Mathematical Modelling and Numerical Simulation with Applications, 2024, 4(2), 165–192

https://dergipark.org.tr/en/pub/mmnsa
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.1416148

RESEARCH PAPER

An approach to stochastic differential equations for long-term forecasting in the presence of α-stable noise: an application to gold prices

Bakary D. Coulibaly [1,*,‡], **Chaibi Ghizlane** [1,‡] **and Mohammed El Khomssi** [1,‡]

[1]Department of Mathematics, Laboratory of Modeling and Mathematical Structures, Sidi Mohamed Ben Abdellah University, Route d'Ilmouzzer, 30000 Fez, Morocco

[*] Corresponding Author
[‡] bakocly@gmail.com (Bakary D. Coulibaly); ghizlane.chaibi@usmba.ac.ma (Chaibi Ghizlane); khomsixmath@yahoo.fr (Mohammed El Khomssi)

Abstract

This article introduces a novel approach to forecasting gold prices over an extended period by leveraging a sophisticated stochastic process. Departing from traditional models, our proposed framework accommodates the non-Gaussian and non-homogeneous nature of gold market dynamics. Rooted in the α-stable distribution, our model captures time-dependent characteristics and exhibits flexibility in handling the distinctive features observed in real gold prices. Building upon prior research, we present a comprehensive methodology for estimating time-dependent parameters and validate its efficacy through simulations. The results affirm the universality of our stochastic model, showcasing its applicability for accurate and robust long-term predictions in gold prices.

Keywords: Stochastic differential equation; modeling α-stable distribution; parameters estimation; forecasting; gold prices; long-term prediction

AMS 2020 Classification: 60G15; 60G52; 60J65; 62M10

1 Introduction

Forecasting metal prices poses a significant challenge due to their intricate dependencies. The volatility and unpredictability of metal prices stem not only from fundamental factors such as supply-demand dynamics but also from macroeconomic conditions and investor sentiment. For mining companies, price assumptions are crucial not only for estimating revenue streams from metal sales but also for determining the optimal extraction plan in mines. This extraction plan forms the foundation for the entire budgeting and planning process. In the mining industry,

➤ **Received:** 07.01.2024 ➤ **Revised:** 29.02.2024 ➤ **Accepted:** 27.05.2024 ➤ **Published:** 30.06.2024

forecasting extends beyond the short term and focuses on a horizon spanning several years, adding complexity to the task. Employing stochastic modeling proves invaluable in comprehensively gauging and understanding the magnitude and likelihood of potential price movements. This becomes a pivotal undertaking for companies, enabling them to formulate effective business strategies in case base-case price scenarios deviate from expectations. The most critical metal risk factors for KGHM, one of the largest mining companies globally, include copper, silver, and gold. In the literature, various approaches have been employed to model mineral commodity prices. While our focus in this paper primarily revolves around the stochastic-based approach, alternative methodologies exist, such as time series modeling [1–6] and econometric-based methods [7, 8]. Utilizing the stochastic approach for forecasting market prices stems from the widespread belief that market fluctuations have random origins [9, 10]. Analyzing real data through continuous-time models involves the discrete-time approximation of the theoretical stochastic process, proving more effective for long-term predictions. One of the classical continuous-time stochastic processes applied in describing financial data is the Ornstein–Uhlenbeck model, introduced by Uhlenbeck and Ornstein [11] as a suitable system for velocity in classical Brownian diffusion. Also known as the Vasicek model [12], the Ornstein–Uhlenbeck process was among the earliest stochastic systems used for term structure. It demonstrates the mean-reversion property, indicating that over time, the process tends towards its long-term mean. This behavior is observable in mineral commodity price data.

The classical Ornstein–Uhlenbeck process follows a Gaussian distribution and is represented by the following stochastic differential equation:

$$dX_t = (\psi_1 + \psi_2 X_t)dt + \delta_1 dB_t, \tag{1}$$

where ψ_1, ψ_2, and δ_1 are constants, and $\{B_t\}_{t\geq 0}$ represents standard Brownian motion. The process defined in Eq. (1) can be viewed as a modification of the random walk in continuous time. It is also recognized as the continuous version of the discrete-time autoregressive model of order 1 (**AR(1)**) time series [13, 14]. However, some authors [9, 15] argue that financial variables exhibit non-Gaussian distributions, emphasizing that assuming a Gaussian distribution of prices is inappropriate. Consequently, in the literature, many researchers propose modifying the process defined in Eq. (1) by using processes other than Brownian motion as noise [16, 17].

In this paper, we follow this approach and replace the standard Brownian motion with a process of stationary independent increments having a α-stable distribution [18, 19]. Models based on the α-stable distribution have been employed to model various phenomena [18, 20].

The second characteristic observed in financial data, in addition to non-Gaussian behavior, is its inhomogeneous nature. Consequently, model (1) with constant coefficients is unsuitable for modeling data with a time-dependent mean and time-dependent scale parameter, especially variance. To address this, various modifications of the classical Ornstein–Uhlenbeck process use time-dependent coefficients instead of constants. Well-known examples include the Ho–Lee [21] and Hull–White [22] models.

In this study, we propose the application of a stochastic model to describe metals' prices, taking into account the aforementioned characteristics of real data. This new model is, in a sense, an extension of the Chan–Karolyi–Longstaff–Sander process based on the α-stable distribution, as described in [23], which has been utilized for currency exchange rate modeling. The model assumes time-dependent coefficients, capturing the crucial property of the analyzed real prices. These time-varying coefficients represent the time-dependent mean and time-dependent scale parameters of the theoretical process, reflecting the observed behavior in real-time series. Furthermore, the adoption of the general class of α-stable distributions appears more suitable than

the Gaussian distribution. The α-stable distribution is more versatile than the Gaussian one, serving as a generalization of the classical distribution. For specific parameter values, it reduces to the normal distribution. Additionally, the α-stable distribution can describe leptokurtic (like Student's t) or platykurtic (like uniform) distributions, depending on parameter values, enhancing its universality. These considerations suggest that the new stochastic model can effectively capture the specific behavior exhibited by real data.
However, the utilization of the stochastic model with time-dependent parameters and a non-Gaussian distribution necessitates employing more sophisticated parameter estimation techniques. While the literature offers various approaches to estimate the parameters of model (1) [24, 25], only a limited number of research papers propose techniques for estimating the time-dependent parameters of stochastic models [26, 27]. Therefore, one of the main objectives of this paper is to present a step-by-step estimation procedure for the proposed stochastic model. Through Monte Carlo simulations, we demonstrate the efficiency of the developed methodology. The applied section of the paper is dedicated to the analysis of real data. We consider three real datasets representing the daily prices of Gold. These analyzed prices are regarded as the main risk factors in the KGHM mining company, making their long-term prediction a crucial task from a risk management perspective.
The remaining part of the paper is structured as follows:
Section 2 provides a brief overview of the main characteristics of the α-stable distribution and introduces the stochastic model with time-dependent parameters, which will be subsequently employed for the description of real data.
In Section 3, we outline a step-by-step procedure for estimating the parameters of the introduced model. This procedure involves more advanced techniques compared to the case of fixed coefficients, and the assumption of the α-stable distribution necessitates non-standard approaches.
Moving on to Section 4, we showcase the efficacy of the new estimation procedure using simulated data. Section 4 is dedicated to the analysis of real data, specifically examining the datasets of gold prices. The obtained results suggest that the proposed stochastic model is versatile and can successfully predict long-term trends in gold prices. The concluding Section 5 summarizes the paper.

2 General stochastic model based on the α-stable distribution

Generally, the discovery of α-stable laws is attributed to [28], in this article, Lévy explores the central limit theorem and notes that when imposing an infinite variance, the limit law is an α-stable law. Lévy then sets out to determine the expression of the Fourier transform of all α-stable probability densities. The probability density of an α-stable distribution is often characterized as a "heavy-tailed" distribution, indicating that the tail of the distribution decreases asymptotically more slowly than the Gaussian law. It also distinguishes itself by an asymmetry coefficient, reflecting the fact that the probability density is not symmetric about its mode, and it exhibits leptokurtic behavior, indicating that most events are situated near the mean.
The concept of stability arises from the fact that any linear combination of α-stable random variables also generates an α-stable law. However, the main obstacle to the use of α-stable laws lies in the lack of an exact analytical expression for their probability density.
Several application domains (such as finance, including stock market, stock market variation, financial returns, etc.) using α-stable distributions are listed in the literature, with detailed bibliographies provided by [23, 29–34]. This category of processes plays a major role and exhibits heavy-tailed distributions. It is involved in stochastic modeling in applied sciences, particularly in financial mathematics, and also in the theoretical motivation for the study of their properties [20, 35].

[28] mathematically described the definition of α-stable laws as an extension of Gaussian laws used in error theory. However, the challenge with the definition of α-stable laws lies in the absence of an analytical expression, except for special cases such as the Gaussian law, the Cauchy law, or the Lévy law. Therefore, α-stable laws remained relatively unknown until the work done by [36] in the 1960s, at a time when financial markets were primarily based on the principles of [37], respecting the three principles of the law of large numbers, the central limit theorem, and the independence of present action from its past. However, these mathematical models proved invalid during financial crises. Then, [36] suggested modeling cotton price variations using an α-stable distribution. Stable laws are now used to represent stock market speculation fluctuations, interest rates, and other aspects of financial markets, providing a robust alternative during crisis periods.

Definition 1 *Let X be a random variable, X is called to be a stable law or $\alpha-$stable distribution random variable if $\forall(a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\exists c > 0$ and $k \in \mathbb{R}$ such that:*

$$aX_1 + bX_2 \overset{d}{=} cX + k, \tag{2}$$

where X_1 and X_2 are two random independent variable copies of X;
$\overset{d}{=}$: designates convergence in distribution.
If $k = 0$ then, the distribution is strictly stable.

Definition 2 *A random variable X is said to have a α- stable distribution if and only if, for any integer $n \geq 1$ and for any family $X_1, X_2, \cdots, X_n$ of i.i.d random variables of the same law as X, $\exists(a_n, b_n) \in \mathbb{R}_+^* \times \mathbb{R}$ such that:*

$$\frac{(X_1 + X_2 + \cdots + X_n) - b_n}{a_n} \overset{d}{=} X. \tag{3}$$

Variables with a Levy-stable distribution have the disadvantage of not having (except in three cases) explicit forms for the probability density and the distribution function.

Definition 3 *A random variable X with a α-stable law is typically described by its characteristic function Δ_X defined on $\mathbb{R}$ by:*

$$\Delta_X(t) = E\left[\exp(itx)\right] = \exp(i\mu t - g_{\alpha,\beta,\sigma}(t)), \tag{4}$$

where

$$g_{\alpha,\beta,\sigma}(t) = \begin{cases} \sigma^\alpha |t|^\alpha \left[1 - i\beta sign(t) \tan\left(\frac{\pi\alpha}{2}\right)\right] & if \quad \alpha \neq 1 \\ \sigma|t|[1 + \frac{2}{\pi} i\beta sign(t) \log|t|] & if \quad \alpha = 1, \end{cases} ; \quad sign(t) = \frac{t}{|t|} = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t = 0 \\ -1 & if \quad t < 0, \end{cases}$$

and having several representations according to the different parameterizations of the stable laws. The most famous of these representations is given in [23, 33].

