Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 54 (3) (2025), 912 – 920
DOI : 10.15672/hujms.1416602

Research Article

Certain observations on local properties of topological spaces

Debraj Chandra*, Nur Alam

Department of Mathematics, University of Gour Banga, Malda-732103, West Bengal, India

Abstract

Let $\mathcal{P}$ be any topological property of a space X. We say that X is $\mathcal{P}$ at $x \in X$ if there exist an open set U and a subspace Y of X satisfying $\mathcal{P}$ such that $x \in U \subseteq Y$. We also say that X is locally $\mathcal{P}$ if X is $\mathcal{P}$ at every point of X. We study this local property and obtain the following results under certain topological assumptions on $\mathcal{P}$.

(1) Every locally $\mathcal{P}$ Hausdorff P-space can be densely embedded in a $\mathcal{P}$ Hausdorff P-space.
(2) If a Hausdorff P-space X is $\mathcal{P}$ at $x \in X$, then $\chi(x, X) \leq \psi(x, X)^\omega$.
(3) For a locally $\mathcal{P}$ Hausdorff P-space X, $w(X) \leq nw(X)^\omega \leq |X|^\omega$.

Besides, few separation like properties are obtained and preservation under certain topological operations are also investigated. Finally we present certain observations on remainders of locally $\mathcal{P}$ spaces.

Mathematics Subject Classification (2020). Primary: 54D20; Secondary: 54B15, 54C10, 54D40, 54D99

Keywords. $\mathcal{P}$ at x, locally $\mathcal{P}$, locally Lindelöf, local properties, remainder

1. Introduction

By a space X we always mean a topological space. All notation and terminology not defined in this paper are given in [9, 19]. This article deals with the local variant of a topological property $\mathcal{P}$. Given a selective property $\mathcal{P}$ its local version have been recently studied in [1,2,8] for the case of Menger, star-Menger, Menger-bounded, Hurewicz-bounded and Rothberger-bounded properties. For the notions of such selective covering properties we refer the reader to consult the papers [11–13].

Let $\mathcal{P}$ be any topological property of a space X. We say that X is a $\mathcal{P}$ space (or, in short X is $\mathcal{P}$) if X has the property $\mathcal{P}$. We now give the main definition of the paper.

Definition 1.1. We say that X is $\mathcal{P}$ at $x \in X$ if there exist an open set U and a subspace Y of X satisfying $\mathcal{P}$ such that $x \in U \subseteq Y$. We also say that X is locally $\mathcal{P}$ if X is $\mathcal{P}$ at every point of X.

*Corresponding Author.
Email addresses: debrajchandra1986@gmail.com (D. Chandra), nurrejwana@gmail.com (N. Alam)
Received: 08.01.2024; Accepted: 03.08.2024

Note that a space X is $\mathcal{P}$ implies X is locally $\mathcal{P}$. In this article we investigate the property $\mathcal{P}$ at x of a space X for arbitrary topological property $\mathcal{P}$. We present reformulations of the local version of any such $\mathcal{P}$ in the context of regular spaces and Hausdorff P-spaces as well (recall that a space is called a P-space if every G_δ set is open). We observe that a locally $\mathcal{P}$ Hausdorff P-space can be densely embedded in a $\mathcal{P}$ Hausdorff P-space. Relations between character and pseudocharacter of a point, and weight and network weight are established in this context. We also obtain some separation like properties. A few intriguing investigations on preservation under certain topological operations are carefully carried out. We also present certain observations on remainders of this local variant.

2. Preliminaries

The weight $w(X)$ of X is the smallest possible cardinality of a base for X and the character $\chi(x, X)$ of a point x in X is the smallest cardinality of a local base for x. A family $\mathcal{N}$ of subsets of X is said to be a network for X if for each $x \in X$ and any neighbourhood U of x there exists a $A \in \mathcal{N}$ such that $x \in A \subseteq U$. The network weight $nw(X)$ of X is defined as the smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network for X. Clearly $nw(X) \leq w(X)$ and $nw(X) \leq |X|$. A family $\mathcal{U}$ of open sets of a T_1 space X is called a pseudobase for X at $x \in X$ if $\cap\mathcal{U} = \{x\}$. The pseudocharacter $\psi(x, X)$ of a point x in a T_1 space X is the smallest cardinality of a pseudobase for X at x.

Recall that a $\mathcal{A} \subseteq P(\mathbb{N})$ is said to be an almost disjoint family if each $A \in \mathcal{A}$ is infinite and for any two distinct elements $B, C \in \mathcal{A}$, $|B \cap C| < \omega$. For an almost disjoint family $\mathcal{A}$, let $\Psi(\mathcal{A}) = \mathcal{A} \cup \mathbb{N}$ be the Isbell-Mrówka space [15]. A space X is said to have the Rothberger property [11, 17] if for each sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence (U_n) such that $U_n \in \mathcal{U}_n$ for each n and $\{U_n : n \in \mathbb{N}\}$ covers X. Note that the Rothberger property is preserved under F_σ subsets, countable unions and continuous mappings [11].

