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$\mathscr{Z}^*$-Tensor on $N(k)$-Contact Metric Manifolds Admitting Ricci Soliton Type Structure

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Abstract

The main goal of this manuscript is to investigate the properties of $N(k)$-contact metric manifolds admitting a $\mathscr{Z}^*$-tensor. We prove the necessary conditions for which $N(k)$-contact metric manifolds endowed with a $\mathscr{Z}^*$-tensor are Einstein manifolds. In this sequel, we accomplish that an $N(k)$-contact metric manifold endowed with a $\mathscr{Z}^*$-tensor satisfying $\mathscr{Z}^*(\mathscr{G}_1,\hat{\xi})\cdot\overset{\star}{\mathscr{R}}=0$ is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or an Einstein manifold. We also prove the condition for which an $N(k)$-contact metric manifold endowed with a $\mathscr{Z}^*$-tensor is a Sasakian manifold. To validate some of our results, we construct a non-trivial example of an $N(k)$-contact metric manifold.

1. Introduction

In 1988, Tanno [1] has initiated the concept of k-nullity distribution of a contact metric manifold. A contact metric manifold with ξ belonging to the k-nullity distribution is said to be $N(k)$-contact metric manifold (briefly, $N(k)$-(CMM)$_{2n+1}$). Blair et al. [2] generalized this idea on a contact manifold with ξ belongs to a (k,μ)-nullity distribution, where k and μ are real constants. In particular, if μ=0, then the (k,μ)-nullity distribution reduces to a k-nullity distribution. For more details see, ([3]- [11]).

The notion of Ricci soliton (RS) on Riemannian manifold $(\Theta,\hat{\mathfrak{g}})$ of dimension m is defined by [12, 13]:

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}+\overset{\star}{\mathscr{S}}+\lambda\hat{\mathfrak{g}}=0, \tag{1.1}$$

where $\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}$ is the Lie derivative of the Riemannian metric $\hat{\mathfrak{g}}$ along the vector field $\overset{\star}{V}$, $\overset{\star}{\mathscr{S}}$ is the Ricci tensor and λ is a real constant. In whole manuscript, an RS is denoted as $(\Theta,\hat{\mathfrak{g}},\overset{\star}{V},\lambda)$. Metrics satisfying (1.1) are interesting and useful in physics and are often referred to as quasi-Einstein metrics [14, 15]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial\hat{\mathfrak{g}}}{\partial t}=-2\overset{\star}{\mathscr{S}}$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling. An RS will be expanding, steady, or shrinking depending on $\lambda>0$, $\lambda=0$ or $\lambda<0$. Ricci solitons have been studied by several authors such as ([16]- [28]).

According to Mantica and Molinari [29], a generalized symmetric $\mathscr{Z}^*$-tensor of type (0, 2) is given by

$$\mathscr{Z}^*=\overset{\star}{\mathscr{S}}+\dot{\varphi}\hat{\mathfrak{g}}, \tag{1.2}$$

where $\dot{\varphi}$ is an arbitrary function. In References ([30]- [36]) various properties of the $\mathscr{Z}^*$-tensor were pointed out. In particular cases, the $\mathscr{Z}^*$-tensor have the several importance on $(\Theta,\hat{\mathfrak{g}})$. For example,

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1. if $\mathscr{Z}^*{}_{ij}$=0 (i.e, $\mathscr{Z}^*$-flat), then $(\Theta,\hat{\mathfrak{g}})$ reduces to an Einstein manifold [37],
2. if $\nabla_k\mathscr{Z}^*{}_{ij}$=$\lambda_k\mathscr{Z}^*{}_{ij}$(i.e., $\mathscr{Z}^*$-recurrent), then $(\Theta,\hat{\mathfrak{g}})$ reduces to a GRR manifold,
3. if $\nabla_k\mathscr{Z}^*{}_{ij}$=$\nabla_i\mathscr{Z}^*{}_{kl}$ (i.e., Codazzi tensor), then we find $\nabla_k\widetilde{\mathscr{R}}_{ij}$-$\nabla_i\widetilde{\mathscr{R}}_{kj}$=$\frac{1}{2(\widetilde{n}-1)}(g_{ij}\nabla_k-g_{kj}\nabla_i)\tau$ [38],
4. the relation between the $\mathscr{Z}^*$-tensor and the energy-stress tensor of Einstein's equations with cosmological constant Γ is $\mathscr{Z}^*{}_{kj}$=$\widetilde{\kappa}\mathscr{T}^*{}_{kj}$ [39], where ϕ=-$\frac{\tau}{2}+\Gamma$ and $\widetilde{\kappa}$ is the gravitational constant. In this case, the $\mathscr{Z}^*$-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution ($\mathscr{Z}^*$=0) determines an Einstein space Γ=$(\frac{(\widetilde{n}-2)}{2\widetilde{n}})\tau$; the conservation of TEM ($\nabla^l\mathscr{T}^*{}_{kl}$=0) gives ($\nabla_j\mathscr{T}^*{}_{kl}$=0) then this space-time gives the conserved energy-momentum density.