The $\alpha-$stable law is thus characterized by four real parameters $\Psi = (\alpha, \beta, \mu, \sigma)$. The parameter α, called characteristic exponent or stability index, is an indicator of the degree of thickness of the tails of the distribution: the smaller it is, the thicker the tails are which corresponds to very large fluctuations. It is the most important parameter, it is between 0 and 2 ($0 < \alpha \leq 2$). Its maximum value $\alpha = 2$, corresponds to a particular stable law: the Gaussian law or normal law. β

is the parameter of dissymmetry, it varies between -1 and 1 ($-1 \leq \beta \leq 1$) and when it is null, the distribution is symmetrical with respect to μ. When α approaches 2, β loses its effect leading to a trend towards the normal distribution. The parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the usual characteristics of position and scale respectively with the remark that for the Gaussian distribution, the standard deviation is $\sigma\sqrt{2}$. A random variable X of stable distribution will be noted, according to [33], by: $X \sim S_\alpha(\beta, \mu, \sigma)$. The three exceptions mentioned above are the very famous Gaussian law $S_2(0, \mu, \sigma)$ and the less known Cauchy's law $S_1(0, \mu, \sigma)$. and Lévy's law $S_{\frac{1}{2}}(1, \mu, \sigma)$. The stable law has an additivity property according to which the sum of two independent stable random variables of the same stability index α is still stable with the same characteristic exponent α. This very interesting property is used in finance to study portfolios where two assets with the same value for α can be considered together. One of the particularities of the stable distribution is that it has infinite variance as soon as α is strictly less than 2. In fact, the moments of order p of $X \sim S_\alpha(\beta, \mu, \sigma)$ are such that for $\alpha = 2$, $E|X|^p < +\infty, \forall p \in \mathbb{N}$.

$$E|X|^p = \begin{cases} < \infty & \text{if} \quad 0 < \alpha < p, \\ = \infty & \text{if} \quad p \geq \alpha. \end{cases}$$

More precisely, it is shown that (see for example [33])

$$\lim_{t \to \infty} t^\alpha \mathbb{P}(X > t) = C_\alpha \frac{1+\beta}{2} \sigma^\alpha; \quad \lim_{t \to \infty} t^\alpha \mathbb{P}(X < -t) = C_\alpha \frac{1-\beta}{2} \sigma^\alpha,$$

where C_α is a constant given by:

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} & \text{if } \alpha = 1, \end{cases}$$

with $\Gamma(\theta)$ is the Euler gamma function defined for $\theta > 0$, by:

$$\Gamma(\theta) = \int_0^{+\infty} x^{\theta-1} e^{-x} dx. \tag{5}$$

Figure 1 illustrates the influence of each parameter of the α-stable distribution on its probability density function (PDF).

We can thus see that the stable law takes into account the distribution tails which are often carriers of essential information, whereas the Gaussian law neglects these tails thus leading to an error which can be fatal for the investor. The disadvantage of the characteristic function 4 is that it is not continuous if $\alpha = 1$ which makes it not adapted to numerical calculations and for these reasons [19] proposed another parameterization called S^0_α which is usable for numerical calculations. simulate stable laws, there is an algorithm developed by [38]. This one allows to generate a law $S_\alpha(\beta, 0, 1)$. To obtain a law $S_\alpha(\beta, \mu, \sigma)$, with $\alpha \in]0, 2]$ and $\beta \in [-1, 1]$.

The parameters α and σ for this generator are very well estimated by the method of [20]. The parameters μ and β are correctly estimated by the method of [20] for small values of β, which is often the case for stock exchange chronicles. A bibliography of methods for estimating the parameters of a α-stable law has been compiled by [39–42]. The PDF of a standard random variable α-stable law in the S^0 representation [43] i.e. $X \sim S^0_\alpha(1, \beta, 0)$.

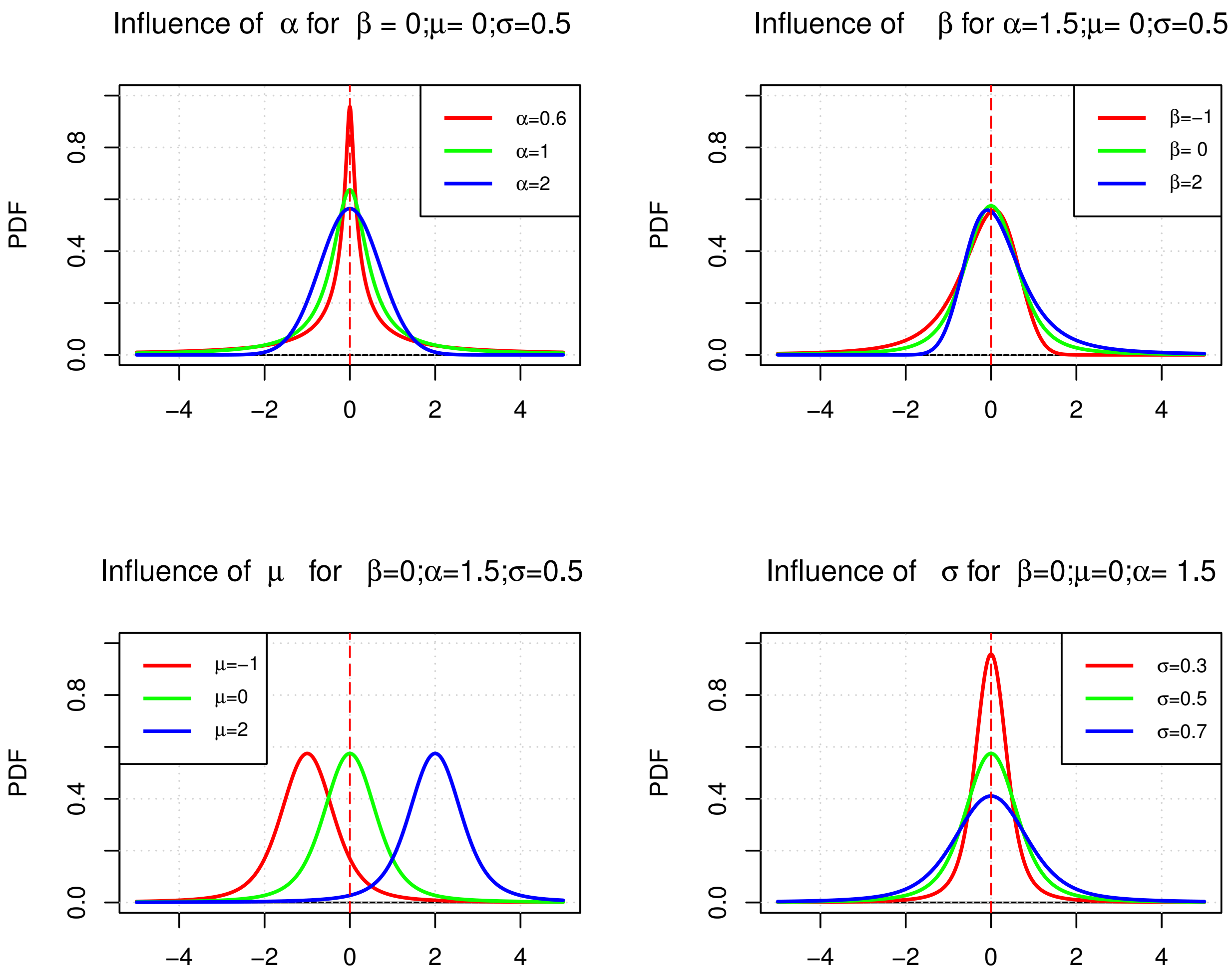

Figure 1. Influences of the parameters of the α-stable distribution on its PDF.

Let's commence the analysis of the stochastic process described by the following stochastic differential equation [44]:

$$dX_t = \psi(X_t, t)dt + \delta(X_t, t)dB_t. \tag{6}$$

In general, $\psi(.)$ and $\delta(.)$ are functions defined as $\psi(.), \delta(.) : \mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$, and $\{B_t\}_{t\geq 0}$ represents standard Brownian motion.
Consequently, $dB_t = B_{t+dt} - B_t$ follows a Gaussian distribution, denoted as $dB_t \sim \mathcal{N}(0, dt)$ [45].
The conditional distribution of the increments of the process defined in Eq. (6) is outlined in **Lemma 2**.
Several well-known examples that conform to Model (6), where the functions $\psi(.)$ and $\delta(.)$ are constant, include Merton [46], Vasicek [12], Brennan–Schwartz ([47]), Dothan [48], and Cox–Ingersoll–Ross [49] processes. Notable models with non-constant functions $\psi(.)$ and $\delta(.)$ comprise Ho–Lee [21], Hull–White [22], and Black–Krasiński [50], as detailed in Table 1.

However, in numerous real-world applications, the Gaussian distribution in model (6) may prove inadequate. Therefore, we propose modifying the model and assuming that the considered process satisfies the subsequent stochastic differential equation:

$$dX_t = \psi(X_t, t)dt + \delta(X_t, t)dS_t. \tag{7}$$

Similar to the previous scenario, in the general case, $\psi(.)$ and $\delta(.)$: $\mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$ are appropriate functions. In this context, we assume that $\{S_t\}$ is a process with stationary independent increments, following the α-stable distribution. In this case, the increment process $\{dS_t\} = \{S_{t+dt} - S_t\}$ constitutes a sequence of independent identically distributed (iid) random variables of the α-stable distribution, with the assumption $\mathbb{E}(dS_t) = 0$ and $\mathbb{E}(dS_t^2) = dt$.
In the subsequent calculations, we assume specific forms for the functions $\psi(.)$ and $\delta(.)$, and ultimately, the analyzed process is described by the stochastic differential equation:

$$dX_t = (\psi_1(t) + \psi_2(t)X_t)dt + (\delta_1(t) + \delta_2(t)X_t)dS_t, \tag{8}$$

for the general functions $\psi_1(.), \psi_2(.), \delta_1(.)$ and $\delta_2(.)$: $[0, T] \longrightarrow \mathbb{R}$. Additionally, we restrict our consideration to the case where $0 < \alpha < 2$. Further constraints on the functions are provided in the subsequent section.

Table 1. Classical models described by Eq. (6)

Model	$\psi(X_t, t)$	$\delta(X_t, t)$
Merton	ψ_1	δ_1
Vasicek	$\psi_1 + \psi_2 X_t$	δ_1
Dothan	$\psi_1 X_t$	$\delta_1 X_t$
Brennan–Schwartz	$\psi_1 + \psi_2 X_t$	$\delta_1 X_t$
Cox–Ingersoll–Ross	$\psi_1 + \psi_2 X_t$	$\delta_1 \sqrt{X_t}$
Ho–Lee	$\psi_1(t)$	$\delta_1(t)$
Hull–White	$\psi_1(t) + \psi_2(t) X_t$	$\delta_1(t)$
Black–Derman–Toy	$\psi_1(t) + \psi_2(t) \ln X_t$	$\delta_1(t)$
Black–Krasiński	$\psi_1(t) + \psi_2(t) X_t \ln X_t$	$\delta_1(t)$

Proposition 1 *Let X_1, X_2 be two random variables α-stable with $X_1 \sim S_\alpha(\beta, \mu, \sigma)$ and $X_2 = \frac{X_1 - \mu}{\sigma^{\frac{1}{\alpha}}}$. For $\alpha \neq 1$ then, we have the following equivalences:*

i. $X_1 \sim S_\alpha(\beta, \mu, \sigma)$;
ii. $X_2 = \frac{X - \mu}{\sigma^{\frac{1}{\alpha}}} \sim S_\alpha(\beta, 0, 1)$.

For $\alpha = 1$ then, we have the following equivalences:

i. $X_1 \sim S_1(\beta, \mu, \sigma)$;
ii. $X_2 = \frac{X_1 - \mu}{\sigma} \sim S_1(\beta, \frac{2}{\pi}\beta \log(\sigma), 1)$.

Proof For $\alpha \neq 1$ then, we have (1)$\Longrightarrow$ (2):

$$\begin{aligned}
\Delta_{X_2}(r) &= E\left[\exp\left(ir\left(\frac{X_1-\mu}{\sigma^{\frac{1}{\alpha}}}\right)\right)\right] = E\left[\exp\left(\frac{irX_1}{\sigma^{\frac{1}{\alpha}}}\right)\exp\left(\frac{-ir\mu}{\sigma^{\frac{1}{\alpha}}}\right)\right] \\
&= \exp\left(\frac{-ir\mu}{\sigma^{\frac{1}{\alpha}}}\right) E\left[\exp\left(\frac{irX_1}{\sigma^{\frac{1}{\alpha}}}\right)\right] = \exp\left(\frac{-ir\mu}{\sigma^{\frac{1}{\alpha}}}\right)\Delta_{X_1}\left(\frac{r}{\sigma^{\frac{1}{\alpha}}}\right) \\
&= \exp\left(\frac{-ir\mu}{\sigma^{\frac{1}{\alpha}}}\right) exp\left(\frac{ir\mu}{\sigma^{\frac{1}{\alpha}}} - \sigma\left|\frac{r}{\sigma^{\frac{1}{\alpha}}}\right|^{\alpha}\left[1 - i\beta sign\left(\frac{r}{\sigma^{\frac{1}{\alpha}}}\right)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right).
\end{aligned}$$

$$\text{Such as } \sigma > 0 \text{ so:} \quad sign\left(\frac{r}{\sigma^{\frac{1}{\alpha}}}\right) = sign(r).sign\left(\frac{1}{\sigma^{\frac{1}{\alpha}}}\right) = sign(r).$$

Subsequently:

$$\begin{aligned}
\Delta_{X_2}(r) &= \exp\left(\frac{-ir\mu}{\sigma^{\frac{1}{\alpha}}}\right) exp\left(\frac{ir\mu}{\sigma^{\frac{1}{\alpha}}} - \sigma\left|\frac{r}{\sigma^{\frac{1}{\alpha}}}\right|^{\alpha}\left[1 - i\beta sign(r)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right) \\
&= \exp\left(-|r|^{\alpha}\left[1 - i\beta sign(r)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right).
\end{aligned}$$

So $Y \sim S_\alpha(\beta, 0, 1)$.
(2)$\Longrightarrow$ (1) is proven in the same way as (1)$\Longrightarrow$ (2).