3. Main results

3.1. The locally $\mathcal{P}$ property

We start by observing that if a property $\mathcal{P}$ implies a property $\mathcal{Q}$, then X is locally $\mathcal{P}$ implies X is locally $\mathcal{Q}$. Note that an uncountable discrete space is locally compact but not Lindelöf. Accordingly for any property $\mathcal{P}$ between compactness and the Lindelöf property the properties $\mathcal{P}$ and locally $\mathcal{P}$ are different. Also note that if $\mathcal{P}$ implies the Lindelöf property and $\mathcal{P}$ is closed under countable unions, then the properties $\mathcal{P}$ and locally $\mathcal{P}$ coincide. So, in this case, to distinguish between the local properties is equivalent to distinguish between the original properties.

A space X is said to be regular with respect to $x \in X$ if for each closed set F not containing x there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$ (or equivalently, for each open set U containing x there exists an open set V containing x such that $\overline{V} \subseteq U$).

Lemma 3.1. *If X is regular with respect to x and $\mathcal{P}$ is inherited by closed subspaces, then the following statements are equivalent.*

(1) *X is $\mathcal{P}$ at x.*
(2) *For every open set V containing x there are an open set U and a subspace Y of X satisfying $\mathcal{P}$ such that $x \in U \subseteq Y \subseteq V$.*
(3) *There is an open set U containing x such that $\overline{U}$ has $\mathcal{P}$.*
(4) *X has a base at x consisting of closed neighbourhoods of x satisfying $\mathcal{P}$.*

Note that every Lindelöf subspace of a Hausdorff P-space is closed.

Lemma 3.2 (Folklore)**.** *If a subspace Y of a Hausdorff P-space X is Lindelöf at any point $y \in Y$, then Y is of the form $U \cap F$ where U is open and F is closed in X.*

Remark 3.3. Let X be a Hausdorff P-space. If $\mathcal{P}$ is inherited by closed subspaces and $\mathcal{P}$ implies the Linelöf property, then X is $\mathcal{P}$ at x implies that X is regular with respect to x.

Theorem 3.4. *Let $\mathcal{P}$ be a property of a space X satisfying that if there exists a point x of X such that the complement of each open neighbourhood of x has $\mathcal{P}$, then X has $\mathcal{P}$. If in addition $\mathcal{P}$ implies the Lindelöf property, and $\mathcal{P}$ is invariant under closed subspaces and countable unions, then every locally $\mathcal{P}$ Hausdorff P-space can be densely embedded in a $\mathcal{P}$ Hausdorff P-space.*

Proof. Consider a locally $\mathcal{P}$ Hausdorff P-space (X,τ). Suppose that X does not satisfy $\mathcal{P}$. Let $X' = X \cup \{p\}$, where $p \notin X$. Clearly $\tau' = \tau \bigcup \{U \subseteq X' : X' \setminus U$ is a $\mathcal{P}$ subspace of $X\}$ is a topology on X'. We now show that X' is a Hausdorff P-space. Choose $x, y \in X'$ such that $x \in X$ and $y \notin X$. Let U be an open set and Y be a $\mathcal{P}$ subspace of X such that $x \in U \subseteq Y$. Thus we obtain two disjoint open sets U and $X' \setminus Y$ in X' with $x \in U$ and $y \in X' \setminus Y$. Hence X' is Hausdorff. Obviously X' is a P-space. Also observe that X is dense in X'. The inclusion mapping $\iota : X \to X'$ is an embedding of X into X'. From the construction of X' we can say that X' satisfies $\mathcal{P}$. □

Theorem 3.5. *Let X be a Hausdorff P-space. If X is Lindelöf at x, then $\chi(x,X) \leq \psi(x,X)^{\omega}$.*

Proof. Let W be a Lindelöf neighbourhood of x. Since $\chi(x,X) = \chi(x,W)$ and $\psi(x,X) = \psi(x,W)$, we can assume that X is Lindelöf. Let $\mathcal{B}$ be a pseudobase for X at x of cardinality $\psi(x,X)$ consisting of closed neighbourhoods of x and let $\mathcal{I}$ be the family all intersections of countable subfamilies of $\mathcal{B}$. If U is a neighbourhood of x, then $X \setminus U \subseteq \bigcup\{X \setminus B : B \in \mathcal{B}\} = \bigcup\{X \setminus B_n : n \in \mathbb{N}\}$, where $B_n \in \mathcal{B}$ for each $n \in \mathbb{N}$. So $x \in I = \bigcap\{B_n : n \in \mathbb{N}\} \subseteq U$. It follows that $\mathcal{I}$ is a base for X at x. Thus $\chi(x,X) \leq \psi(x,X)^{\omega}$ because $|\mathcal{I}| \leq \psi(x,X)^{\omega}$. □

Corollary 3.6. *Let $\mathcal{P}$ be any property stronger than the Lindelöf property and X be a Hausdorff P-space. If X is $\mathcal{P}$ at x, then $\chi(x,X) \leq \psi(x,X)^{\omega}$.*

Lemma 3.7 ([2, Lemma 3.1])**.** *Let X be a Hausdorff P-space.*

(1) *There exists a continuous bijective mapping of X onto a Hausdorff P-space Y such that $w(Y) \leq nw(X)^{\omega}$.*
(2) *Moreover if X is Lindelöf, then $w(X) \leq nw(X)^{\omega}$.*