A new curvature tensor $\overset{\star}{\mathscr{Q}}$ of type $(1,3)$ on $(\Theta,\hat{\mathfrak{g}})$, $n>2$ is defined as

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3=\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3-\frac{\breve{\Psi}}{n-1}[\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2], \tag{1.3}$$

is known as $\mathscr{Q}$-curvature tensor [40], where $\breve{\Psi}$ is an arbitrary scalar function. If $\breve{\Psi}$=$\frac{\overset{\star}{\breve{\kappa}}}{n}$, where $\overset{\star}{\breve{\kappa}}$ is the scalar curvature, then $\mathscr{Q}$-curvature tensor reduces to concircular curvature tensor $\overset{\star}{\mathscr{C}}$ [41]. For more details about $\mathscr{Q}$-curvature tensor, see [42,43]).

With the help of (1.1) and (1.2), we define:

Definition 1.1. *A Riemannian metric $\hat{\mathfrak{g}}$ is called a $\mathscr{Z}^*$-soliton if*

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}+\mathscr{Z}^*+\lambda\hat{\mathfrak{g}}=0, \tag{1.4}$$

where $\mathfrak{L}$ is the Lie derivative and λ a real scalar. If $\overset{\star}{V}$ is the gradient of f, $\mathscr{Z}^*$-soliton is referred to as a gradient $\mathscr{Z}^*$-soliton and then equation (1.4) simplifies to

$$\nabla^2 f+\mathscr{Z}^*+\lambda\hat{\mathfrak{g}}=0,$$

where the Hessian of the function f is $\nabla^2 f$.
As per above sequel, we obtain some results by using the $\mathscr{Z}^*$-tensor on $N(k)$-(CMM)$_{2n+1}$ with (RS)$_{2n+1}$. After the introduction, Section 2, deals with some basic concept of $N(k)$-(CMM)$_{2n+1}$. We also examine $N(k)$-(CMM)$_{2n+1}$ with conditions $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{Z}^*$=0, $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}}$=0, $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0 and $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}}$=0 in the Sections 3, 4, 5 and 6, respectively. In Section 7, we categorized $N(k)$-(CMM)$_{2n+1}$ which satisfy the conditions $\overset{\star}{\mathscr{Q}}.h$=0, $h.\overset{\star}{\mathscr{Q}}$=0. In the Section 8, we deal with $\mathscr{Z}^\star$-recurrent on $N(k)$-(CMM)$_{2n+1}$. Finally, an appropriate example establishes the existence of a $\mathscr{Z}^*$-soliton on a $N(k)$-(CMM)$_3$ which validates some of our results.

2. Preliminaries

A contact metric manifold $(\Theta,\hat{\mathfrak{g}})$ of dimension $m(=2n+1),(n>1)$ is a quadruple $(\hat{\varphi},\hat{\zeta},\hat{\eta},\hat{\mathfrak{g}})$, where $\hat{\varphi}$ is a (1, 1)-tensor field, $\hat{\zeta}$ is a vector field, $\hat{\eta}$ is a 1-form on $(\Theta,\hat{\mathfrak{g}})$ and $\hat{\mathfrak{g}}$ is a Riemannian metric, such that

$$\hat{\varphi}^2\mathscr{G}_1=-\mathscr{G}_1+\hat{\eta}(\mathscr{G}_1)\hat{\zeta},\quad \hat{\eta}(\hat{\zeta})=1,\quad \hat{\varphi}\hat{\zeta}=0,\quad \hat{\eta}\circ\hat{\varphi}=0, \tag{2.1}$$

$$\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_1,\hat{\varphi}\mathscr{G}_2)=\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)-\hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2), \tag{2.2}$$

$$\hat{\mathfrak{g}}(\mathscr{G}_1,\hat{\varphi}\mathscr{G}_2)=-\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_1,\mathscr{G}_2),\quad \hat{\mathfrak{g}}(\mathscr{G}_1,\hat{\zeta})=\hat{\eta}(\mathscr{G}_1) \tag{2.3}$$

for all vector field $\mathscr{G}_1,\mathscr{G}_2\in\Gamma(\Theta)$. On $(\Theta,\hat{\mathfrak{g}})$, a $(1,1)$-tensor field h is defined by h=$\frac{1}{2}\mathfrak{L}_{\hat{\zeta}}\hat{\varphi}$, which is symmetric and satisfies (see [44,45])

$$h\hat{\varphi}=-\hat{\varphi}h,\ Tr.h=Tr.,\ \hat{\varphi}h=0,\ h\hat{\zeta}=0,$$

$$\nabla_{\mathscr{G}_1}\hat{\zeta}=-\hat{\varphi}\mathscr{G}_1-\hat{\varphi}h\mathscr{G}_1, \tag{2.4}$$

$$\hat{\mathfrak{g}}(h\mathscr{G}_1,\mathscr{G}_2)=\hat{\mathfrak{g}}(\mathscr{G}_1,h\mathscr{G}_2), \tag{2.5}$$

$$\hat{\eta}(h\mathscr{G}_1)=0. \tag{2.6}$$

In 1995, Blair et al. introduced the notion of $N(k,\mu)$-(CMM), for real numbers k and μ as a distribution [2,46]

$$\begin{aligned}N(k,\mu):p\to N_p(k,\mu)=[&\mathscr{G}_3\in T_p\Theta:\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3\\ &=(kl+\mu h)(\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2)].\end{aligned}$$