For $\alpha = 1$ then, (1)$\Longrightarrow$ (2) we have:

$$\begin{aligned}
\Delta_{X_2}(r) &= E\left[\exp\left(ir\left(\frac{X_1-\mu}{\sigma}\right)\right)\right] = E\left[\exp\left(\frac{irX_1}{\sigma}\right)\exp\left(\frac{-ir\mu}{\sigma}\right)\right] \\
&= \exp\left(\frac{-ir\mu}{\sigma}\right) E\left[\exp\left(\frac{irX_1}{\sigma}\right)\right] \\
&= \exp\left(\frac{-ir\mu}{\sigma}\right)\exp\left(\frac{ir\mu}{\sigma} - \sigma\left|\frac{r}{\sigma}\right|\left(1 + i\frac{2}{\pi}\beta sign\left(\frac{r}{\sigma}\right)\log\left|\frac{r}{\sigma}\right|\right)\right) \\
&= \exp\left(-|r|\left[1 + i\frac{2}{\pi}\beta sign(r)\log\left(\left|\frac{r}{\sigma}\right|\right)\right]\right) \\
&= \exp\left(i\frac{2}{\pi}\beta|r|sign(r)\log(\sigma) - |r|\left[1 + i\frac{2}{\pi}\beta sign(r)\log(|r|)\right]\right) \\
&= \exp\left(i\frac{2}{\pi}\beta\log(\sigma)r - |r|\left[1 + i\frac{2}{\pi}\beta sign(r)\log(|r|)\right]\right).
\end{aligned}$$

So $X_2 = \frac{X_1-\mu}{\sigma^{\frac{1}{\alpha}}} \sim S_\alpha(\beta, \frac{2}{\pi}\beta\log(\sigma), 1)$.

(2)$\Longrightarrow$ (1) is demonstrated in the same way as (1)$\Longrightarrow$ (2). This completes the proof of this Proposition 1.

Lemma 1 *Let X_1 and X_2 two random variables α-stable.*
For $\alpha \neq 1$, if $X_1 \sim S_\alpha(\beta, 0, 1)$ and $X_2 \sim S_\alpha(\beta, \mu, \sigma)$. Then,

$$\sigma X_1 + \mu \overset{d}{=} X_2.$$

Proof Let us define $X_1 \sim S_\alpha(\beta, 0, 1)$ and $X_2 \sim S_\alpha(\beta, \mu, \sigma)$. We will show that:

$$\mathcal{P}(\sigma X_1 + \mu < r) = \mathcal{P}(X_2 < r). \tag{9}$$

If f is the probability density function of the α-stable random variable X_1 then, we have:

$$f(r, \alpha, \beta, \mu, \sigma) = \frac{1}{\sigma} f\left(\frac{r-\mu}{\sigma}, \alpha, \beta, 0, 1\right). \tag{10}$$

We use the formula from Eq. (10) for the PDF of α-stable distribution given in Eq. (9):

$$\begin{aligned}
\mathcal{P}(\sigma X_1 + \mu < r) &= \mathcal{P}\left(X_1 < \frac{r-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\frac{r-\mu}{\sigma}} f(t, \alpha, \beta, 0, 1) dt \\
&= \int_{-\infty}^{r} \frac{1}{\sigma} f\left(\frac{t-\mu}{\sigma}, \alpha, \beta, 0, 1\right) dt \\
&= \int_{-\infty}^{r} f(t, \alpha, \beta, \mu, \sigma)\, dt \\
&= \mathcal{P}(X_2 < r).
\end{aligned}$$

Lemma 2 *For the stochastic process $\{X_t\}$ as defined in Eq. (6), the increment $dX_t = X_{t+dt} - X_t$ follows the subsequent relationship:*

$$dX_t \mid X_t \sim \mathcal{N}\left(\psi(X_t, X_t)dt, \delta^2(X_t, t)dt\right).$$

Proof Initially, we will demonstrate that:

$$\mathbb{E}(dX_t \mid X_t) = \psi(X_t, t)dt; \quad \text{and } Var(dX_t \mid X_t) = \delta^2(X_t, X_t)dt.$$

To establish this, we will leverage the properties of standard Brownian motion, where $\mathbb{E}(dB_t) = 0$ and $\mathbb{E}(dB_t^2) = dt$. Consequently, we derive:

$$\begin{aligned}
\mathbb{E}(dX_t \mid X_t) &= \mathbb{E}(\psi(X_t, t)dt + \delta(X_t, t)dB_t \mid X_t) \\
&= \mathbb{E}(\psi(X_t, t)d_t | X_t) + \mathbb{E}(\delta(X_t, t)dB_t \mid X_t) \\
&= \psi(X_t, t)dt + \delta(X_t, t)\mathbb{E}(dB_t) \\
&= \psi(X_t, t)dt.
\end{aligned}$$

The second moment of $dX_t \mid X_t$ is expressed as:

$$\begin{aligned}
\mathbb{E}(dX_t^2 \mid X_t) &= \mathbb{E}((\psi(X_t, t)dt + \delta(X_t, t)dB_t)^2 \mid X_t) \\
&= \mathbb{E}(\psi^2(X_t, t)dt^2 \mid X_t) + 2\mathbb{E}(\psi(X_t, t)dt\delta(X_t, t)dB_t \mid X_t) + \mathbb{E}(\delta^2(X_t, t)dB_t^2 \mid X_t) \\
&= \psi^2(X_t, t)dt^2 + 2\psi(X_t, t)dt\delta(X_t, t)\mathbb{E}(dB_t \mid X_t) + \delta^2(X_t, t)\mathbb{E}(dB_t^2 \mid X_t) \\
&= \psi^2(X_t, t)dt^2 + \delta^2(X_t, t)dt.
\end{aligned}$$

Thus, the variance of $dX_t \mid X_t$ can be expressed as:

$$\begin{aligned} Var(dX_t \mid X_t) &= \mathbb{E}(dX_t^2 \mid X_t) - [\mathbb{E}(dX_t \mid X_t)]^2 \\ &= \psi^2(X_t, t)dt^2 + \delta^2(X_t, t)dt - [\psi(X_t, t)dt]^2 \\ &= \delta^2(Xt, t)dt. \end{aligned}$$

Due to the Gaussian distribution of dB_t and the property $\mathbb{E}(X + c) = \mathbb{E}(X) + c$ for any random variable X, as well as $Var(cX) = c^2 Var(X)$, we can express $dX_t \mid X_t$ as:

$$dX_t \mid X_t \sim \mathcal{N}\left(\psi(X_t, t)dt, \delta^2(X_t, t)dt\right).$$

3 Estimation of the parameters for general model based on α-stable distribution

In this section, we outline a method for estimating the parameters of the stochastic process described in Eq. (8).

Let's assume we have a vector of realizations of the stochastic process given by Eq. (8), denoted as $X_0, X_2, \cdots, X_n$, with corresponding time points $t_0, t_1, \cdots, t_n$, such that $\forall_{j\in\{1,2,\cdots,n\}}$

$$t_j - t_{j-1} = \Theta.$$

For the sake of simplicity, we assume $\Theta = 1$.

Consequently, we represent the increments of the observed data as $y_0, y_1, \cdots, y_{n-1}$, where $y_j = X_{j+1} - X_j$ for $j = 0, 1, \cdots, n-1$. To achieve this, we initially transform Eq. (8) into its discrete form.

$$\begin{aligned} y_j &= X_{j+1} - X_j \\ &= \psi_1(t_j) + \psi_2(t_j)X_j + (\delta_1(t_j) + \delta_2(t_j)X_j)S_j; \quad j = 0, 1, \cdots, n-1. \end{aligned} \tag{11}$$

In this context, $\{s_j\}$ represents a time series of independent and identically distributed (iid) random variables following the α-stable distribution $S_\alpha(\beta, \mu = 0, \sigma = 1)$.

In this paper, we employ the local regression approach [51], following a similar methodology as in [27], to derive estimates for the functions $\psi_1(.)$ and $\psi_2(.)$ within Model (8). We make the assumption that $\psi_1(.) \in \mathcal{C}^{d_1^\psi}$ and $\psi_2(.) \in \mathcal{C}^{d_2^\psi}$, allowing them to be expanded into Taylor's polynomials [52] at every time point $t^* \in \{t_0, t_1, \cdots, t_{n-1}\}$ of degrees d_1^ψ and d_2^ψ, respectively:

$$\psi_l(t_j) = \sum_{k=0}^{d_l^\psi} \frac{\psi_l^{(k)}(t^*)}{k!}(t_j - t^*)^k + R_{d_l^\psi}(t_j); \quad l = 1, 2, \tag{12}$$

where $R_{d_l^\psi}(.)$ represents Peano's remainder, which we, in further considerations, neglect. After expanding (12) and consolidating constants for common t_j^k, we arrive at the following approximation:

$$\psi_l(t_j) \approx \sum_{k=0}^{d_l^\psi} {}_k\psi_l t_j^k; \quad l = 1, 2. \tag{13}$$

To obtain ${}_k\psi_l$ estimates for all t_j in the vicinity of ρ from Eq. (13), we formulate the loss function as a weighted sum of squared errors. It's important to emphasize that $\{{}_k\psi_l\}$ is estimated independently for each time point $t^* \in \{t_0, t_1, \cdots, t_{n-1}\}$. Deriving from Eq. (11), we derive the following:

$$\begin{aligned} S_j &= \frac{y_j - (\psi_1(t_j) + \psi_2(t_j)X_j)}{\delta_1(t_j) + \delta_2(t_j)X_j} \\ &\approx \frac{y_j - \left(\sum_{k=0}^{d_1^\psi} {}_k\psi_1 t_j^k + \sum_{k=0}^{d_2^\psi} {}_k\psi_2 t_j^k X_j\right)}{\delta_1(t_j) + \delta_2(t_j)X_j} \\ &=: \tilde{S}_j; \quad j = 0, 1, \cdots, n-1. \end{aligned} \tag{14}$$

In this paper, we posit that the loss function, utilized in the estimation algorithm for each $t^* \in \{t_0, t_1, \cdots, t_{n-1}\}$, adopts the following form:

$$G^*_{\tilde{\Omega}}\left(\{X_j\}, \{t_j\}; \{{}_k\psi_l\}\right) = \sum_{j=0}^{n-1} \tilde{S}_j^2 K_{\rho^\psi, \rho_r^\psi}\left(t_j - t^*\right) + \eta\left(\sum_{k=0}^{d_1^\psi} {}_k\psi_1^2 + \sum_{k=0}^{d_2^\psi} {}_k\psi_2^2\right), \tag{15}$$

with $\tilde{\Omega} = (t^*, \delta, \eta, d_1^\psi, d_2^\psi, \rho^\psi, \rho_r^\psi)$.
The first component of the loss function, specifically $\tilde{S}_j^2 K_{\rho^\psi, \rho_r^\psi}\left(t_j - t^*\right)$, is linked to the fact that the estimators are fitted locally (rather than globally). Additionally, akin to Ridge regression [53], we have incorporated into the loss function a second component.
Tikhonov regularization [54] (with parameter η). This regularization compensates for the potentially non-unique solution and high variance of the estimators. In this paper, we utilized a single-valued parameter η; however, it can be replaced with a vector $\{\eta_j\}$. This substitution results in improved estimates but requires the entire vector $\{\eta_j\}$ to be determined. In this paper, we suggest using the asymmetric kernel function $K_{\rho,\rho_r}(.)$ in Eq. (15), defined as follows:

$$K_{\rho,\rho_r}(t) = \frac{2K\left(\frac{t}{\rho - \rho_r}\right)\mathbb{1}_{t \le 0} + K\left(\frac{t}{\rho_r}\right)\mathbb{1}_{t > 0}}{\rho}. \tag{16}$$

In this context, "ρ" represents the width of the kernel function $K_{\rho,\rho_r}(.)$, denoting the distance from the left root to the right, while "ρ_r" represents the distance to the right root from 0. This specific form of the kernel provides the flexibility to strike a balance between the traditional symmetric and causal kernel functions, resulting in estimators with reduced variance. The parameters $\rho^\psi, \rho_r^\psi, d_1^\psi, d_2^\psi$ and η in the estimation process are referred to as hyperparameters. In practical applications, three commonly utilized kernel functions $K(.)$ in Eq. (16) are [27, 51, 55, 56]:

- Gaussian kernel: $K(t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2})$;
- Epanechnikov kernel: $K(t) = \frac{3}{4}(1 - t^2)\mathbb{1}_{t \in (-1,1)}$;
- Tricube kernel: $K(t) = \frac{70}{81}(1 - |t|^3)^3 \mathbb{1}_{t \in (-1,1)}$.