Theorem 3.8. *For a locally Lindelöf Hausdorff P-space X, $w(X) \leq nw(X)^{\omega}$.*

Proof. Assume that $nw(X) = \kappa$. Let $\mathcal{N}$ be a network for X such that $|\mathcal{N}| = \kappa$. For each $x \in X$ pick an open set V_x in X containing x with $\overline{V_x}$ is Lindelöf. Later for each $x \in X$ choose a $A_x \in \mathcal{N}$ with $x \in A_x \subseteq V_x$. It follows that there exists a collection $\{A_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{N}$ such that for each $\alpha \in \Lambda$, $\overline{A_\alpha}$ is Lindelöf and $X = \bigcup_{\alpha\in\Lambda} A_\alpha$. For each $\alpha \in \Lambda$ one can easily obtain an open set V_α in X with $\overline{A_\alpha} \subseteq V_\alpha$ and $\overline{V_\alpha}$ is Lindelöf. By Lemma 3.7(2), $w(\overline{V_\alpha}) \leq \kappa^{\omega}$ because $nw(\overline{V_\alpha}) \leq nw(X) = \kappa$. Thus $w(V_\alpha) \leq \kappa^{\omega}$, i.e. there is a base $\mathcal{B}_\alpha$ for V_α such that $|\mathcal{B}_\alpha| \leq \kappa^{\omega}$. Since $\mathcal{B} = \bigcup_{\alpha\in\Lambda} \mathcal{B}_\alpha$ is a base for X and $|\mathcal{B}| \leq \kappa^{\omega}$, we get $w(X) \leq \kappa^{\omega}$, i.e. $w(X) \leq nw(X)^{\omega}$. □

Corollary 3.9. *Let $\mathcal{P}$ be any property stronger than the Lindelöf property and X be a Hausdorff P-space. If X is locally $\mathcal{P}$, then $w(X) \leq nw(X)^{\omega} \leq |X|^{\omega}$.*

In this connection we mention the classical result of F. Galvin, given in [10]. If X is a Lindelöf space, then X is a P-space if and only if X is a γ-set. If $\mathcal{P}$ lies between Lindelöf and γ-set, then any P-space X is locally $\mathcal{P}$ if and only if X is locally Lindelöf.

3.2. Separation like properties

Theorem 3.10. *If $\mathcal{P}$ is preserved under closed subspaces and countable unions and X is a regular locally $\mathcal{P}$ space, then for each Lindelöf subspace L of X and each $x \in X \setminus L$ there exists a subset $B \subseteq X$ satisfying $\mathcal{P}$ such that $L \subseteq B \subseteq X \setminus \{x\}$.*

Proof. For each $y \in L$ choose an open set U_y such that $x \notin U_y$. By Lemma 3.1, we get an open subset V_y and a $\mathcal{P}$ subspace B_y of X such that $y \in V_y \subseteq B_y \subseteq U_y$. Then $\{V_y : y \in L\}$ is a cover of L by open sets in X and hence there is a countable subfamily $\{V_{y_n} : n \in \mathbb{N}\}$ that covers L. Thus $B = \bigcup_{n\in\mathbb{N}} B_{y_n}$ is a $\mathcal{P}$ subspace of X such that $L \subseteq B \subseteq X \setminus \{x\}$. □

Corollary 3.11. *If $\mathcal{P}$ implies the Lindelöf property and is preserved under closed subspaces and countable unions, and X is a locally $\mathcal{P}$ Hausdorff P-space, then for each Lindelöf subspace L of X and each $x \in X \setminus L$ there exists a closed subset $B \subseteq X$ satisfying $\mathcal{P}$ such that $L \subseteq B \subseteq X \setminus \{x\}$.*

Theorem 3.12. *If $\mathcal{P}$ is preserved under closed subspaces and finite unions and X is a regular locally $\mathcal{P}$ space, then for each compact subspace C of X and each $x \in X \setminus C$ there exists a closed subset $B \subseteq X$ satisfying $\mathcal{P}$ such that $C \subseteq B \subseteq X \setminus \{x\}$.*

Theorem 3.13. *If $\mathcal{P}$ implies the Lindelöf property and is preserved under closed subspaces and finite unions, and X is a regular locally $\mathcal{P}$ space, then for each compact subspace C and each open subset V of X with $C \subseteq V$ there exists a closed subset $B \subseteq X$ satisfying $\mathcal{P}$ such that $C \subseteq B \subseteq V$. Moreover there exists a continuous function $f : X \to [0,1]$ satisfying $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $X \setminus B$.*

Proof. For each $x \in C$ choose an open set U_x such that $x \in U_x \subseteq \overline{U_x} \subseteq V$ and $\overline{U_x}$ satisfies $\mathcal{P}$. Since C is compact, we get a finite subset $F \subseteq C$ such that $C \subseteq \bigcup_{x\in F} U_x$. Thus $B = \bigcup_{x\in F} \overline{U_x}$ is a closed $\mathcal{P}$ (hence normal) subspace of X with $C \subseteq B \subseteq V$. Observe that $B \setminus \mathrm{Int}(B)$ and C are disjoint closed subsets of B. Since B is normal, there exists a continuous function $g : B \to [0,1]$ with $g(x) = 0$ for all $x \in C$ and $g(x) = 1$ for all $x \in B \setminus \mathrm{Int}(B)$. We define a continuous function $f : X \to [0,1]$ by $f(x) = g(x)$ if $x \in B$ and $f(x) = 1$ otherwise. This completes the proof. □

