{n \in \mathbb{N} : N_C(t_n - \eta, m) \leq 1 - \xi\} \subseteq \{n \in \mathbb{N} : N_C(s_n - \eta, m) \leq 1 - \xi\} \cup \{n \in \mathbb{N} : t_n \neq s_n\}.$$

Since $N_C - lims_n = \eta$, so the set $\{n \in \mathbb{N} : N_C(s_n - \eta, m) \leq 1 - \xi\}$ involves at most finitely many terms. Also by supposition, $\delta_\theta(\{n \in \mathbb{N} : t_n \neq s_n\}) = 0$. So,

$$\delta_\theta(\{n \in \mathbb{N} : N_C(t_n - \eta, m) \leq 1 - \xi\}) = 0.$$

Hence, we acquire $S_\theta - limt_n = \eta(N_C)$.

Theorem 2.8. Let (t_n) and (p_n) be sequences in *FCNS* $(V, N_C, *)$. Then

a) If $I_\theta - limt_n = \eta(N_C)$ and $I_\theta - limp_n = \lambda(N_C)$, then $I_\theta - lim(t_n + p_n) = \eta + \lambda(N_C)$.

b) If $I_\theta - limt_n = \eta(N_C)$ and k be any real number, then If $I_\theta - limkt_n = k\eta(N_C)$.

Proof. a) Let $\xi \in (0,1)$. By Remark 1.6 [**6**], we can select $\xi_0 \in (0,1)$ such that

$$(1 - \xi_0) * (1 - \xi_0) > 1 - \xi. \quad (1)$$

For $m \in E$ with $\theta_E \prec m$, put

$$A(\xi, m) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in I_r} N_C(t_n + p_n - (\eta + \lambda), m) \leq 1 - \xi\right\},$$

$$K_1(\xi_0, m) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in I_r} N_C(t_n - \eta, m) \leq 1 - \xi_0\right\},$$

$$K_2(\xi_0, m) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in I_r} N_C(p_n - \lambda, m) \leq 1 - \xi_0\right\}.$$

By assumption $K_1(\xi_0, m) \in I$ and $K_2(\xi_0, m) \in I$. Since I is an ideal, $K(\xi, m) = K_1(\xi_0, m) \cup K_2(\xi_0, m) \in I$ and $K^c(\xi, m) \in F(I)$. We have to indicate that $K^c(\xi, m) \subseteq A^c(\xi, m)$. Let $w \in K^c(\xi, m)$. Then, we get

$$\frac{1}{h_r}\sum_{n \in I_r} N_C\left(t_w - \eta, \frac{m}{2}\right) > 1 - \xi_0 \text{ and } \frac{1}{h_r}\sum_{n \in I_r} N_C\left(p_w - \lambda, \frac{m}{2}\right) > 1 - \xi_0.$$

Since N_C is a fuzzy cone norm and by (2.1),

$$\frac{1}{h_r}\sum_{n \in I_r} N_C(t_w + p_w - (\eta + \lambda), m) \geq \frac{1}{h_r}\sum_{n \in I_r} N_C\left(t_w - \eta, \frac{m}{2}\right) * \frac{1}{h_r}\sum_{n \in I_r} N_C\left(p_w - \lambda, \frac{m}{2}\right) > (1 - \xi_0) * (1 - \xi_0) > 1 - \xi.$$

Then, we obtain $w \in A^c(\xi, m)$. Since $K^c(\xi, m) \in F(I)$, we get $A^c(\xi, m) \in F(I)$. Therefore, $I_\theta - lim(t_n + p_n) = \eta + \lambda(N_C)$.

b) Case-(1) $k = 0$, then it is clear.