Because of their compact support, the Epanechnikov and tricube kernels are commonly employed in modeling financial data problems [27, 51, 56]. In our applications, we opted for the tricube kernel.
To streamline the calculations, the initial step of the estimation procedure involves treating the $\delta_1(.)$ and $\delta_2(.)$ functions in model (8) as known. We utilize an iterative method to derive estimates,

commencing with a predefined starting condition:

$$\zeta_{j,\eta}^{(0)} := \hat{\delta}_1(t_j) + \hat{\delta}_2(t_j)X_j \equiv 1. \tag{17}$$

Nevertheless, the optimal values for d_1^ψ, d_2^ψ (refer to Eq. (13)), as well as the kernel widths ρ^ψ, ρ_r^ψ, remain unknown. We determine the optimal values for hyperparameters $\rho^\psi, \rho_r^\psi, d_1^\psi, d_2^\psi$ and η (refer to Eq. (15)) by selecting those that result in the lowest mean squared error (MSE) statistics:

$$MSE_Y = \sum_{j=0}^{n-1}\left(y_j - \left(\sum_{k=0}^{d_1^\psi} {}_k\hat{\psi}_1 t_j^k + \sum_{k=0}^{d_2^\psi} {}_k\hat{\psi}_2 t_j^k X_j\right)\right)^2 \omega_j, \tag{18}$$

$$MSE_X = \sum_{j=1}^{n}\left(X_j - X_0 - \sum_{h=1}^{j}\left(\sum_{k=0}^{d_1^\psi} {}_k\hat{\psi}_1 t_h^k + \sum_{k=0}^{d_2^\psi} {}_k\hat{\psi}_2 t_h^k X_h\right)\right)^2 \omega_j, \tag{19}$$

and the Augmented Dickey–Fuller test statistic [57] (where the null hypothesis assumes the presence of a unit root in the time series data) for the vector

$$\left\{ y_j - \left(\sum_{k=0}^{d_1^\psi} {}_k\hat{\psi}_1 t_j^k + \sum_{k=0}^{d_2^\psi} {}_k\hat{\psi}_2 t_j^k X_j\right)\right\}.$$

The weights $\{\omega_j\}$ in Eqs. (18) and (19) are computed using the exponential smoothing method [58]. Once the optimal values for hyperparameters are determined, we can express the loss function $G^*(\cdot)$ defined in Eq. (15) using matrices:

$$Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}; \quad \Psi = \begin{pmatrix} {}_0\psi_1 \\ {}_1\psi_1 \\ {}_2\psi_1 \\ \vdots \\ {}_{d_1^\psi}\psi_1 \\ {}_0\psi_2 \\ {}_1\psi_2 \\ \vdots \\ {}_{d_2^\psi}\psi_2 \end{pmatrix}; \quad T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{n-1} \\ t_0^2 & t_1^2 & \cdots & t_{n-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{d_1^\psi} & t_1^{d_1^\psi} & \cdots & t_{n-1}^{d_1^\psi} \\ X_0 & X_1 & \cdots & X_{n-1} \\ t_0X_0 & t_1X_1 & \cdots & t_{n-1}X_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{d_2^\psi}X_0 & t_1^{d_2^\psi}X_1 & \cdots & t_{n-1}^{d_2^\psi}X_{n-1} \end{pmatrix};$$

$$K_{t^*} = \begin{pmatrix} \frac{K_{\rho^\psi,\rho_r^\psi}(t_0-t^*)}{\left(\zeta_{0,\eta}^{(0)}\right)^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{K_{\rho^\psi,\rho_r^\psi}(t_1-t^*)}{\left(\zeta_{1,\eta}^{(0)}\right)^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{K_{\rho^\psi,\rho_r^\psi}(t_{n-1}-t^*)}{\left(\zeta_{n-1,\eta}^{(0)}\right)^2} \end{pmatrix}.$$

Then, the loss function from Eq. (15) takes the form:

$$G^* = (Y - T'\Psi)'K_{t^*}(Y - T'\Psi) + \eta\Psi'\Psi; \tag{20}$$

which we minimize Eq. (20) with respect to the vector Ψ:

$$\frac{\partial G^*}{\partial \Psi} = -2TK_{t^*}(Y - T'\Psi) + 2\eta I\Psi = 0. \tag{21}$$

Then,

$$(TK_{t^*}T' + \eta I)\Psi = TK_{t^*}Y \Longrightarrow \hat{\Psi} = (TK_{t^*}T' + \eta I)^{-1}TK_{t^*}Y. \tag{22}$$

With the estimation of $\psi_1(.)$ and $\psi_2(.)$ functions from model (8), we can proceed to estimate the functions $\delta_1(.)$ and $\delta_2(.)$. In a manner similar to the estimation of $\psi_1(.)$ and $\psi_2(.)$ functions, we will employ Taylor's polynomials [52] to approximate the functions $\delta_1(.)$ and $\delta_2(.)$ from model (8):

$$\delta_l(t_j) \approx \sum_{k=0}^{d_l^\delta} {}_k\delta_l t_j^k, l = 1, 2. \tag{23}$$

Subsequently, the parameters $\{{}_k\delta_l\}$ can be determined through the maximum likelihood method [51]. Leveraging properties established in Lemma 1 and considering the independence and identically distributed (iid) nature of $\{S_j\}$, the log-likelihood function can be expressed as:

$$l^*_{\tilde{\Omega}_\delta}(\tilde{\Omega}_{par}) = \sum_{j=0}^{n-1} \ln\left(f\left(\hat{e}_j; \sum_{k=0}^{d_1^\delta} {}_k\delta_1 t_j^k + \sum_{k=0}^{d_2^\delta} {}_k\delta_2 t_j^k, \alpha, \beta, 0, \sigma\right)\right) K_{\rho^\delta,\rho_r^\delta}\left(t_j - t^*\right); \tag{24}$$

$$\text{with: } \tilde{\Omega}_\delta = (t^*, \delta, \eta, d_1^\delta, d_2^\delta, \rho^\delta, \rho_r^\delta); \quad \tilde{\Omega}_{par} = (\{\hat{e}_j\}, \{t_j\}; \{{}_k\delta_l\}, \alpha, \beta, \sigma)).$$

where $\hat{e}_j$ is derived by transforming Eq. (11) in the following manner:

$$\hat{e}_j := y_j - (\hat{\psi}_1(t_j) + \hat{\psi}_2(t_j)X_j) \approx (\delta_1(t_j) + \delta_2(t_j)X_j)S_j; j = 0, 1, 2, \cdots, n-1. \tag{25}$$

In this scenario, additional optimal hyperparameters, namely d_j^δ ($j = 1, 2$), ρ^δ and ρ_r^δ, must be determined. We suggest employing the Breusch–Pagan test statistic [59] with the null hypothesis that the variance is independent of descriptive (independent) variables, indicating homoscedasticity in the time series. We aim to identify the set of hyperparameters d_j^δ ($j = 1, 2$), ρ^δ and ρ_r^δ that minimizes the test statistic calculated for the time series:

$$\{\hat{S}_j\} = \frac{y_j - \left(\sum_{k=0}^{d_1^\psi} {}_k\hat{\psi}_1 t_j^k + \sum_{k=0}^{d_2^\psi} {}_k\hat{\psi}_2 t_j^k X_j\right)}{\hat{\delta}_1(t_j) + \hat{\delta}_2(t_j)X_j}. \tag{26}$$

Once hyperparameters are determined, we optimize the log-likelihood function (Eq. 24) with respect to $\{{}_k\delta_j\}$ (as defined in Eq. 23) and the unknown parameters α, β, σ associated with residuals. Since there is no analytical solution for maximizing the function (Eq. 24), numerical algorithms are required to find the function's maximum. To streamline the computations, we exploit the invariance property of maximum likelihood estimators [23, 30, 60, 61].

Note that $\alpha \in]0, 2]$, $\beta \in [-1, 1]$, and $\sigma > 0$. We aim to maximize the function $l^*(.)$ (Eq. 24) with respect to the parameters: ${}_k\delta_j \in \mathbb{R}$ ($j = 1, 2$; and $k = 0, \cdots, d_j^\delta$); $\hat{\alpha} \in]0, 2]$, $\hat{\beta} \in [-1, 1]$, and $\hat{\sigma} \in \mathbb{R}_+^*$. Optimization can be achieved using a broader and more straightforward class of algorithms, such as the Broyden–Fletcher–Goldfarb–Shanno algorithm [62].

The initial proposition of $\zeta_{j,\delta}^{(0)} \equiv 1$ (refer to Eq. (17)) can be highly questionable, particularly in cases of evident heteroskedasticity in time series.

To address this concern, we employ an iterative method for estimating $\psi_1(.), \psi_2(.), \delta_1$ and $\delta_2(.)$. In the subsequent step of the estimation, we set:

$$\left\{\zeta_{j,\psi}^{(1)}\right\} = \{\hat{\psi}_1(t_j) + \hat{\psi}_2(t_j)X_j\}; \quad \text{and} \quad \left\{\zeta_{j,\delta}^{(1)}\right\} = \{\hat{\delta}_1(t_j) + \hat{\delta}_2(t_j)X_j\};$$

and repeat the entire estimation procedure until the changes in the estimated functions become negligible, that is, until:

$$\exists_j \left\|\zeta_{j,\psi}^{(\gamma)} - \zeta_{j,\psi}^{(\gamma-1)}\right\| > \epsilon_\psi \quad \text{or} \quad \exists_j \left\|\zeta_{j,\delta}^{(\gamma)} - \zeta_{j,\delta}^{(\gamma-1)}\right\| > \epsilon_\delta,$$

where ϵ_ψ and ϵ_δ are defined thresholds, and γ represents the current iteration number (or after a specified number of iterations).

After estimating the $\psi_1(.), \psi_2(.), \delta_1$ and $\delta_2(.)$ functions, we can ultimately estimate the global parameters of residuals $\{\hat{S}_j\}$ (defined in Eq. (26)) modeled by the α-stable distribution. In the previous steps of the estimation procedure, only local estimates of the parameters are obtained. We find $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\sigma}$ by numerically maximizing the likelihood function with respect to $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\sigma}$ refer to [30, 60]:

$$L\left(\{\hat{S}_j\}, \hat{\alpha}, \hat{\beta}, \hat{\sigma}\right) = \prod_{j=0}^{n-1} \frac{1}{\sigma} f(\hat{S}_j, \alpha, \beta, 0, 1). \tag{27}$$

An algorithm describing the parameter estimation procedure is shown in Figure 2.