Theorem 3.14. *If $\mathcal{P}$ implies the Lindelöf property and is preserved under closed subspaces and countable unions, and X is a locally $\mathcal{P}$ Hausdorff P-space, then for each Lindelöf subspace L and each open subset V of X with $L \subseteq V$ there exists a closed subset $B \subseteq X$ satisfying $\mathcal{P}$ such that $L \subseteq B \subseteq V$. Moreover there exists a continuous function $f : X \to [0,1]$ satisfying $f(x) = 0$ for all $x \in L$ and $f(x) = 1$ for all $X \setminus B$.*

3.3. Preservation under certain topological operations

Observe that if $\mathcal{P}$ is preserved under F_σ (respectively, closed, clopen) subsets and if a space X is $\mathcal{P}$ at $x \in X$, then any F_σ (respectively, closed, clopen) subset of X containing x is also $\mathcal{P}$ at x. If X is regular and $\mathcal{P}$ is preserved under closed subsets, then X is $\mathcal{P}$ at x implies any locally closed subset of X containing x is also $\mathcal{P}$ at x. Moreover if $\mathcal{P}$ is preserved under closed subsets, then a locally closed subset of a regular $\mathcal{P}$ space need not be $\mathcal{P}$, the one point compactification of an uncountable discrete space is a counter example to it.

Note that if $\mathcal{P}$ is preserved under continuous mappings, then continuous image of a locally $\mathcal{P}$ space need not be locally $\mathcal{P}$. If $\mathcal{P}$ is the Rothberger property, then the identity mapping $i : X \to Y$ is continuous, where $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the usual topology, but Y is not locally $\mathcal{P}$. Recall from [14] that a surjective continuous mapping $f : X \to Y$ is called

(1) weakly perfect if f is closed and for each $y \in Y$, $f^{-1}(y)$ is Lindelöf.

(2) bi-quotient if $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open sets in X for some $y \in Y$, then $\{f(U) : U \in \mathcal{U}\}$ has a finite subset that covers some open set containing y in Y.

Clearly open continuous surjective (and also perfect) mappings are bi-quotient.

Theorem 3.15. *Let $\mathcal{P}$ be invariant under continuous mappings and countable unions. If $f : X \to Y$ is a weakly perfect mapping and X is $\mathcal{P}$ at every point of $f^{-1}(f(x))$ for some $x \in X$, then Y is $\mathcal{P}$ at $f(x)$.*

Proof. Choose $y = f(x)$ and $w \in f^{-1}(y)$. Let U_w be an open and Z_w be a $\mathcal{P}$ subspace of X such that $w \in U_w \subseteq Z_w$. Then $\{U_w : w \in f^{-1}(y)\}$ is a cover of $f^{-1}(y)$ by open sets in X. Thus we get a set $\{w_n : n \in \mathbb{N}\} \subseteq f^{-1}(y)$ such that $f^{-1}(y) \subseteq \bigcup_{n\in\mathbb{N}} U_{w_n}$ and $f^{-1}(y) \subseteq \bigcup_{n\in\mathbb{N}} Z_{w_n}$. Observe that $Y \setminus f(X \setminus \bigcup_{n\in\mathbb{N}} U_{w_n})$ is an open subset and $f(\bigcup_{n\in\mathbb{N}} Z_{w_n})$ is a $\mathcal{P}$ subspace of Y with $y \in Y \setminus f(X \setminus \bigcup_{n\in\mathbb{N}} U_{w_n}) \subseteq f(\bigcup_{n\in\mathbb{N}} Z_{w_n})$. Hence the result. □

Theorem 3.16. *Let $\mathcal{P}$ be invariant under continuous mappings and finite unions. If $f : X \to Y$ is a bi-quotient mapping and X is $\mathcal{P}$ at every point of $f^{-1}(f(x))$ for some $x \in X$, then Y is $\mathcal{P}$ at $f(x)$.*

Proof. Choose $y = f(x)$ and $w \in f^{-1}(y)$. Let U_w be an open and Z_w be a $\mathcal{P}$ subspace of X such that $w \in U_w \subseteq Z_w$. Then $\{U_w : w \in f^{-1}(y)\}$ is a cover of $f^{-1}(y)$ by open sets in X. Then we get a finite set $\{w_i : 1 \le i \le k\} \subseteq f^{-1}(y)$ and an open set $V \subseteq Y$ containing y such that $V \subseteq \bigcup_{i=1}^{k} f(U_{w_i})$. One can readily observe that $y \in \operatorname{Int} f(\bigcup_{i=1}^{k} U_{w_i})$ and $f(\bigcup_{i=1}^{k} Z_{w_i})$ is a $\mathcal{P}$ subspace of Y such that $\operatorname{Int} f(\bigcup_{i=1}^{k} U_{w_i}) \subseteq f(\bigcup_{i=1}^{k} Z_{w_i})$. Hence Y is $\mathcal{P}$ at y. □