If μ=0, the (k,μ)-nullity distribution reduces to k-nullity distributions and defined as [1,47]

$$N(k):p\to N_p(k)=[\mathscr{G}_3\in T_p\Theta:\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3 \\ =k\{\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2\}], \tag{2.7}$$

where k is constant. In particular, if k=1, then $(\Theta,\hat{\mathfrak{g}})$ is Sasakian and if k=0, then $(\Theta,\hat{\mathfrak{g}})$ is locally isometric to the product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for n=1 [2,47]. In $N(k)$-(CMM)$_{2n+1}$, we have

$$h^2=(k-1)\hat{\varphi}^2,\quad k\leq 1, \tag{2.8}$$

$$(\nabla_{\mathscr{G}_1}\hat{\varphi})\mathscr{G}_2=\hat{\mathfrak{g}}(\mathscr{G}_1+h\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}-\hat{\eta}(\mathscr{G}_2)(\mathscr{G}_1+h\mathscr{G}_1),$$

$$\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}=k[\hat{\eta}(\mathscr{G}_2)\mathscr{G}_1-\hat{\eta}(\mathscr{G}_1)\mathscr{G}_2], \tag{2.9}$$

$$\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\hat{\zeta})\mathscr{G}_2=k[\hat{\eta}(\mathscr{G}_2)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}], \tag{2.10}$$

$$\overset{\star}{\mathscr{R}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2=k[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}-\hat{\eta}(\mathscr{G}_2)\mathscr{G}_1], \tag{2.11}$$

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)=2(n-1)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)+2(n-1)\hat{\mathfrak{g}}(h\mathscr{G}_1,\mathscr{G}_2)+2(nk-(n-1))\hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2), \tag{2.12}$$

$$\overset{\star}{\mathscr{S}}(\hat{\varphi}\mathscr{G}_1,\hat{\varphi}\mathscr{G}_2)=\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)-2nk\hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2)-4(n-1)\hat{\mathfrak{g}}(h\mathscr{G}_1,\mathscr{G}_2),$$

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\hat{\zeta})=2nk\hat{\eta}(\mathscr{G}_1), \tag{2.13}$$

$$\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_2)=(2nk+\varphi)\hat{\eta}(\mathscr{G}_2), \tag{2.14}$$

$$\mathscr{Z}^*(\hat{\zeta},\hat{\zeta})=(2nk+\varphi) \tag{2.15}$$

for any vector field $\mathscr{G}_1,\mathscr{G}_2\in\Gamma(\Theta)$.
Now, we recall some propositions, which will be used later on as follows:

Lemma 2.1 ([48]). *A contact metric manifold $\Theta(\hat{\varphi},\hat{\zeta},\hat{\eta},\hat{\mathfrak{g}})$ fulfills the criteria $\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}$=0 for all $\mathscr{G}_1,\mathscr{G}_2$ is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.*

3. $N(k)$-(CMM)$_{2n+1}$ Admitting $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{Z}^*$=0

The condition $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{Z}^*$=0 on $(\Theta,\hat{\mathfrak{g}})$ implies that

$$\mathscr{Z}^*(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2,\mathscr{G}_3)+\mathscr{Z}^*(\mathscr{G}_2,\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_3)=0. \tag{3.1}$$

Using (1.3), (2.11), (2.14), and (2.15) in (3.1), we obtain

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[(2nk+\varphi)\hat{\eta}(\mathscr{G}_3)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)+(2nk+\varphi)\hat{\eta}(\mathscr{G}_2)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)-\hat{\eta}(\mathscr{G}_2)\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_3)-\hat{\eta}(\mathscr{G}_3)\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_2)]=0. \tag{3.2}$$

Putting $\mathscr{G}_3=\hat{\zeta}$ in (3.2) and using (1.2), (2.3) and (2.14), we find

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)]=0,$$

which implies that either k=$\frac{\breve{\Psi}}{2n}$, or $\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)$=$2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)$. If $k\neq\frac{\breve{\Psi}}{2n}$, then one can get

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)=2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2). \tag{3.3}$$

So, we have:

Theorem 3.1. *An $N(k)$-$(CMM)_{2n+1}$ admitting $\mathscr{L}^*$-tensor fulfills the criteria $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0 is an Einstein manifold provided $k\neq\frac{\breve{\Psi}}{2n}$.*