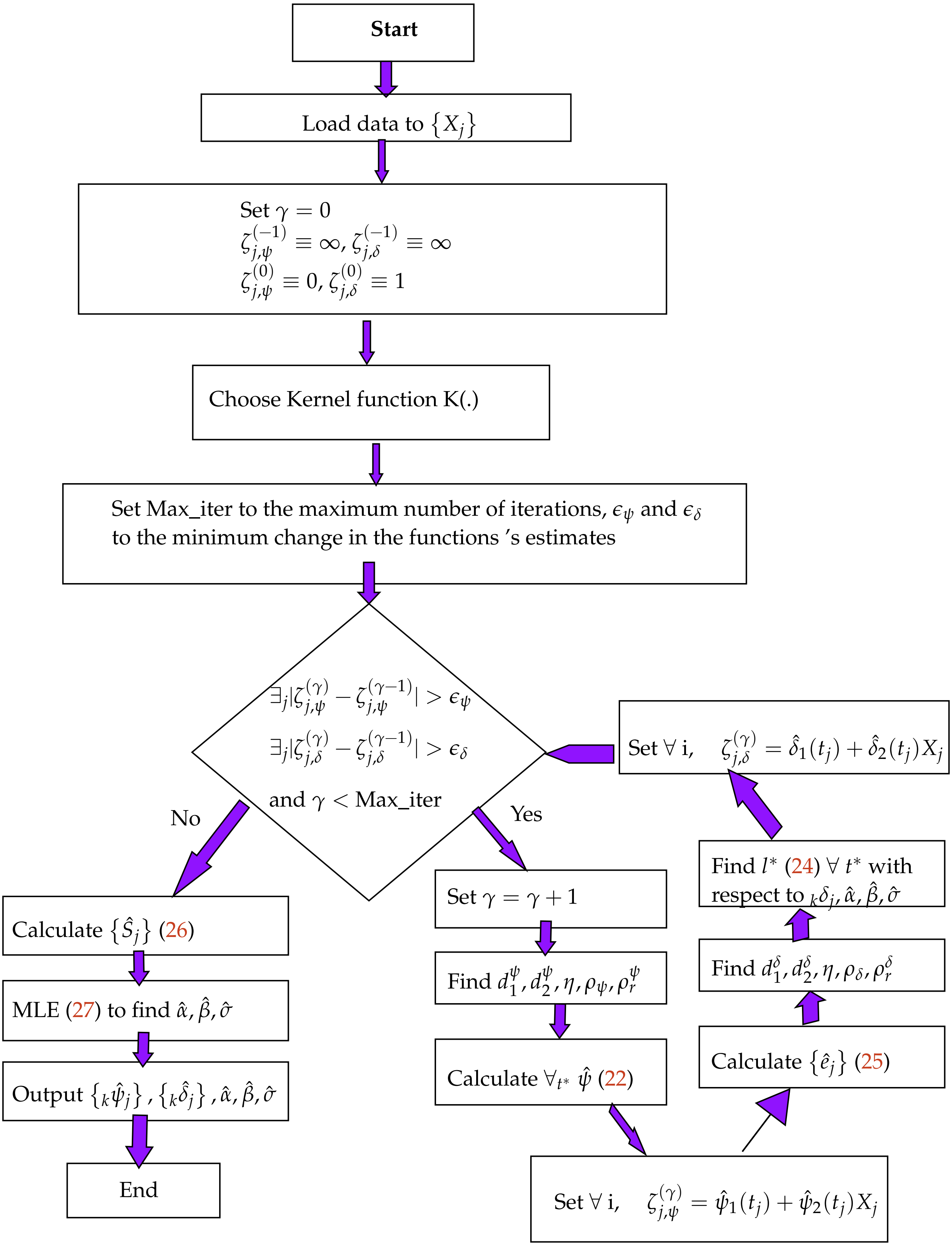

Figure 2. Algorithm describing the parameter estimation procedure

4 Numerical simulation

Simulated data analysis

Utilizing the methodology outlined in Section 3, we evaluate the efficacy of the estimation procedure through the analysis of simulated data. Employing Euler's method, Runge Kutta's method and Milstein's method [63], we simulate the trajectory of the process governed by the stochastic differential equation:

$$dX_t = (0.1 + 0.025t - 0.015X_t)dt + (0.03 + 0.001t)dS_t, \tag{28}$$

under the assumption: $dS_t \sim S_\alpha(\beta, \mu, \sigma)$ with $\alpha = 1.8, \beta = 0.8, \mu = 0$ and $\sigma = 1$.
The illustration of the $\{dS_t\}$ and $\{X_t\}$ processes is depicted in Figure 3 as an exemplary representation. Model parameters are intentionally selected in a manner that allows them to be, in some sense, comparable to the parameters derived from real data.

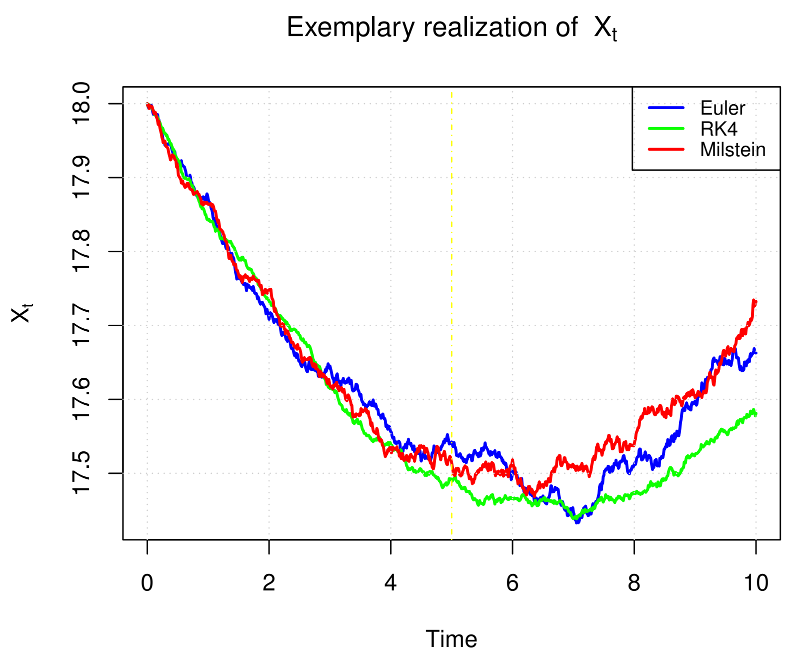

Figure 3. The illustrative realizations of the stochastic process defined by Eq. (28) with residuals following the α-stable distribution

Applying the outlined methodology, we have computed the $\psi_1(.)$, $\psi_2(.)$, $\delta_1(.)$, and $\delta_2(.)$ functions based on model (8). In Figure 4 and Figure 5, we display both the estimated functions and the theoretical counterparts from Model (28). It is evident from the observation that the estimates align well with the theoretical functions. The estimated parameters for the α-stale distribution (using the method for estimating [20]) are $\hat{\alpha} = 1.84385$, $\hat{\beta} = 0.7672$, $\hat{\mu} = 0$, and $\hat{\sigma} = 0.9873$ demonstrating close proximity to the theoretical parameters $\alpha = 1.8$, $\beta = 0.8$, $\mu = 0$ and $\sigma = 1$. Additionally, the Kolmogorov–Smirnov test [64] resulted in a statistic $K = 0.00586$ and a p-value of 0.930, leading to the conclusion that the model parameters have been accurately estimated.
Moreover, we have conducted Monte-Carlo simulations [65] for the process defined by Eq. (28). Specifically, we have generated 100 realizations of the process and applied the estimation methodology outlined in the preceding section. Subsequently, we have obtained estimates

for $[0.05, 0.25, 0.5, 0.75, 0.95]$ quantiles of the $\psi_1(.)$, $\psi_2(.)$, $\delta_1(.)$, and $\delta_2(.)$ functions' estimators.

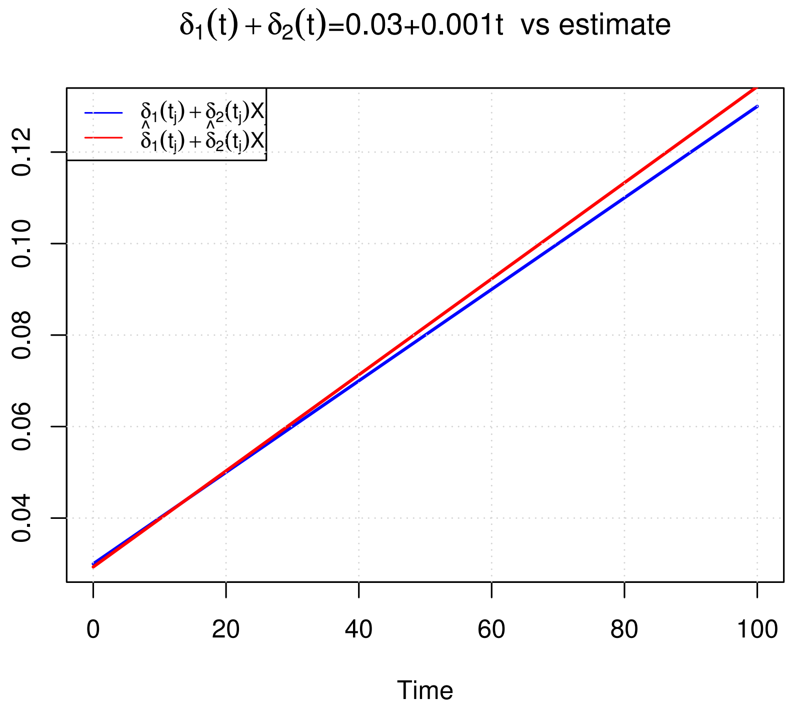

Figure 4. Comparison of the theoretical functions $\delta_1(t) + \delta_2(t) = 0.03 + 0.001t$ with its estimate

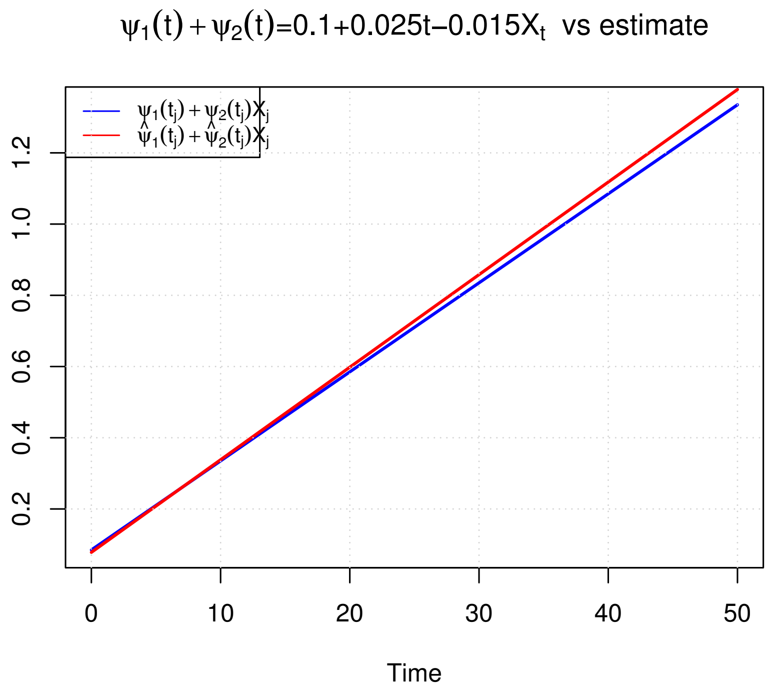

Figure 5. Comparison of the theoretical functions $\psi_1(t) + \psi_2(t)x_t = 0.1 + 0.025t - 0.015x_t$ with its estimate

The results are illustrated in Figure 6 and Figure 7. In both Figure 6 and Figure 7, we observe that the initial points of the functions' estimators exhibit significant variance, attributed to the limited number of samples employed in the estimation process.