Corollary 3.17. *Let $\mathcal{P}$ be invariant under continuous mappings and finite unions. If $f : X \to Y$ is a perfect mapping and X is $\mathcal{P}$ at every point of $f^{-1}(f(x))$ for some $x \in X$, then Y is $\mathcal{P}$ at $f(x)$.*

Also observe that if $\mathcal{P}$ is invariant under continuous mappings and $f : X \to Y$ is an open continuous mapping from X onto Y, and if X is $\mathcal{P}$ at x, then Y is $\mathcal{P}$ at $f(x)$. It follows that if $\mathcal{P}$ is invariant under continuous mappings and closed subsets, and if $f : X \to Y$ is an injective closed continuous mapping and Y is $\mathcal{P}$ at $y \in f(X)$, then X is $\mathcal{P}$ at $f^{-1}(y)$. If we replace 'injective closed continuous mapping' by 'open continuous mapping', then the result does not hold. For example, take $\mathcal{P}$ as the Rothberger property and consider the projection mapping $p_1 : X \to X_1$ $X = X_1 \times X_2$, where $X_1 = \Psi(\mathcal{A})$ is a Ψ-space and $X_2 = \mathbb{R}$ is the set of reals.

Theorem 3.18. *Let $\mathcal{P}$ be such that the collection of all $\mathcal{P}$ subspaces of a space covers the space. If $\mathcal{P}$ is invariant under continuous mappings, then for a space X the following assertions are equivalent.*

(1) *A subset U is open in X provided that $U \cap Y$ is open in Y for every $\mathcal{P}$ subspace Y of X.*
(2) *A subset F is closed in X provided that $F \cap Y$ is closed in Y for every $\mathcal{P}$ subspace Y of X.*
(3) *X is a quotient image of some locally $\mathcal{P}$ space.*

Proof. (1) ⇒ (3). If $\{Y_\alpha : \alpha \in \Lambda\}$ is the collection of all $\mathcal{P}$ subspaces of X, then $\bigoplus_{\alpha\in\Lambda} Y_\alpha$ is locally $\mathcal{P}$. Observe that $f : \bigoplus_{\alpha\in\Lambda} Y_\alpha \to X$ given by $f(x, \alpha) = x$ is a quotient mapping.

(3) ⇒ (1). Let Z be a locally $\mathcal{P}$ space and $q : Z \to X$ be a quotient mapping. Consider a set $U \subseteq X$ with $U \cap Y$ is open in Y for each $\mathcal{P}$ subspace Y of X. Pick $x \in q^{-1}(U)$, an open set V and a $\mathcal{P}$ subspace Y of Z such that $x \in V \subseteq Y$. Since $q(Y)$ is a $\mathcal{P}$ subspace of X, $U \cap q(Y)$ is open in $q(Y)$ and $U \cap q(Y) = W \cap q(Y)$ for some open set W in X. It follows that $x \in q^{-1}(W) \cap V$. Since $q^{-1}(W) \cap V$ is open in Z with $q^{-1}(W) \cap V \subseteq q^{-1}(U)$, $q^{-1}(U)$ is open in Z and so U is open in X. Hence Z satisfies (1). □

Let $\mathcal{P}$ be such that if (x_n) is a sequence in a space X convergent to some $x \in X$, then the subspace $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ satisfies $\mathcal{P}$. Note that every sequential space satisfies each of the conditions of Theorem 3.18 for such $\mathcal{P}$. Next we observe that a quotient image of a locally $\mathcal{P}$ space need not be locally $\mathcal{P}$.

Example 3.19. Let $\mathcal{P}$ be the Rothberger property. The space $X = \bigoplus_{\alpha<\omega_1}[0,1]$ is a quotient image of some locally $\mathcal{P}$ space by Theorem 3.18 (as X is a sequential space), but X is not locally $\mathcal{P}$.

Let $X = \bigcup_{\alpha\in\Lambda} X_\alpha$. If for some $\alpha \in \Lambda$, X_α is an open subspace of X such that X_α is $\mathcal{P}$ at $x \in X_\alpha$, then X is $\mathcal{P}$ at x. Note that if $\mathcal{P}$ is the Rothberger property, then $[0,\omega_1) = \bigcup_{\alpha<\omega_1}[0,\alpha)$ does not satisfy $\mathcal{P}$, on the other hand for each $\alpha < \omega_1$, $[0,\alpha)$ satisfies $\mathcal{P}$. If $\mathcal{P}$ is preserved under closed subsets, then the topological sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ is $\mathcal{P}$ at (x,α) for some $\alpha \in \Lambda$ if and only if X_α is $\mathcal{P}$ at x. Similarly this result need not hold for $\mathcal{P}$ spaces if $\mathcal{P}$ is the Rothberger property. The space $Y = \bigoplus_{\alpha<\omega_1} L$ does not satisfy the Rothberger property, where L is a Lusin set (i.e. an uncountable subset of reals whose intersection with every first category set of reals is countable).

Let $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ and $x \in X$. We use $\Lambda(x)$ to denote the collection of all $\alpha \in \Lambda$ such that $x \in X_\alpha$.