Corollary 3.2. *An $N(k)$-$(CMM)_{2n+1}$ admitting $\mathscr{L}^*$-tensor satisfying the condition $\overset{\star}{\mathscr{C}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0 is an Einstein manifold provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.*

Again from (1.2), (1.4) and (3.3), we have

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)+[2nk+\dot{\varphi}+\lambda]\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)=0. \tag{3.4}$$

Taking $\mathscr{G}_1$=$\mathscr{G}_2$=e_i in (3.4) and summing over i, $(1\leq i\leq 2n+1)$, we get

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}(e_i,e_i)+[2nk+\dot{\varphi}+\lambda]\hat{\mathfrak{g}}(e_i,e_i)=0$$

which is equivalent to

$$div(\overset{\star}{V})+[2nk+\dot{\varphi}+\lambda](2n+1)=0. \tag{3.5}$$

If $\overset{\star}{V}$ is solenoidal that is, $div(\overset{\star}{V})$=0, then (3.5) reduces to

$$\lambda=-(2nk+\dot{\varphi}).$$

Also if $\overset{\star}{V}$=$grad(f)$. So from (3.5), we yield

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1),$$

where $\nabla(f)$ is the Laplacian of smooth function f. Thus we conclude:

Corollary 3.3. *An $N(k)$-$(CMM)_{2n+1}$ admitting gradient $\mathscr{L}^*$-soliton fulfills the criteria $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0, then*

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1)$$

provided $k\neq\frac{\breve{\Psi}}{2n}$.

Corollary 3.4. *An $N(k)$-$(CMM)_{2n+1}$ with $\mathscr{L}^*$-soliton satisfies the condition $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0, where $\overset{\star}{V}$ is solenoidal, then the soliton is increasing, stable, or reducing depending on $\dot{\varphi}<-2nk$, $\dot{\varphi}$=$2nk$, or $\dot{\varphi}>2nk$.*

4. $N(k)$-(CMM)$_{2n+1}$ With $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}}$=0

The condition $(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}})(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4$=0 on (Θ,g) implies that

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{Q}}(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_4=0. \tag{4.1}$$

Also from (2.7) and (1.3) we have

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2], \tag{4.2}$$

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)\mathscr{G}_3=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}-\hat{\eta}(\mathscr{G}_3)\mathscr{G}_2], \tag{4.3}$$

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-\hat{\eta}(\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1], \tag{4.4}$$

$$\overset{\star}{\mathscr{Q}}(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4-\hat{\eta}(\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4], \tag{4.5}$$

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_3)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4-\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4], \tag{4.6}$$

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}-\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1]. \tag{4.7}$$

Using (4.3), (4.4), (4.5), (4.6) and (4.7) in (4.1), we get

$$\begin{aligned}\left(k-\frac{\breve{\Psi}}{2n}\right)&[\hat{\mathfrak{g}}(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-\hat{\eta}(\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4\\&+\hat{\eta}(\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4+\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4\\&-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}+\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1]=0.\end{aligned} \tag{4.8}$$

Taking the inner product of (4.8) with $\hat{\zeta}$ and using (4.1), (4.3), we find

Theorem 4.1. *An $N(k)$-$(CMM)_{2n+1}$ always fulfills the condition $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}}=0$, provided $k\neq\frac{\breve{\Psi}}{2n}$.*

Corollary 4.2. *An $N(k)$-$(CMM)_{2n+1}$ always satisfy the condition $\overset{\star}{\mathscr{C}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{C}}=0$, provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.*

5. $N(k)$-(CMM)$_{2n+1}$ Satisfying $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0

Let the condition $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4$=0 holds on $(\Theta,\hat{\mathfrak{g}})$. Then we have

$$\begin{aligned}&\mathscr{Z}^*(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-\mathscr{Z}^*(\hat{\zeta},\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1-\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4+\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4\\&-\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4+\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4-\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}+\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1=0.\end{aligned}\tag{5.1}$$

Using (1.2) and (2.14) in (5.1), we get

$$\begin{aligned}&\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}+\dot{\varphi}\hat{\mathfrak{g}}((\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-(2nk+\dot{\varphi})\hat{\eta}(\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4\\&-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4\\&+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1=0.\end{aligned}\tag{5.2}$$

Taking inner product of (5.2) with $\hat{\zeta}$ and using (4.2) and (4.3), we obtain

$$\begin{aligned}\left(k-\frac{\breve{\Psi}}{2n}\right)[&\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)+\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_3)\hat{\eta}(\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\hat{\eta}(\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)\\&+(2nk+\dot{\varphi})\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\hat{\eta}(\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)-(2nk+\dot{\varphi})\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)]=0.\end{aligned}\tag{5.3}$$

For, fix $\mathscr{G}_3=\hat{\zeta}$ in (5.3) and using (2.3), we get

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)-2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)]=0.$$