Case-(2) $|k| > 1$: For $m \in E$ with $\theta_E \prec m$, take

$$A(\xi, m) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in I_r} N_C(t_n - \eta, m) \leq 1 - \xi\right\},$$

$$B(\xi, m) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in I_r} N_C(k(t_n - \eta), m) \leq 1 - \xi\right\}.$$

Since N_C is a fuzzy cone norm,

$$N_C(k(t_n - \eta), m) = N_C\left(t_n - \eta, \frac{m}{|k|}\right) \quad (2)$$

Since N_C is an nondecreasing function and $\frac{m}{|k|} \leq m$ for $|k| > 1$,

$$N_C\left(t_n - \eta, \frac{m}{|k|}\right) \leq N_C(t_n - \eta, m). \quad (3)$$

As $I_\theta - limt_n = \eta(N_C)$, $A(\xi, m) \in I$. By (2.2) and (2.3), it follows that $B(\xi, m) \subseteq A(\xi, m)$. Then, we have $B(\xi, m) \in I$.

Case-(3) $|k| < 1$ and $k \neq 0$. For each $\xi \in (0,1)$ and $m \in E$,

$$K(\xi,m)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi\right\},$$

$$M(\xi,m)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(k(t_n-\eta),m)>1-\xi\right\}.$$

Since N_C is a fuzzy cone norm, we obtain

$$N_C\left(t_n-\eta,\frac{m}{|k|}\right)\geq N_C(t_n-\eta,m)*N_C\left(0,\frac{m}{|k|}-m\right)=N_C(t_n-\eta,m). \quad (4)$$

As $I_\theta-limt_n=\eta(N_C)$, $K(\xi,m)\in F(I)$. By (2.2) and (2.4), it follows that $K(\xi,m)\subseteq M(\xi,m)$. Then, we have $M(\xi,m)\in F(I)$.

Theorem 2.9. If $\theta-limt_n=\eta(N_C)$, then $I_\theta-limt_n=\eta(N_C)$.

Proof.Let $\theta-limt_n=\eta(N_C)$. Then, for every $\xi\in(0,1)$ and for any $m\in E$ with $\theta_E\prec m$, there exists $r_1\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi$$

for all $r\geq r_1$. Therefore, we obtain

$$T=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\leq 1-\xi\right\}\subseteq\{1,2,\ldots,n_0-1\}.$$

If we accept I as admissible ideal, we get $T\in I$. $I_\theta-limt_n=\eta(N_C)$.

Theorem 2.10. Let I be an be an admissible ideal and (t_n) be a sequence in *FCNS* $(V,N_C,*)$. If each subsequence of (t_n) is I_θ-convergent to η *w.r.t* fuzzy cone norm on V, then (t_n) is I_θ-convergent to η.

Proof. Assume that (t_n) is not I_θ-convergent to η. Then, there exists $\xi\in(0,1)$ and $m\in E$ with $\theta_E\prec m$ such that

$$A(\xi,m)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\geq 1-\xi\right\}\notin I.$$

Since I is an admissible ideal, $A(\xi,m)$ must be an infinite set. Let $A(\xi,m)=\{n_1<n_2<\cdots<n_k<\cdots\}$. Let $y_k=t_{n_k}$ for $k\in\mathbb{N}$ which is not I_θ-convergent to η. This is a contradiction.

The following example shows that the converse of Theorem 2.10 may not be true, in general.

Example 2.2. Let $E=\mathbb{R}^2$. Then, $P=\{(b_1,b_2): b_1, b_2\geq 0\}\subseteq E$ is a normal cone. Let $V=\mathbb{R}$, $u*v=uv$ and N_C: $V\times E\to[0,1]$ contemplated by $N_C(t,m)=e^{-\frac{|t|}{\|m\|}}$ for all $t\in V$ and $m\in E$ with $\theta_E\prec m$. Let $I=\{A\subseteq\mathbb{N}:\delta(A)=0\}$. Identify a sequence (t_n) in V,

$$t_n=\begin{cases}1, & \text{if } k=n^2,\\ 0, & \text{otherwise.}\end{cases}$$

Then, for every $\xi\in(0,1)$ and $m\in E$ with $\theta_E\prec m$, the set

$$A(\xi,m)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-0,m)\leq 1-\xi\right\}$$

will be a finite set. Consequently $A(\xi,m)\in I$, i.e., $I_\theta-limt_n=0(N_C)$. But $(t_{km})=(1)$, subsequence of (t_n) is not I_θ-convergent to 0.