Theorem 3.20. *Consider $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ with each X_α is closed in X. Let $\{X_\alpha : \alpha \in \Lambda\}$ be locally finite in X and $x \in X$. If $\mathcal{P}$ is invariant under continuous mappings and finite unions, and if X_α is $\mathcal{P}$ at x for all $\alpha \in \Lambda(x)$, then X is $\mathcal{P}$ at x.*

Proof. Clearly $Y = \bigoplus_{\alpha\in\Lambda} X_\alpha$ is $\mathcal{P}$ at (x,α) for all $\alpha \in \Lambda(x)$ because X_α is $\mathcal{P}$ at x for all $\alpha \in \Lambda(x)$. Let $f : Y \to X$ be defined by $f(y,\alpha) = y$ and for each α, $\varphi_\alpha : X_\alpha \to Y$ be defined by $\varphi_\alpha(y) = (y,\alpha)$. Observe that for each closed F in Y, $f(F) = \bigcup_{\alpha\in\Lambda} \varphi_\alpha^{-1}(F)$ is closed in X. Let $y \in X$. Since $\{X_\alpha : \alpha \in \Lambda\}$ is locally finite in X, there exists an open set V containing y such that V intersects only finitely many members of it, say $X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_k}$. It is easy to see that $f^{-1}(y) = \bigoplus_{\{\alpha_i:1\le i\le k\}}\{y\}$. Thus f is perfect. By Corollary 3.17, X is $\mathcal{P}$ at x. □

A similar result in the context of P-spaces can be observed by using Lemma 3.21.

Lemma 3.21 (Folklore)**.** *For any locally countable family $\{X_\alpha : \alpha \in \Lambda\}$ of closed sets in a P-space X, $\bigcup_{\alpha\in\Lambda} X_\alpha$ is closed.*

Theorem 3.22. *Consider $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ with each X_α closed in X. Let $\{X_\alpha : \alpha \in \Lambda\}$ be locally countable in X and $x \in X$. Suppose that $\mathcal{P}$ is invariant under continuous mappings and countable unions. If X is a P-space and X_α is $\mathcal{P}$ at x for all $\alpha \in \Lambda(x)$, then X is $\mathcal{P}$ at x.*

Theorem 3.23. *Let $\mathcal{P}$ be preserved under closed subsets, continuous mappings and finite unions, and let $\mathcal{P}$ imply the Lindelöf property. Then a regular space X is both locally $\mathcal{P}$ and locally metrizable if and only if X is bi-quotient image of some locally $\mathcal{P}$ metrizable space.*

Proof. If X is both locally $\mathcal{P}$ and locally metrizable, then by Lemma 3.1, X has a basis consisting of closed $\mathcal{P}$ neighbourhoods. It follows that X has a cover $\{X_\alpha : \alpha \in \Lambda\}$ with each X_α metrizable closed $\mathcal{P}$ subspace. We can obtain a metrizable locally $\mathcal{P}$ space $Y = \bigcup_{\alpha\in\Lambda} Y_\alpha$ such that Y_α's are pairwise disjoint metrizable open $\mathcal{P}$ subspaces of Y and for each α Y_α is homeomorphic to X_α. For each α let $h_\alpha : Y_\alpha \to X_\alpha$ be a homeomorphism. Clearly the function $f : Y \to X$ given by $f(y) = h_\alpha(y)$ for $y \in Y_\alpha$ is bi-quotient.

Conversely let Y be a locally $\mathcal{P}$ metrizable space and $g : Y \to X$ be a bi-quotient mapping. Then X is locally $\mathcal{P}$ by Theorem 3.16. Let $\mathcal{U} = \{U_y : y \in Y\}$ be an open cover of Y with $y \in U_y \subseteq Z_y$ and Z_y is $\mathcal{P}$. Pick a $x \in X$. Then we get a finite set

$\{U_{y_i} : 1 \leq i \leq k\} \subseteq \mathcal{U}$ and an open set U in X with $x \in U \subseteq \bigcup_{i=1}^{k} g(U_{y_i})$. Clearly $\bigcup_{i=1}^{k} Z_{y_i}$ is metrizable $\mathcal{P}$, i.e. second countable. Thus $g(\bigcup_{i=1}^{k} Z_{y_i})$ is a regular second countable $\mathcal{P}$ space because the second countability is preserved under bi-quotient mappings and hence $g(\bigcup_{i=1}^{k} Z_{y_i})$ is metrizable. Thus X is locally metrizable □

Theorem 3.24. *Let $\mathcal{P}$ be preserved under closed subsets and continuous mappings, and let $\mathcal{P}$ imply the Lindelöf property. Then a regular space X is both locally $\mathcal{P}$ and locally metrizable if and only if X is open continuous image of some locally $\mathcal{P}$ metrizable space.*

The following facts can be easily verified.

(1) If $\mathcal{P}$ is closed under finite products, then X is $\mathcal{P}$ at x and Y is $\mathcal{P}$ at y imply $X \times Y$ is $\mathcal{P}$ at (x, y).