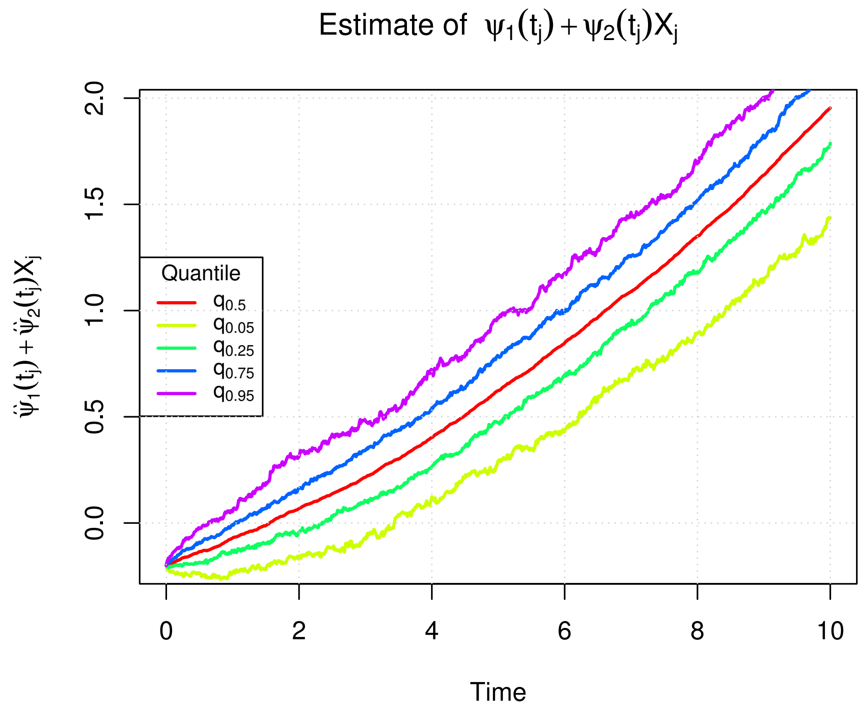

Figure 6. Estimation of $\psi_1(.)$ and $\psi_2(.)$ functions

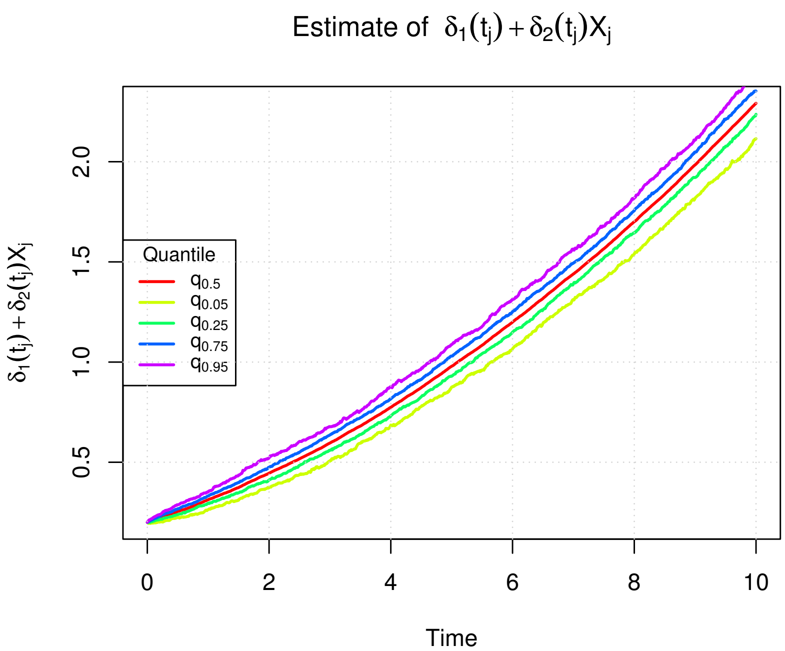

Figure 7. Estimation of $\delta_1(.)$ and $\delta_2(.)$ functions

Additionally, we depicted box-plots of the estimated parameters for the α-stable distribution, as shown in Figure 8. For each estimated parameter, we note that the medians closely align with the theoretical values, and the variance of the estimated parameters is minimal.

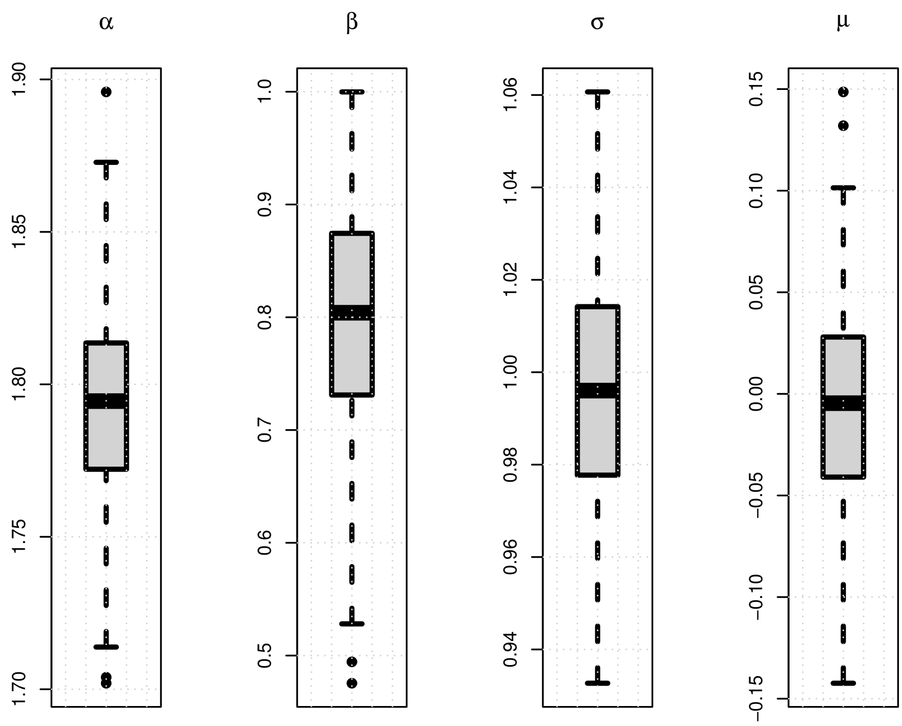

Figure 8. Box-plots of α-stable distribution's parameters' estimates for 100 Monte Carlo simulations

Real data analysis

In this section, we examine the real-time series that depict the prices of metals, specifically focusing on the price of gold. Our analysis aims to showcase the acceptability of the proposed model (designated as Model (8)), which is based on the α-stable distribution, for all the examined time series. Additionally, we present the outcomes of long-term predictions derived from the model we have developed.

Furthermore, in the process of estimation, we set Max_{iter} to be 2, indicating two iterations of estimation. We employed the tricube as the kernel function $K(.)$ in Eq. (16). It is important to emphasize that the actual data pertaining to metals' prices is utilized solely for illustrating the introduced methodology in this context. We posit that the versatility of the proposed model extends beyond metals' prices and can be effectively applied to real data originating from diverse domains.

We examine the time series associated with the gold price, comprising a dataset with 4274 observations spanning from january 01, 2007, to december 22, 2023. Figure 9 visually represents the considered data. Observing the non-stable variance apparent in the observation vector, we address this issue by transforming the data through the Box-Cox transformation [49]:

$$\forall_t \quad X_t = ln(X_t^*).$$

Here, X_t^* represents the vector of gold price data. The transformed vector is illustrated in Figure 9. The dataset is partitioned into a training time series, covering the period from the start of 2007 to the conclusion of December 31, 2018 (utilized for model parameter estimation), and a testing time series, spanning from 2019 to December 22, 2023 (utilized for model validation). The training time series comprises 3020 observations, while the validation time series consist of 1254 observations. In the initial phase, we determine the optimal hyperparameters, namely d_1^ψ, d_2^ψ, ρ_ψ, ρ_ψ^r and η, essential for minimizing the loss function (15). Following the approach outlined in Section 3, we employ MSEx (19), MSEy (18), and the Augmented Dickey–Fuller test statistic. We obtain weights $\{\omega_j\}$ (for statistics MSEx and MSEy, as per Eqs. (18) and (19)) using the exponential smoothing method [58] with a smoothing parameter $\phi = 8 \times 10^{-4}$. The calculation of weights $\{\omega_j\}$ is based on the following formula:

$$\omega_j = \frac{1-\exp(-\phi)}{1-\exp(-n\phi)} \exp(\phi(j-n)); \quad j = 1, \ldots, n. \tag{29}$$

Utilizing specified hyperparameters $(d_1^\psi = 0, d_2^\psi = 1, \rho_\psi = 750, \rho_r^\psi = 237.5, \eta = 0.7)$, we proceed to estimate the $\psi_1(.)$ and $\psi_2(.)$ functions based on Model (8) employing Eq. (22).

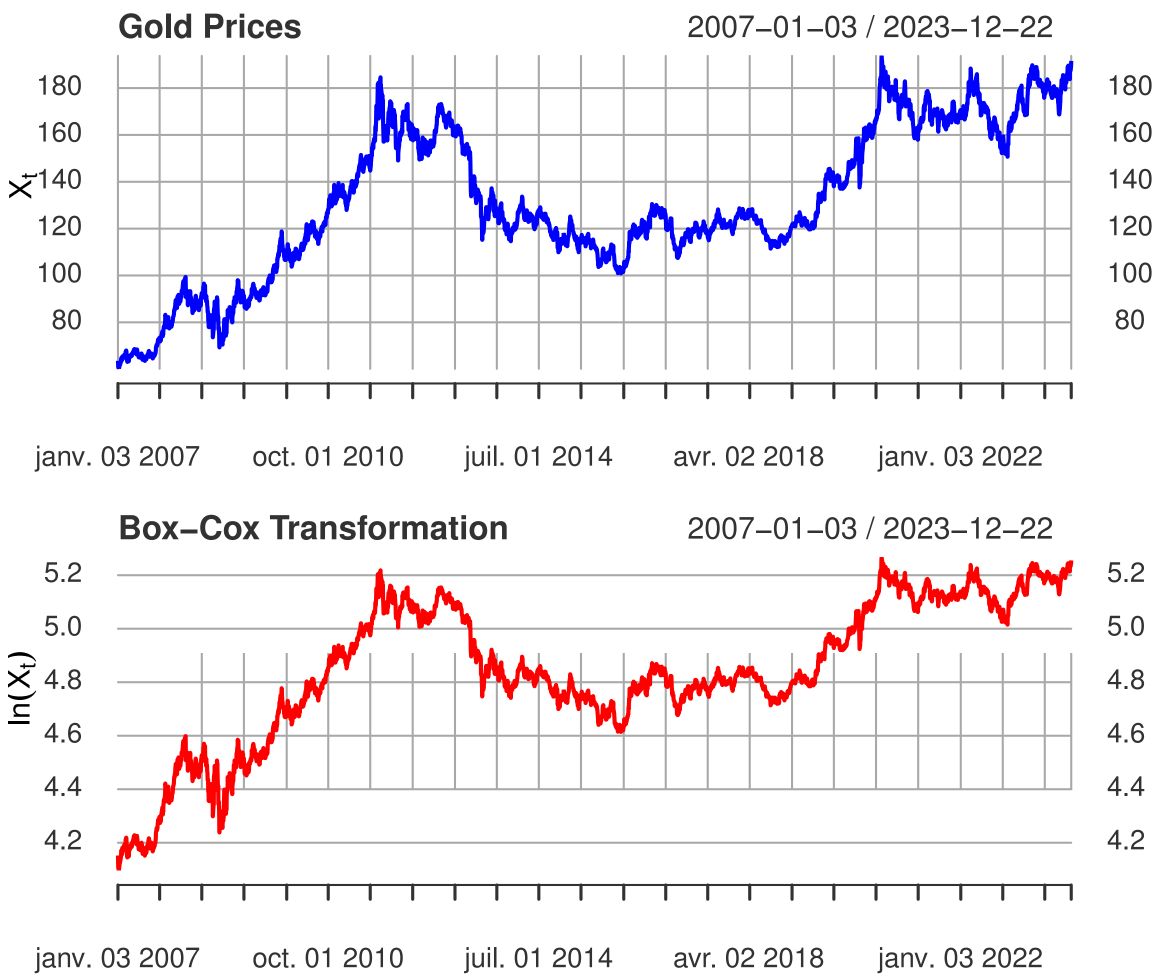

Figure 9. Evolution of Gold prices from January 01, 2007 to December 22, 2023

Table 2. Descriptive statistics Gold prices from January 01, 2007 to December 22, 2023

Data	Values						
	Min.	1st Qu.	Median	Mean	3rd Qu.	Max	S.d
Gold	60.14	113.26	125.17	131.04	161.10	193.74	31.79816

Table 3. α-Stable law parameters extracted from the GOLD prices data

Data	α-Stable law parameters			
	α	β	σ	μ
GOLD	1.85150999	0.76486510	93.81960439	0.07620218