So, we mention the result:

Theorem 5.1. *An $N(k)$-$(CMM)_{2n+1}$ admitting $\mathscr{Z}^*$-tensor satisfying the criteria $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0, is an Einstein manifold provided $k\neq\frac{\breve{\Psi}}{2n}$.*

Corollary 5.2. *An $N(k)$-$(CMM)_{2n+1}$ admitting $\mathscr{Z}^*$-tensor satisfying the condition $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{C}})$=0, is an Einstein manifold provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.*

Likewise Section 3, we state the followings:

Corollary 5.3. *If a gradient $\mathscr{Z}^*$-soliton $(g,\overset{\star}{V},\lambda)$ on $N(k)$-$(CMM)_{2n+1}$ satisfies the criteria $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{C}})$=0, then*

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1)$$

provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.

Corollary 5.4. *If a gradient $\mathscr{Z}^*$-soliton $(g,\overset{\star}{V},\lambda)$ on $N(k)$-$(CMM)_{2n+1}$ satisfies the condition $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{C}})$=0, where $\overset{\star}{V}$ is solenoidal, then the soliton is increasing, stable, or reducing depending on $\dot{\varphi}<-2nk$, $\dot{\varphi}$=$2nk$, or $\dot{\varphi}>2nk$.*

Corollary 5.5. *An $N(k)$-$(CMM)_{2n+1}$ admits gradient $\mathscr{Z}^*$-soliton $(g,\overset{\star}{V},\lambda)$ fulfills the criteria $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0, where $\overset{\star}{V}$ is the gradient of a smooth function f, then we have*

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1)$$

provided $k\neq\frac{\breve{\Psi}}{2n}$.

Corollary 5.6. *An $N(k)$-$(CMM)_{2n+1}$ with gradient $\mathscr{Z}^*$-soliton $(g,\overset{\star}{V},\lambda)$ satisfying the condition $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0, where $\overset{\star}{V}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $\dot{\varphi}<-2nk$, $\dot{\varphi}$=$2nk$, or $\dot{\varphi}>2nk$.*

6. $N(k)$-(CMM)$_{2n+1}$ Satisfying $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}}$=0

We suppose that $(\Theta,\hat{\mathfrak{g}})$ satisfies the below the condition

$$(\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3=0, \tag{6.1}$$

which implies that

$$(\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3=((\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3, \tag{6.2}$$

where the endomorphism $(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\mathscr{G}_4)\mathscr{G}_5$ is defined as

$$(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\mathscr{G}_4)\mathscr{G}_5=\mathscr{Z}^*(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_1-\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_5)\mathscr{G}_4. \tag{6.3}$$

Now, from (6.2) we have

$$\begin{aligned}(\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3&=((\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3-\overset{\star}{\mathscr{R}}((\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3\\&-\overset{\star}{\mathscr{R}}(\mathscr{G}_4,(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\mathscr{G}_5)\mathscr{G}_3-\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\mathscr{G}_3.\end{aligned} \tag{6.4}$$

Also, in view of (6.1), (6.3) and (6.4) we get

$$\begin{aligned}&\mathscr{Z}^*(\hat{\zeta},\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3)\mathscr{G}_1-\mathscr{Z}^*(\mathscr{G}_1,\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3)\hat{\zeta}-\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_4)\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_5)\mathscr{G}_3+\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{R}}(\hat{\zeta},\mathscr{G}_5)\mathscr{G}_3\\&-\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_5)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_1)\mathscr{G}_3+\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_5)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\hat{\zeta})\mathscr{G}_3-\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_3)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_1+\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\hat{\zeta}=0.\end{aligned} \tag{6.5}$$

Using (1.2), (2.9), (2.10), (2.12) and (2.14) in (6.5) and then taking the inner product with $\hat{\zeta}$, we obtain

$$\begin{aligned}&k[-\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_5)\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)-\varphi\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_5)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)+\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_5)\\&+\varphi\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_5)-(2nk+\varphi)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_5)+(2nk+\varphi)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_5)]=0.\end{aligned} \tag{6.6}$$

Putting $\mathscr{G}_5=\hat{\zeta}$ in (6.6), we get

$$k[-\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)+2nk\hat{\eta}(\mathscr{G}_3)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)]=0. \tag{6.7}$$

Again putting $\mathscr{G}_3=\hat{\zeta}$ in (6.7), we find

$$k[-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)+2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)]=0,$$

which implies that either k=0 or,

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)=2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4).$$

Now, if k=0, then in view of (2.9) and Proposition 2.1, we state the following results:

Theorem 6.1. *If an $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{Z}^*$-tensor fulfills the criteria $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}}$=0, then $(\Theta,\hat{\mathfrak{g}})$ is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or the manifold is an Einstein.*

Corollary 6.2. *A $\mathscr{Z}^*$-soliton $(g,\overset{\star}{V},\lambda)$ on locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$, is reducing, stable or increasing depending upon the sign of scalar curvature.*