(2) Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are such that if X is $\mathcal{P}$ and Y is $\mathcal{Q}$, then $X \times Y$ is $\mathcal{P}$. Then X is $\mathcal{P}$ at x and Y is $\mathcal{Q}$ at y imply $X \times Y$ is $\mathcal{P}$ at (x, y).

(3) If $\mathcal{P}$ is invariant under continuous mappings and if the Cartesian product $\prod_{\alpha \in \Lambda} X_\alpha$ is $\mathcal{P}$ at x, then each X_α is $\mathcal{P}$ at $p_\alpha(x)$ where for each $\alpha \in \Lambda$, $p_\alpha : \prod_{\alpha \in \Lambda} X_\alpha \to X_\alpha$ is the projection mapping.

Also the following result can be obtained.

Proposition 3.25. *If $\mathcal{P}$ is invariant under continuous mappings and the Cartesian product $\prod_{\alpha \in \Lambda} X_\alpha$ is $\mathcal{P}$ at some point x, then X_α is $\mathcal{P}$ for all but finitely many α.*

Let $\mathcal{P}$ be invariant under continuous mappings. Then it is immediate from the above result that if $\prod_{\alpha \in \Lambda} X_\alpha$ is locally $\mathcal{P}$, then X_α is $\mathcal{P}$ for all but finitely many α. But the converse of this result does not hold. If $\mathcal{P}$ is the Rothberger property, then the Cantor space 2^ω is the product of ω copies of $\mathcal{P}$ spaces, whereas 2^ω is not $\mathcal{P}$.

3.4. Remainders and locally Lindelöf spaces

In this section it is assumed that every space is Tychonoff. For any compactification bX of X, $bX \setminus X$ is called a remainder of X. Recall from [2, 4] that a space X is called a p-space if in any (in some) compactification bX of X if for each $n \in \mathbb{N}$ there is a collection $\mathcal{U}_n$ of open sets in bX such that for each $x \in X$, $x \in \bigcap_{n \in \mathbb{N}} \bigcup\{U \in \mathcal{U}_n : x \in U\} \subseteq X$. Every metrizable space is a p-space (see [3, 5]) and every closed subspace of a p-space is a p-space (see [2]). A space X is said to be a Lindelöf Σ-space [16] if it is a continuous image of a Lindelöf p-space. An s-space [6] is a space which has a countable open source [6] in any (in some) compactification of it. Also recall that every Lindelöf p-space is an s-space [6] and any remainder of a Lindelöf p-space is also a Lindelöf p-space (see [5, Theorem 2.1]). Let Y be a subspace of X. Then X has the property $\mathcal{P}$ outside of Y whenever each closed set $F \subseteq X$ with $Y \cap F = \emptyset$ has the property $\mathcal{P}$.

Theorem 3.26. *If Y is a remainder of a locally Lindelöf p-space X, then Y is a Lindelöf p-space outside of K (hence an s-space outside of K) for some compact subset K of it.*

Proof. Let bX be a compactification of X such that $Y = bX \setminus X$. Since X is a locally Lindelöf p-space, we get an open cover $\mathcal{U}$ of X with $\overline{U}^X$ Lindelöf for each $U \in \mathcal{U}$. For each $U \in \mathcal{U}$ let V_U be an open set in bX with $V_U \cap X = U$. If $W = \bigcup\{V_U : U \in \mathcal{U}\}$, then W is open in bX with $X \subseteq W$ and $K = bX \setminus W$ is compact with $K \subseteq Y$. We claim that Y is a Lindelöf p-space outside of K. Pick a closed set $F \subseteq Y$ with $K \cap F = \emptyset$. Observe that $\overline{F}^{bX} \subseteq W$ and consequently we get a finite set $\{V_{U_i} : 1 \leq i \leq k\} \subseteq \{V_U : U \in \mathcal{U}\}$ such that $\overline{F}^{bX} \subseteq \bigcup_{i=1}^{k} V_{U_i}$. Clearly $C = \bigcup_{i=1}^{k} \overline{U_i}^X$ is a Lindelöf p-space and $Z = \overline{C}^{bX}$ is a compactification of C. Thus $Z \cap Y$ is a Lindelöf p-space because it is the remainder of C in Z. It is easy to see that F is a closed subset of $Z \cap Y$. Consequently F is a Lindelöf p-space and the proof is now complete. □

Corollary 3.27. *Let $\mathcal{P}$ imply the Lindelöf property. If Y is a remainder of a locally $\mathcal{P}$ p-space X, then Y is a Lindelöf p-space outside of K (hence an s-space outside of K) for some compact subset K of it.*

We call a space X homogeneous if for any $x, y \in X$ there is a homeomorphism $f : X \to X$ with $f(x) = y$.