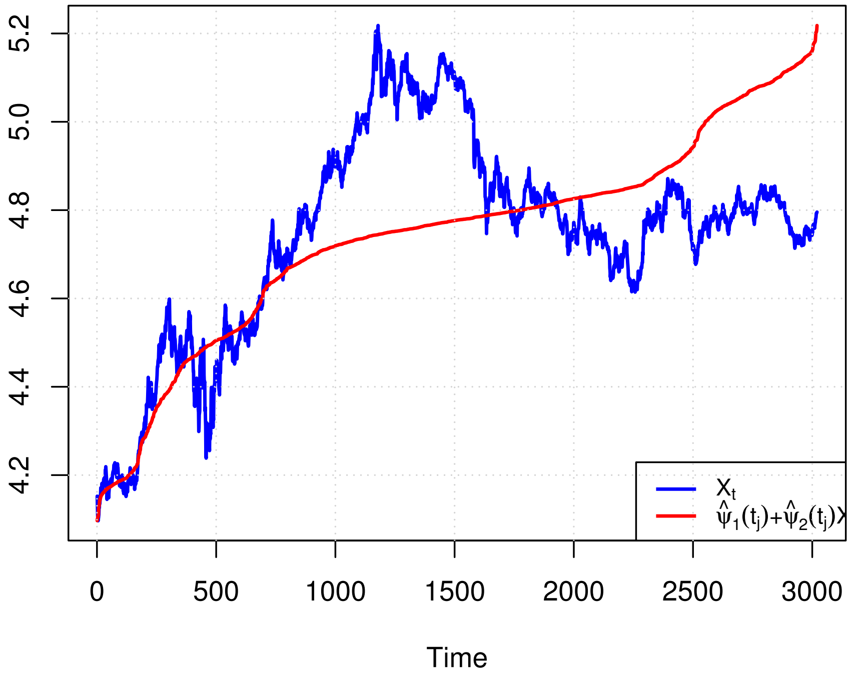

Figure 10. Comparison of estimated trend line to the examined time series from the first iteration of estimation of $\psi_1(.)$ and $\psi_2(.)$ functions

The resulting estimates are depicted in Figure 10, demonstrating a well-fitted alignment with the observed data. However, it is noteworthy that such a fit may suggest potential overfitting. The presence of heteroskedasticity in the time series introduces complexities, as changes in the process's variance during the initial iteration may be erroneously attributed to the $\psi_1(.)$ and $\psi_2(.)$ functions. To mitigate this challenge, we leverage an iterative estimation method, as discussed in Section 3.

We computed the vector $\{\hat{e}_j\}$ using formula (25) and determined the optimal hyperparameters for estimating the $\delta_1(.)$ and $\delta_2(.)$ functions as follows: $d_1^{\delta} = 0, d_2^{\delta} = 0, \rho_{\delta} = 1000, \rho_r^{\delta} = 10$. The optimal estimates were obtained by maximizing the function (24).

Subsequently, based on these estimates, their composition was calculated and visually presented in Figure 11, alongside a series of $\{\hat{e}_j\}$ for comparative analysis. The results reveal that the composition of the estimates for $\delta_1(.)$ and $\delta_2(.)$ functions serves as a reliable approximation of the standard deviation of the observed time series.
Continuing, we iterate through the previously outlined steps, incorporating:

$$\zeta^{(1)}_{j,\delta} = \hat{\delta}_1(t_j) + \hat{\delta}_1(t_j)X_j.$$

The ensuing results are as follows:

i. The chosen hyperparameters for estimating $\psi_1(.)$ and $\psi_2(.)$ functions: $d_1^{\psi} = 0, d_2^{\psi} = 0, \rho_{\psi} = 937.5, \rho_r^{\psi} = 1.25, \eta = 8 \times 10^{-3}$;

ii. The resulting estimates for $\psi_1(.)$ and $\psi_2(.)$ functions (refer to Figure 12);

iii. Selected hyperparameters for estimating $\delta_1(.)$ and $\delta_2(.)$ functions $d_1^{\delta} = 1, d_2^{\delta} = 0, \rho_{\delta} = 1437.5, \rho_r^{\delta} = 60$;

iv. The estimated $\delta_1(.)$ and $\delta_2(.)$ functions (see Figure 13).

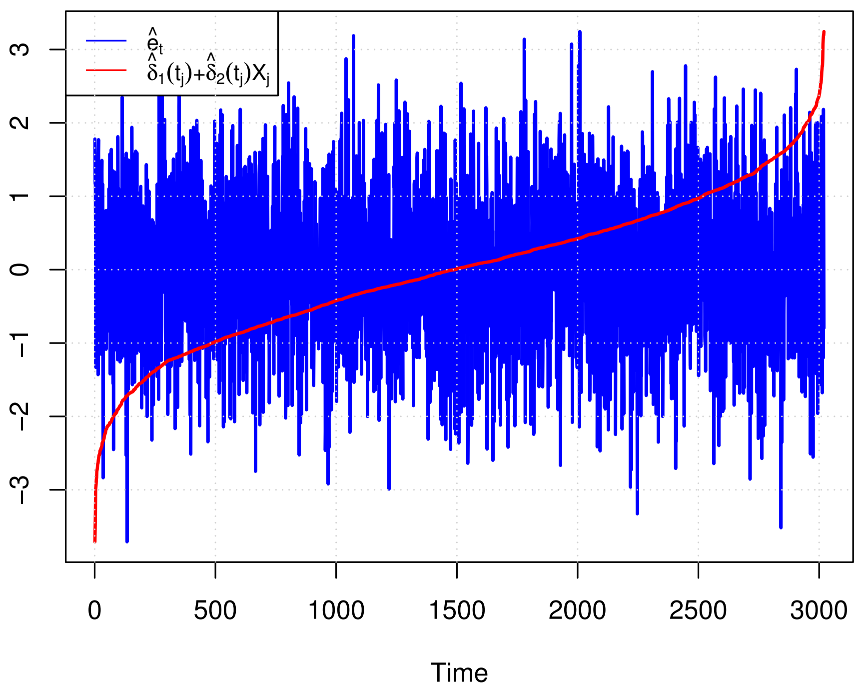

Figure 11. Comparison of composition $\{\delta_1(t_j) + \delta_2(t_j)X_j\}$ with $\{\hat{e}_j\}$ series from the first iteration of estimation of $\delta_1(.)$ and $\delta_2(.)$ functions

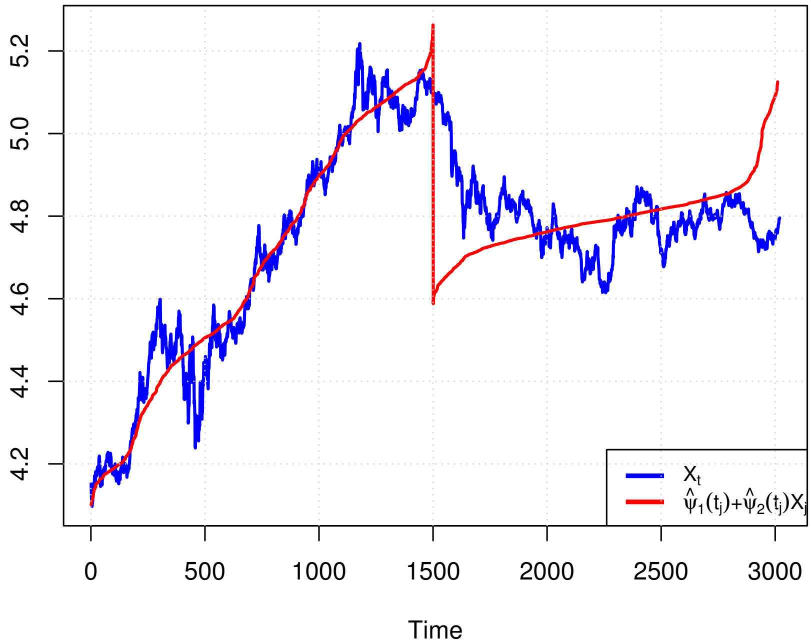

Figure 12. Comparison of the estimated trend line to the examined time series from the second iteration of estimation of $\psi_1(.)$ and $\psi_2(.)$ functions

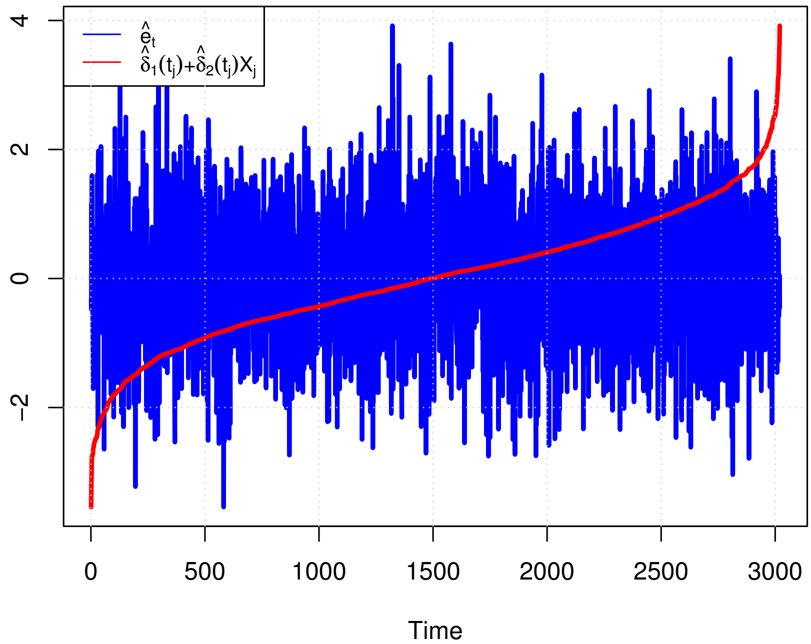

Figure 13. Comparison of composition $\{\delta_1(t_j) + \delta_2(t_j)X_j\}$ with $\{\hat{e}_j\}$ series from the second iteration of estimation of $\delta_1(.)$ and $\delta_2(.)$ functions

5 Conclusion

In this study, our focus has been on modeling the prices of metals for long-term predictions, specifically addressing the significant risk factors associated with metals, such as the price of gold, relevant to the KGHM mining company. We have delved into the analysis of a general time-inhomogeneous stochastic process grounded in the α-stable distribution. This model serves as an extension of the classical Ornstein–Uhlenbeck process and the CKLS model previously investigated in our prior work [66].

Within the examined model, we incorporate time-dependent parameters and exhibit non-Gaussian behavior, aligning with the observed characteristics in metals' prices—namely, time-dependent features (mean and scale) and a heavy-tailed (non-Gaussian) distribution. Consequently, the proposed stochastic model is anticipated to outperform classical models with fixed coefficients and Gaussian behavior.

The primary objective of this research has been to introduce a model with time-dependent coefficients based on the α-stable distribution and to propose a novel estimation procedure. Through Monte Carlo simulations, we have demonstrated the effectiveness of the proposed estimation algorithm in describing data. Furthermore, to underscore the universality of the proposed stochastic process, we have applied Model to actual data related to metals' prices, using them to illustrate the new methodology.

It is essential to note that while we have utilized metals' prices for illustration, the generality and universality of this model extend beyond financial data description. We recognize significant potential for applying the proposed model to datasets where key characteristics, such as mean or scale, undergo temporal changes, coupled with the presence of non-Gaussian behavior within the observation vector.

Declarations

Use of AI tools

The authors wish to explicitly state that they have not employed any Artificial Intelligence (AI) tools or methodologies during the conception, research, drafting, or any phase of the creation of this article. The entire content has been meticulously crafted by the authors without reliance on AI assistance or automation.

Data availability statement

The real dataset used in this work is available online and free of charge. The R codes for the simulation study are available from the author on reasonable request.

Ethical approval (optional)

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

Funding

No funding was obtained for this study.

Author's contributions

C.D.B.: Conceptualization, Methodology, Software, Validation, Data Curation, Writing - Original. C.G. and M.E.K.: Writing - Review & Editing, Supervision. All authors have read and agreed to the published version of the manuscript.

Acknowledgements

The authors express their sincere gratitude to the associate editor and referees for their thorough review and valuable comments and suggestions, which significantly contributed to enhancing the quality of the paper.