Theorem 2.11. If there is an I_θ-convergent sequence (t_n) in *FCNS* $(V,N_C,*)$ such that $\{n\in\mathbb{N}:t_n\neq s_n\}\in I$, then (s_n) is also I_θ-convergent in V.

Proof. Suppose that $I_\theta-limt_n=\eta(N_C)$ and $\{n\in\mathbb{N}: t_n\neq s_n\}\in I$. Then, for each $\xi\in(0,1)$ and $m\in E$ with $\theta_E\prec m$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\leq 1-\xi\right\}\in I.$$

For every $\xi\in(0,1)$ and $m\in E$ with $\theta_E\prec m$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(s_n-\eta,m)\leq 1-\xi\right\}\subseteq\{n\in\mathbb{N}:t_n\neq s_n\}$$

$$\cup\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\leq 1-\xi\right\}.$$

As both the sets of right-hand side of the above relation is in I, therefore we have that

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(s_n-\eta,m)\leq 1-\xi\right\}\in I.$$

Hence, $I_\theta-lims_n=\eta(N_C)$.

Definition 2.5. Let V be a *FCNS* and (t_n) be a sequence in V. Then, (t_n) is called to be I^*-convergent to η in V if there exists a subset

$$J=\{k_1<k_2<\cdots<k_n<\cdots\}\subseteq\mathbb{N}$$

such that $J'=\{r\in\mathbb{N}: k_n\in I_r\}\in F(I)$ and $\theta-\lim_{n\to\infty}t_{k_n}=\eta(N_C)$ for each $m\in E$with $\theta_E\prec m$. In this case, we denote $I_\theta^*-limt_n=\eta(N_C)$.

Theorem 2.12. Let V be a FCNS, I be an admissible ideal and $t=(t_n)$ in V. If $I^*-limt_n=\eta(N_C)$, then $I_\theta-limt_n=\eta(N_C)$.

Proof. Assume that $I^*-limt_n=\eta(N_C)$. Then, there is a subset

$$J=\{k_1<k_2<\cdots<k_n<\cdots\}\subseteq\mathbb{N}$$

such that $J'=\{r\in\mathbb{N}: k_n\in I_r\}\in F(I)$ and there exists $k_0\in\mathbb{N}$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{n\in I_r}N_C(t_{k_n}-\eta,m)=1$$

for all $k\geq k_0$. Let $A=\{k_1<k_2<\cdots<k_n\}$. I is an admissible ideal and $A\in I$. Since $J'\in F(I)$, there is a set $B\in I$ such that $J'=\mathbb{N}/B$.

$$R(\xi,m)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\leq 1-\xi\right\}\subset A\cup B.$$

By the definition of ideal $A\cup B\in I$ and $R(\xi,m)\in I$. Therefore, we obtain $I_\theta - limt_n=\eta(N_C)$.

Definition 2.6. Let V be a *FCNS* and (t_n) be a sequence in V.

a) An element $\eta\in V$ is called to be lacunary *I-limit point of* $t=(t_n)$ if there is a set $J=\{p_1<p_2<\cdots<p_n<\cdots\}\subseteq\mathbb{N}$ such that the set $J'=\{r\in\mathbb{N}:p_n\in I_r\}\notin I$ and $\theta-\lim_{n\to\infty}t_{p_n}=\eta(N_C)$ for every $m\in E$ with $\theta_E\prec m$.

b) An element $\eta\in V$ is called to be lacunary I-cluster point of $t=(tn)$ if for any $m\in E$ with $\theta_E\prec m$ and $\xi\in(0,1)$, we get

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi\right\}\notin I.$$

The set of I_θ-limit points of (tn) is denoted by $\mathcal{L}_{N_C}^{I_\theta}(t)$ and the set of I_θ-cluster points of $(tn)$is denoted by $Cl_{N_C}^{I_\theta}(t)$ in $(V,N_C,*)$.

Theorem 2.13. For each sequence $t=(tn)$ in *FCNS*, we have $\mathcal{L}_{N_C}^{I_\theta}(t)\subseteq Cl_{N_C}^{I_\theta}(t)$.