Lemma 3.28 ([18]). *A finite union of closed s-spaces is an s-space.*

Theorem 3.29. *Every homogeneous remainder of a locally Lindelöf p-space is an s-space.*

Proof. Let Y be a homogeneous remainder of a locally Lindelöf p-space X. Then we get a compact set $K \subseteq Y$ such that Y is a Lindelöf p-space outside of K (see Theorem 3.26). The case is trivial when $Y = K$. Suppose that $K \subsetneqq Y$. Since $Y \setminus K$ is open in Y for every $y \in Y \setminus K$, we get an open subset U_y of Y such that $y \in U_y \subseteq \overline{U_y}^Y \subseteq Y \setminus K$ and $\overline{U_y}^Y$ is a Lindelöf p-space. Pick $x \in Y$. Let $y \in Y \setminus K$ be fixed. Since Y is homogeneous, there exists a homeomorphism $f : Y \to Y$ such that $f(y) = x$. Then we can obtain an open set $U_y \subseteq Y$ such that $y \in U_y$ and $\overline{U_y}^Y$ is a Lindelöf p-space. Thus $V_x = f(U_y)$ is an open subset of Y with $x \in V_x$ and $\overline{V_x}^Y$ is a Lindelöf p-space, i.e. an s-space. Consequently we have a finite set $\{x_i : 1 \leq i \leq k\} \subseteq Y$ such that $K \subseteq \bigcup_{i=1}^k V_{x_i}$. Obviously $Y \setminus \bigcup_{i=1}^k V_{x_i}$ is an s-space. By Lemma 3.28, $Y = (\bigcup_{i=1}^k \overline{V_{x_i}}^Y) \cup (Y \setminus \bigcup_{i=1}^k V_{x_i})$ is an s-space. □

Corollary 3.30. *Let $\mathcal{P}$ imply the Lindelöf property. Every homogeneous remainder of a locally $\mathcal{P}$ p-space is an s-space.*

Lemma 3.31 ([7, Theorem 2.7]). *Any (some) remainder of an s-space in a compactification of it is a Lindelöf Σ-space.*

Theorem 3.32. *If a locally Lindelöf p-space X has a homogeneous remainder, then $X = L \cup Z$ for some closed Lindelöf Σ-subspace L and open locally compact subspace Z.*

Proof. Let bX be a compactification of X such that $Y = bX \setminus X$ is homogeneous. Then Y is an s-space (see Theorem 3.29). Since $bY = \overline{Y}^{bX}$ is a compactification of Y and $L = bY \cap X$ is a closed subset of X, $L = bY \setminus Y$ and hence L is a Lindelöf Σ-space (see Lemma 3.31). Obviously $Z = bX \setminus bY$ is a locally compact subspace of X and $X = L \cup Z$. □

Corollary 3.33.

(1) *Let $\mathcal{P}$ imply the Lindelöf property. If a locally $\mathcal{P}$ p-space X has a homogeneous remainder, then $X = L \cup Z$ for some closed Lindelöf Σ-subspace L and open locally compact subspace Z.*
(2) *Let $\mathcal{P}$ imply the Lindelöf property. If a locally $\mathcal{P}$ p-space X that is nowhere locally compact has a homogeneous remainder, then X is a Lindelöf Σ-space.*

Acknowledgements

The authors are thankful to the referee(s) for numerous useful comments and suggestions that considerably improved the presentation of the paper.

References

[1] N. Alam and D. Chandra, *On certain localized version of uniform selection principles*, Filomat **36** (20), 6855-6865, 2022.
[2] N. Alam and D. Chandra, *On localization of the Menger property*, Quaest. Math. **46** (6), 1069-1092, 2023.

[3] A.V. Arhangel'skii, *A class of spaces which contains all metric and all locally compact spaces*, Mat. Sb. (N.S.) **67** (109), 55-88, 1965, English translation: Amer. Math. Soc. Transl. **92**, 1-39, 1970.
[4] A.V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys **21** (4), 115-162, 1966.
[5] A.V. Arhangel'skii, *Remainders in compactifications and generalized metrizability properties*, Topology Appl. **150**, 79-90, 2005.
[6] A.V. Arhangel'skii and M.M. Choban, *Some generalizations of the concept of a p-space*, Topology Appl. **158**, 1381-1389, 2011.
[7] A.V. Arhangel'skii, *A generalization of Čech-complete spaces and Lindelöf* Σ*-spaces*, Comment. Math. Univ. Carolin. **54** (2), 121-139, 2013.
[8] D. Chandra and N. Alam, *On localization of the star-Menger selection principle*, Hacet. J. Math. Stat. **50** (4), 1155-1168, 2021.
[9] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
[10] J. Gerlits and Zs. Nagy, *Some properties of* $C(X)$*, I*, Topology Appl. **14** (2), 151-161, 1982.
[11] W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki, *The combinatorics of open covers (II)*, Topology Appl. **73**, 241-266, 1996.
[12] Lj.D.R. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen **55**, 421-431, 1999.
[13] Lj.D.R. Kočinac, *Selection principles in uniform spaces*, Note Mat. **22** (2), 127-139, 2003.
[14] E.A. Michael, *Bi-quotient maps and cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble) **18** (2), 287-302, 1968.
[15] S. Mrówka, *On completely regular spaces*, Fund. Math. **41**, 105-106, 1954.
[16] K. Nagami, Σ*-spaces*, Fund. Math. **65**, 169-192, 1969.
[17] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69**, 31-62, 1996.
[18] H. Wang and W. He, *On remainders of locally* s*-spaces*, Topology Appl. **278**, 107231, 2020.
[19] S. Willard, *General Topology*, Addison Wesley Publishing Co., 1970.