References

[1] Din, A., Sabbar, Y. and Wu, P. A novel stochastic Hepatitis B virus epidemic model with second-order multiplicative α-stable noise and real data. *Acta Mathematica Scientia*, 44, 752-788, (2024). [CrossRef]

[2] Zhang, G.P. Time series forecasting using a hybrid ARIMA and neural network model. *Neurocomputing*, 50, 159–175, (2003). [CrossRef]

[3] Nisar, K.S. and Sabbar, Y. Long-run analysis of a perturbed HIV/AIDS model with antiretroviral therapy and heavy-tailed increments performed by tempered stable Lévy jumps. *Alexandria Engineering Journal*, 78, 498-516, (2023). [CrossRef]

[4] Sabbar, Y., Khan, A., Din, A. and Tilioua, M. New method to investigate the impact of independent quadratic α-stable Poisson jumps on the dynamics of a disease under vaccination strategy. *Fractal and Fractional*, 7(3), 226, (2023). [CrossRef]

[5] Sabbar, Y. Asymptotic extinction and persistence of a perturbed epidemic model with different intervention measures and standard Lévy jumps. *Bulletin of Biomathematics*, 1(1), 58-77, (2023). [CrossRef]

[6] Ru, Y. and Ren, H.J. Application of ARMA model in forecasting aluminum price. in: Applied Mechanics and Materials, Vol. 155, Trans Tech Publ, pp. 66-71, (2012). [CrossRef]

[7] Rossen, A. What are metal prices like? Co-movement, price cycles and long-run trends. *Resources Policy*, 45, 255–276, (2015). [CrossRef]

[8] Haque, M.A., Topal, E. and Lilford, E. Iron ore prices and the value of the Australian dollar. *Mining Technology*, 124(2), 107-120, (2015). [CrossRef]

[9] Cortez, C.T., Saydam, S., Coulton, J. and Sammut, C. Alternative techniques for forecasting mineral commodity prices. *International Journal of Mining Science and Technology*, 28(2), 309-322, (2018). [CrossRef]

[10] Lee, J., List, J.A. and Strazicich, M.C. Non-renewable resource prices: Deterministic or stochastic trends?. *Journal of Environmental Economics and Management*, 51(3), 354–370, (2006). [CrossRef]

[11] Uhlenbeck, G.E. and Ornstein, L.S. On the theory of the Brownian motion. *Physical Review*, 36(5), 823, (1930). [CrossRef]

[12] Vasicek, O. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2), 177-188, (1977). [CrossRef]

[13] Wylomanska, A. The dependence structure for symmetric α-stable CARMA (p, q) processes. In Proceedings, *Workshop on Cyclostationary Systems and Their Applications (CSTA)*, pp. 189-206, Springer, (2014, February).

[14] Brockwell, P.J. Continuous-time ARMA processes. *HandBook of Statistics*, 19, 249-276, (2001). [CrossRef]

[15] Tully, E. and Lucey, B.M. A power GARCH examination of the gold market. *Research in International Business and Finance*, 21(2), 316-325, (2007). [CrossRef]

[16] Wyłomańska, A. Measures of dependence for Ornstein–Uhlenbeck process with tempered stable distribution. *Acta Physica Polonica B*, 42(10), 2049-2062, (2011). [CrossRef]

[17] Obuchowski, J. and WYŁOMANSKA, A. The Ornstein–Uhlenbeck process with non-Gaussian structure. *Acta Physica Polonica B*, 44(5), 1123-1136, (2013). [CrossRef]

[18] Nolan, J.P. Computational aspects of stable distributions. *Wiley Interdisciplinary Reviews: Computational Statistics*, 14(4), e1569, (2021). [CrossRef]

[19] Zolotarev V.M., On representation of stable laws by integrals. *Selected Translation in Mathematical Statistics and Probability*, 6, 84-88, (1966).

[20] McCulloch, J.H. Simple consistent estimators of stable distribution parameters. *Communications in Statistics-Simulation and Computation*, 15(4), 1109–1136, (1986). [CrossRef]

[21] Ho, T.S.Y., Lee, S.B. Term structure movements and pricing interest rate contingent claims. *The Journal of Finance*, 41(5), 1011-1029, (1986). [CrossRef]

[22] Hull, J. and White, A. Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3(4), 573-592, (1990). [CrossRef]

[23] Nolan, J.P. *Univariate Stable Distributions: Models for Heavy Tailed Data*. Springer: Switzerland, (2020). [CrossRef]

[24] Zhang, S. and Zhang, X. A least squares estimator for discretely observed Ornstein-Uhlenbeck processes driven by symmetric α-stable motions. *Annals of the Institute of Statistical Mathematics*, 65, 89-103, (2013). [CrossRef]

[25] Hu, Y. and Long, H. Parameter estimation for Ornstein–Uhlenbeck processes driven by α-stable Lévy motions. *Communications on Stochastic Analysis*, 1(2), 175-192, (2007). [CrossRef]

[26] Cui, H. Estimation in partial linear EV models with replicated observations. *Science in China Series A: Mathematics*, 47, 144, (2004). [CrossRef]

[27] Fan, J., Jiang, J., Zhang, C. and Zhou, Z. Time-dependent diffusion models for term structure dynamics. *Statistica Sinica*, 13(4), 965-992, (2003). [CrossRef]

[28] Lévy P. Théorie des erreurs. La loi de Gauss et les lois exceptionnelles. *Bulletin de la Société Mathématique de France*, 52, 49-85, (1924). [CrossRef]

[29] McCulloch, J.H. 13 Financial applications of stable distributions. *Handbook of Statistics*, 14, 393-425, (1996). [CrossRef]

[30] Mittnik, S., Rachev, S.T., Doganoglu, T. and Chenyao, D. Maximum likelihood estimation of stable Paretian models. *Mathematical and Computer Modelling*, 29(10-12), 275–293, (1999). [CrossRef]

[31] Nolan, J.P. Modeling financial data with stable distributions. In *Handbook of Heavy Tailed Distributions in Finance* (pp. 105–130). North-Holland, Holland: Elsevier, (2003). [CrossRef]

[32] Rachev, S.T. and Mittnik, S. *Stable Paretian models in finance.* New York: Wiley: (2000).

[33] Samorodnitsky, G., Taqqu, M.S. and Linde, R.W. *Stable non-gaussian random processes: stochastic models with infinite variance.* New York; London: Chapman & Hall, (1994).

[34] Zolotarev, A. *One-Dimensional Stable Distributions.* USA: American Mathematical Society, Providence, (1986).

[35] Fama, E.F. and Roll, R. Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association*, 66, 331–338, (1971). [CrossRef]

[36] Mandelbrot, B.B. The variation of certain speculative prices. In: *Fractals and Scaling in Finance.* New York: Springer, (1997). [CrossRef]

[37] Bachelier, L. *Théorie de la spéculation.* Annales scientifiques de l'École Normale Supérieure, Serie 3, Vol 17, 21-86, (1900). [CrossRef]

[38] Chambers, J.M., Mallows, C.L. and Stuck, B.W. A method for simulating stable random variables. *Journal of the American Statistical Association*, 71, 340–344, (1976). [CrossRef]

[39] Koutrouvelis, I.A. An iterative procedure for the estimation of the parameters of stable laws. *Communications in Statistics-Simulation and Computation*, 10, 17-28, (1981). [CrossRef]

[40] Koutrouvelis, I.A. Regression-type estimation of the parameters of stable laws. *Journal of the American Statistical Association*, 75, 918–928, (1980). [CrossRef]

[41] Press, S.J. Estimation in univariate and multivariate stable distributions. *Journal of the American Statistical Association*, 67, 842–846, (1972). [CrossRef]

[42] Weron, R. *Performance of the estimators of stable law parameters.* Hugo Steinhaus Center, Wroclaw University of Technology, HSC Research Reports, HSC/95/01, (1995).

[43] Nolan, J.P. Numerical calculation of stable densities and distribution functions. *Communications in Statistics. Stochastic Models*, 13(4), 759–774, (1997). [CrossRef]

[44] Weron, A. and Weron, R. *Inzynieria finansowa.* HSC Books: Wydawnictwo Naukowo-Techniczne, Warszawa, (1998).

[45] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion* (Vol. 293). Springer: Berlin, pp. 14-39, (1991).

[46] Merton, R.C. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1), 141-183, (1973). [CrossRef]

[47] Brennan, M.J. and Schwartz, E.S. An equilibrium model of bond pricing and a test of market efficiency. *Journal of Financial and Quantitative Analysis*, 17(3), 301-329, (1982). [CrossRef]

[48] Dothan, L.U. On the term structure of interest rates. *Journal of Financial Economics*, 6(1), 59-69, (1978). [CrossRef]

[49] Cox, J.C., Ingersoll Jr., J.E. and Ross, S.A. A theory of the term structure of interest rates. *Econometrica*, 53(2), 385-407, (1985).

[50] Black, F. and Karasinski, P. Bond and option pricing when short rates are lognormal. *Financial Analysts Journal*, 47(4), 52-59, (1991). [CrossRef]

[51] Hastie, T., Friedman, J. and Tibshirani, R. *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. New York: Springer, (2009). [CrossRef]

[52] Marsden, J. and Weinstein, A. *Calculus II*, Springer: New York, (1985).

[53] Saleh, A.M.E., Arashi, M. and Tabatabaey, S.M.M. *Statistical Inference for Models with Multivariate T-Distributed Errors*. John Wiley & Sons: USA, (2014).

[54] Cont, R. *Encyclopedia of Quantitative Finance.* (Vol. 4). John Wiley & Sons: USA, (2010).

[55] W Cleveland, W.S. Robust locally weighted regression and smoothing scatterplots. *Journal of the American Statistical Association*, 74(368), 829-836, (1979). [CrossRef]

[56] Jaditz, T. and Riddick, L.A. Time-series near-neighbor regression. *Studies in Nonlinear Dynamics & Econometrics*, 4(1), 35-44, (2000). [CrossRef]

[57] Elliott, G., Rothenberg, T.J. and Stock, J.H. Efficient tests for an autoregressive unit root. *Econometrica*, 64(4), 813-836, (1996). [CrossRef]

[58] Pozzi, F., Di Matteo, T. and Aste, T. Exponential smoothing weighted correlations. *The European Physical Journal B*, 85(175), 1-21, (2012). [CrossRef]

[59] Breusch, T.S. and Pagan, A.R. The lagrange multiplier test and its applications to model specifications in econometrics. *The Review of Economic Studies*, 47(1), 239-253, (1980). [CrossRef]

[60] Nolan, J.P. Maximum likelihood estimation of stable parameters. In *Levy processes: Theory and applications*. (pp. 379–400). Boston: Birkhauser, (2001).

[61] Tan, P. and Drossos, C. Invariance properties of maximum likelihood estimators. *Mathematics Magazine*, 48, 37-41, (1975). [CrossRef]

[62] Shanno, D.F. Conditioning of quasi-Newton methods for function minimization. *Mathematics of Computation*, 24(111), 647-656, (1970).

[63] Fox, L. and Mayers, D.F. *Numerical Solution of Ordinary Differential Equations*. Chapman and Hall: London, (1987). [CrossRef]

[64] Stephens, M.A. EDF statistics for goodness of fit and some comparisons. *Journal of the American Statistical Association* , 69, 730-737, (1974). [CrossRef]

[65] Brandimarte, P. *HandBook in Monte Carlo Simulation: Applications in Financial Engineering, Risk Management, and Economics*. John Wiley & Sons: USA,(2014).

[66] Sikora, G., Michalak, A., Bielak, Ł., Miśta, P. and Wyłomańska, A. Stochastic modeling of currency exchange rates with novel validation techniques. *Physica A: Statistical Mechanics and its Applications*, 523, 1202-1215, (2019). [CrossRef]

Mathematical Modelling and Numerical Simulation with Applications (MMNSA)
(https://dergipark.org.tr/en/pub/mmnsa)

How to cite this article: Coulibaly, B.D., Ghizlane, C. & Khomssi, M.E. (2024). An approach to stochastic differential equations for long-term forecasting in the presence of α-stable noise: an application to gold prices. *Mathematical Modelling and Numerical Simulation with Applications*, 4(2), 165-192. https://doi.org/10.53391/mmnsa.1416148