7. $N(k)$-(CMM)$_{2n+1}$ Equipped With $\overset{\star}{\mathscr{Q}}.h=0, h.\overset{\star}{\mathscr{Q}}$=0

The condition $(\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2).h)\mathscr{G}_3$=0 on $(\Theta,\hat{\mathfrak{g}})$ implies that

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)h\mathscr{G}_3-h(\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3)=0 \tag{7.1}$$

for any $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3\in\Gamma(\Theta)$. Putting $\mathscr{G}_1=\hat{\zeta}$ in (7.1), we have

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)h\mathscr{G}_3-h(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)\mathscr{G}_3)=0. \tag{7.2}$$

Using (2.6), (4.3) in (7.2), we obtain

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_2,h\mathscr{G}_3)\hat{\zeta}+\hat{\eta}(\mathscr{G}_3)h\mathscr{G}_2]=0. \tag{7.3}$$

Replacing $\mathscr{G}_3$ by $h\mathscr{G}_3$ in (7.3) and using (2.1), (2.2), (2.6), (2.8), we obtain

$$-(k-\frac{\breve{\Psi}}{2n})(k-1)\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_2,\hat{\varphi}\mathscr{G}_3)=0$$

and hence

$$\left(k-\frac{\breve{\Psi}}{2n}\right)(k-1)d\hat{\eta}(\hat{\varphi}\mathscr{G}_2,\mathscr{G}_3)=0,$$

which implies that either k=1, or $(k-\frac{\breve{\Psi}}{2n})d\hat{\eta}(\hat{\varphi}\mathscr{G}_2,\mathscr{G}_3)$=0. Thus we state:

Theorem 7.1. *If an $N(k)$-(CMM)$_{2n+1}$ satisfies the criteria $\overset{\star}{\mathscr{Q}}.h$=0, then $(\Theta, \hat{\mathfrak{g}})$ is Sasakian manifold provided $k \neq \frac{\breve{\Psi}}{2n}$.*

Next, we assume that $N(k)$-(CMM)$_{2n+1}$ fits the criteria $(h.\overset{\star}{\mathscr{Q}})(\mathscr{G}_1, \mathscr{G}_2)\mathscr{G}_3$=0, that is

$$h(\overset{\star}{\mathscr{Q}}(\mathscr{G}_1, \mathscr{G}_2)\mathscr{G}_3) - \overset{\star}{\mathscr{Q}}(h\mathscr{G}_1, \mathscr{G}_2)\mathscr{G}_3 - \overset{\star}{\mathscr{Q}}(\mathscr{G}_1, h\mathscr{G}_2)\mathscr{G}_3 - \overset{\star}{\mathscr{Q}}(\mathscr{G}_1, \mathscr{G}_2)h\mathscr{G}_3 = 0 \tag{7.4}$$

for any $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3 \in \Gamma(\Theta)$. Putting $\mathscr{G}_1 = \hat{\zeta}$ in (7.4) and using $h\hat{\zeta} = 0$, we are leads to

$$h(\overset{\star}{\mathscr{Q}}(\hat{\zeta}, \mathscr{G}_2)\mathscr{G}_3) - \overset{\star}{\mathscr{Q}}(\hat{\zeta}, h\mathscr{G}_2)\mathscr{G}_3 - \overset{\star}{\mathscr{Q}}(\hat{\zeta}, \mathscr{G}_2)h\mathscr{G}_3 = 0. \tag{7.5}$$

Using (2.5), (2.6), (4.3) in (7.5), we find

$$-2(k - \frac{\breve{\Psi}}{2n})\hat{\mathfrak{g}}(h\mathscr{G}_2, \mathscr{G}_3) = 0. \tag{7.6}$$

Replacing $\mathscr{G}_2$ by $h\mathscr{G}_2$ in (7.6) and by making use of (2.1), (2.2), (2.6), (2.8), the equation (7.6) reduces to

$$2(k-1)(k - \frac{\breve{\Psi}}{2n})\hat{\mathfrak{g}}(\hat{\phi}\mathscr{G}_2, \hat{\phi}\mathscr{G}_3) = 0.$$

So, we conclude the results as:

Theorem 7.2. *If an $N(k)$-(CMM)$_{2n+1}$ satisfying the condition $h.\overset{\star}{\mathscr{Q}}$=0, then the $(\Theta, \hat{g})$ is Sasakian manifold, provided $k \neq \frac{\breve{\Psi}}{2n}$.*

In view of Theorem 7.1 and Theorem 7.2, we turn up the below outcome:

Corollary 7.3. *In an $N(k)$-(CMM)$_{2n+1}$ with $k \neq \frac{\breve{\Psi}}{2n}$, we have $\overset{\star}{\mathscr{Q}}.h = h.\overset{\star}{\mathscr{Q}}$.*