Proof. Let $\eta\in\mathcal{L}_{N_C}^{I_\theta}(t)$. Then, there exists a set $J=\{p_1<p_2<\cdots<p_n<\cdots\}\subseteq\mathbb{N}$ such that the set $J'=\{r\in\mathbb{N}:p_n\in I_r\}\notin I$ and $\theta-\lim_{n\to\infty}t_{p_n}=\eta(N_C)$ for every $m\in E$ with $\theta_E\prec m$.

Let $\xi\in(0,1)$ and $m\in E$. By hypothesis, there is an integer $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{n\in I_r}N_C\left(t_{p_n}-\eta,m\right)>1-\xi$$

for all $r\geq r_0$. Thus, we get

$$H=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi\right\}\supseteq J'/\{p_1<p_2<\cdots<p_{n_0}\}.$$

Now, with I being admissible, we must have

$$J'/\{p_1<p_2<\cdots<p_{n_0}\}\notin I$$

and as such

$$H=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi\right\}\notin I$$

Hence, $\eta\in Cl_{N_C}^{I_\theta}(t)$.

Theorem 2.14. If $I_\theta - limt_n=\eta(N_C)$, then $\mathcal{L}_{N_C}^{I_\theta}(t)=Cl_{N_C}^{I_\theta}(t)=\{\eta\}$.

Proof. Presume that $I_\theta - limt_n=\eta(N_C)$. Then, for every $\xi\in(0,1)$ and $m\in E$ with $\theta_E\prec m$ the set

$$A=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\leq 1-\xi\right\}\in I$$

and so

$$A^c=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi\right\}\notin I,$$

and $\eta\in Cl_{N_C}^{I_\theta}(t_n)$. We suppose that $Cl_{N_C}^{I_\theta}(t_n)=\{\gamma\}$ where $\eta\neq\gamma$. By Definition 2.6, for every $\xi\in(0,1)$ and $m\in E$ with $\theta_E\prec m$the sets

$$P=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)>1-\xi\right\}\notin I.$$

$$R=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\gamma,m)>1-\xi\right\}\notin I.$$

For $\eta\neq\gamma$, we get $P\cap R=\emptyset$. By hypothesis,

$$P^c=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-\eta,m)\leq 1-\xi\right\}\in I,$$

so we get $R\in I$, which contradictions to $P\notin I$. Therefore, $Cl_{N_C}^{I_\theta}(t_n)=\{\eta\}$.

On the other hand, by hypothesis, Theorem 2.13 and Definition 2.6, we obtain $\eta\in Cl_{N_C}^{I_\theta}(t_n)$. By previous theorem, we get $\mathcal{L}_{N_C}^{I_\theta}(t_n)=Cl_{N_C}^{I_\theta}(t_n)=\{\eta\}$.

Theorem 2.15. Take I as an admissible ideal. For each sequence $t=(tn)$in *FCNS*, the set $Cl_{N_C}^{I_\theta}(t_n)$ is closed in V *w.r.t* the topology induced by the fuzzy cone norm N_C.

Proof. Let $q\in\overline{Cl_{N_C}^{I_\theta}(t_n)}$. Then, we obtain $B(q,r,m)\cap Cl_{N_C}^{I_\theta}(t_n)\neq\emptyset$ where $m\in E$with $\theta_E\prec$ *mand* $r\in(0,1)$ *Let* $p\in B(q,r,m)\cap Cl_{N_C}^{I_\theta}(t_n)$. Select $r_0\in(0,1)$ such that $B(p,r_0,m)\subset B(q,r,m)$. We get

$$G=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-q,m)>1-r\right\}$$

$$\supseteq\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{n\in I_r}N_C(t_n-p,m)>1-r_0\right\}=H.$$

Since $p\in Cl_{N_C}^{I_\theta}(t_n)$, $H\notin I$, and so $G\notin I$. Hence, $q\in Cl_{N_C}^{I_\theta}(t_n)$.