8. $\mathscr{Z}^*$-Recurrent on $N(k)$-(CMM)$_{2n+1}$

For $\mathscr{Z}^*$-recurrent on $(\Theta, \hat{\mathfrak{g}})$, we get

$$(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{Z}^*)(\mathscr{G}_4, \mathscr{G}_5) = \hat{\eta}(\mathscr{G}_1)\mathscr{Z}^*(\mathscr{G}_4, \mathscr{G}_5). \tag{8.1}$$

Since, we have

$$(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{Z}^*)(\mathscr{G}_4, \mathscr{G}_5) = \mathscr{G}_1\mathscr{Z}^*(\mathscr{G}_4, \mathscr{G}_5) - \mathscr{Z}^*(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_4, \mathscr{G}_5) - \mathscr{Z}^*(\mathscr{G}_4, \overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_5). \tag{8.2}$$

With the help of (8.1) and (8.2) we yield

$$\mathscr{G}_1\mathscr{Z}^*(\mathscr{G}_4, \mathscr{G}_5) - \mathscr{Z}^*(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_4, \mathscr{G}_5) - \mathscr{Z}^*(\mathscr{G}_4, \overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_5) = \hat{\eta}(\mathscr{G}_1)\mathscr{Z}^*(\mathscr{G}_4, \mathscr{G}_5). \tag{8.3}$$

Fix $\mathscr{G}_4 = \mathscr{G}_5 = \hat{\zeta}$ in (8.3) and using (2.1), (2.4), (2.14) and (2.15), we obtain

$$\mathscr{G}_1(2nk + \dot{\varphi}) = \hat{\eta}(\mathscr{G}_1)(2nk + \dot{\varphi}).$$

We state the following:

Theorem 8.1. *In a $\mathscr{Z}^*$-recurrent $N(k)$-(CMM)$_{2n+1}$, we have*

$$\mathscr{G}_1(2nk + \dot{\varphi}) = \hat{\eta}(\mathscr{G}_1)(2nk + \dot{\varphi}),$$

for all $\mathscr{G}_1 \in \Gamma(\Theta)$.

A $\mathscr{Z}^*$-recurrent manifold is $\mathscr{Z}^*$-symmetric if and only if the 1-form $\hat{\eta}$ is zero. So we notice:

Corollary 8.2. *In a $\mathscr{Z}^*$-symmetric $N(k)$-(CMM)$_{2n+1}$, $2nk+\dot{\varphi}$=constant.*

Corollary 8.3. *If an $N(k)$-(CMM)$_{2n+1}$ is $\mathscr{Z}^*$-recurrent and if $2nk+\dot{\varphi}$ is constant, then either $2nk+\dot{\varphi}$=0 or, $(\Theta, \hat{\mathfrak{g}})$ reduces to a $\mathscr{Z}^*$-symmetric.*

Finally, we consider $\mathscr{Z}^*$-soliton with $\overset{\star}{V} = \hat{\zeta}$ on $N(k)$-(CMM)$_{2n+1}$. Then from (1.4), we have

$$\mathfrak{L}_{\hat{\zeta}}\hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2) + 2\mathscr{Z}^*(\mathscr{G}_1, \mathscr{G}_2) + 2\lambda\hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2) = 0. \tag{8.4}$$

Using (2.3) and (2.4), we find

$$\mathfrak{L}_{\hat{\zeta}}\hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2) = -2\hat{\mathfrak{g}}(\hat{\phi}h\mathscr{G}_1, \mathscr{G}_2). \tag{8.5}$$

Now using (1.2), (8.5) in (8.4), we obtain

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1, \mathscr{G}_2) = -\hat{\mathfrak{g}}(h\mathscr{G}_1, \hat{\phi}\mathscr{G}_2) - (\dot{\varphi} + \lambda)\hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2). \tag{8.6}$$

In view of (2.13) and (8.6) we have

$$[2nk + \dot{\varphi} + \lambda]\eta(\mathscr{G}_1) = 0,$$

which implies that

$$\lambda = -(2nk + \dot{\varphi}).$$

As per above, we mention the result:

Theorem 8.4. *If an $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{Z}^*$-soliton, then we have*
(i) $\mathscr{Z}^*$-soliton is expanding if $\dot{\varphi} < -2nk$
(ii) $\mathscr{Z}^*$-soliton is shrinking if $\dot{\varphi} > -2nk$
(iii) $\mathscr{Z}^*$-soliton is steady if $\dot{\varphi} = -2nk$

Corollary 8.5. *A $\mathscr{Z}^*$-symmetric $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{Z}^*$-soliton is always shrinking.*

Corollary 8.6. *A $\mathscr{Z}^*$-soliton on $\mathscr{Z}^*$-recurrent $N(k)$-(CMM)$_{2n+1}$ is always steady if $2nk+\dot{\varphi}$=constant.*