Theorem 2.16. Let $t=(t_n)$ be a sequence in *FCNS* $(V,N_C,*)$. The following situations are equivalent.

a) $\eta\in\mathcal{L}_{N_C}^{I_\theta}(t)$.

b) There are two sequences $w=(w_n)$ and $q=(q_n)$ in V such that $t=w+q$ and $\theta-limw=\xi$ and $\{r\in\mathbb{N}:n\in I_r,q_n\neq\theta\}\in I$, where θ indicates the zero element of V.

Proof. Presume that (a) holds. Then, there are J and J' are as above such that $J'\notin I$ and $\theta-limt_{p_n}=\eta$.

Take the sequences w and q as follows:

$$w_n = \begin{cases} t_n, & \text{if } n \in I_r \text{ such that } r \in J' \\ \eta, & otherwise \end{cases}$$

and

$$q_n = \begin{cases} \theta, & \text{if } n \in I_r \text{ such that } r \in J' \\ t_n - \eta, & otherwise. \end{cases}$$

It suffices to think the case $n \in I_r$ such that $r \in \mathbb{N}/J'$. Then, for each $\xi \in (0,1)$ and for any $m \in E$ with $\theta_E \prec m$, we have

$$\frac{1}{h_r} \sum_{n \in I_r} N_C(w_n - \eta, m) > 1 - \xi.$$

Hence, $\theta - limw_n = \eta$. Now,

$$\{r \in \mathbb{N}: n \in I_r, q_n \neq \theta\} \subset \mathbb{N}/J'.$$

Then, $\mathbb{N}/J' \in I$, and so

$$\{r \in \mathbb{N}: n \in I_r, q_n \neq \theta\} \in I.$$

Now, assume that (b) holds. Let $J' = \{r \in \mathbb{N}: n \in I_r, q_n = \theta\}$. Then, obviously $J' \in F(I)$ and so it is an infinite set. Construct the set $J = \{p_1 < p_2 < \cdots < p_n < \cdots\} \subseteq \mathbb{N}$ such that $p_n \in I_r$ and $q_{p_n} = \theta$. Since $t_{p_n} = w_{p_n}$ and $\theta - limw_n = \eta$ we get $\theta - limt_{p_n} = \eta$. This completes the proof of the theorem.

Definition 2.7. A sequence $t = (t_n)$ in V is said to be lacunary I-Cauchy *w.r.t* fuzzy cone norm on V, if for every $\xi \in (0,1)$ and $m \in E$ with $\theta_E \prec m$, there is $s \in \mathbb{N}$ providing that

$$\left\{ r \in \mathbb{N}: \frac{1}{h_r} \sum_{n \in I_r} N_C(t_n - t_s, m) > 1 - \xi \right\} \in F(I).$$

Definition 2.8. A sequence $t = (t_n)$ in V is said to be lacunary Cauchy *w.r.t* fuzzy cone norm on V, if there is set $J = \{p_1 < p_2 < \cdots < p_n < \cdots\} \subseteq \mathbb{N}$ such that the set

$$J' = \{r \in \mathbb{N}: p_n \in I_r\} \in F(I)$$

and the subsequence (t_{p_n}) is a lacunary Cauchy sequence *w.r.t* fuzzy cone norm on V.

Theorem 2.17. If a sequence $t = (t_n)$ in V lacunary Cauchy *w.r.t* fuzzy cone norm, then it is lacunary I- Cauchy *w.r.t* the same.

Theorem 2.18. If a sequence $t = (t_n)$ in V lacunary Cauchy *w.r.t* fuzzy cone norm, then there is a subsequence of t which is ordinary Cauchy *w.r.t* the same.

Theorem 2.19. If a sequence $t = (t_n)$ in V lacunary I^*-Cauchy *w.r.t* fuzzy cone norm, then it is lacunary I-Cauchy *w.r.t* the same.

CONCLUSION

This paper extends existing theories on convergence in FCNS to encompass lacunary ideal convergence. The introduction of the novel concept of lacunary convergence, along with the examination of lacunary I-limit points and lacunary I-cluster points, enhances our understanding of convergence patterns in FCNS. Additionally, the exploration of lacunary Cauchy and lacunary I-Cauchy sequences, as well as the study of their relationships, contributes valuable insights to the theoretical framework of FCNS.

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CONFLICT OF INTEREST

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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