9. Example

Let a 3-dimensional manifold $\Theta = \{(r,s,t) \in \mathfrak{R}^3 : (r,s,t) \neq 0\}$, where (r,s,t) are standard coordinates in $\mathfrak{R}^3$. Let $(\vartheta_1, \vartheta_2, \vartheta_3)$ be the orthogonal system of vector fields at each point of Θ, defined as

$$\vartheta_1 = e^t \frac{\partial}{\partial r}, \ \vartheta_2 = e^t \frac{\partial}{\partial s}, \ \vartheta_3 = -\frac{\partial}{\partial t}$$

and

$$[\vartheta_1, \vartheta_2] = 0, \ [\vartheta_1, \vartheta_3] = \vartheta_1, \ [\vartheta_2, \vartheta_3] = \vartheta_2.$$

Let, we define the metric $\hat{\mathfrak{g}}$ as follows

$$\hat{\mathfrak{g}}_{ij} = \left\{ \begin{array}{ll} 0, & i \neq j = 1,2,3. \\ 1, & i = j \end{array} \right\}.$$

If $\hat{\eta}$ the 1-form have the significance

$$\hat{\eta}(\mathscr{G}_1) = \hat{\mathfrak{g}}(\mathscr{G}_1, \vartheta_1)$$

for any $\mathscr{G}_1 \in \Gamma(\Theta)$. Let $\hat{\varphi}$ be the $(1,1)$-tensor field defined by

$$\hat{\varphi}\,\vartheta_1 = 0, \qquad \hat{\varphi}\,\vartheta_2 = -\vartheta_3, \qquad \hat{\varphi}\,\vartheta_3 = \vartheta_2.$$

Making use of the linearity of $\hat{\varphi}$ and $\hat{\mathfrak{g}}$ we have

$$\hat{\eta}(\vartheta_1) = 1,$$
$$\hat{\varphi}^2(\mathscr{G}_1) = -\mathscr{G}_1 + \hat{\eta}(\mathscr{G}_1)\vartheta_1,$$
$$\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_1, \hat{\varphi}\mathscr{G}_2) = \hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2) - \hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2),$$

for any $\mathscr{G}_1, \mathscr{G}_2 \in \Gamma(\Theta)$. Thus for ϑ_1=$\hat{\zeta}$ the structure $(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{\mathfrak{g}})$ leads to a contact metric structure in $\mathfrak{R}^3$. We recall the Koszul's formula

$$2\hat{\mathfrak{g}}(\nabla_{\mathscr{G}_1}\mathscr{G}_2, \mathscr{G}_3) = \mathscr{G}_1(\hat{\mathfrak{g}}(\mathscr{G}_2, \mathscr{G}_3)) + \mathscr{G}_2(\hat{\mathfrak{g}}(\mathscr{G}_3, \mathscr{G}_1)) - \mathscr{G}_3(\hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2)) - \hat{\mathfrak{g}}(\mathscr{G}_1, [\mathscr{G}_2, \mathscr{G}_3]) - \hat{\mathfrak{g}}(\mathscr{G}_2, [\mathscr{G}_1, \mathscr{G}_3]) + \hat{\mathfrak{g}}(\mathscr{G}_3, [\mathscr{G}_1, \mathscr{G}_2]).$$

Making use Koszul's formula we have:

$$\left\{ \begin{array}{l} \nabla_{\vartheta_1}\vartheta_1 = -\vartheta_3, \quad \nabla_{\vartheta_1}\vartheta_2 = 0, \quad \nabla_{\vartheta_1}\vartheta_3 = \vartheta_1, \\ \nabla_{\vartheta_2}\vartheta_2 = 0, \quad \nabla_{\vartheta_2}\vartheta_3 = \vartheta_2, \quad \nabla_{\vartheta_3}\vartheta_1 = 0, \\ \nabla_{\vartheta_3}\vartheta_3 = 0, \quad \nabla_{\vartheta_2}\vartheta_1 = 0, \quad \nabla_{\vartheta_3}\vartheta_2 = 0. \end{array} \right.$$

Also we recall the following formula

$$\nabla_{\mathscr{G}_1}\vartheta_1 = -\hat{\varphi}\mathscr{G}_1 - \hat{\varphi}h\mathscr{G}_1.$$

Using above formula, one can easily calculate

$$h\vartheta_2 = -\vartheta_2, \ h\vartheta_3 = -\vartheta_3, \ h\vartheta_1 = 0.$$

The non-vanishing component of $\overset{\star}{\mathscr{R}}$ as follows:

$$\left\{ \begin{array}{lll} \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_1)\vartheta_1 = \vartheta_2, & \overset{\star}{\mathscr{R}}(\vartheta_3, \vartheta_1)\vartheta_1 = \vartheta_3, & \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_1)\vartheta_1 = \vartheta_2, \\ \overset{\star}{\mathscr{R}}(\vartheta_1, \vartheta_2)\vartheta_2 = \vartheta_1, & \overset{\star}{\mathscr{R}}(\vartheta_1, \vartheta_3)\vartheta_3 = \vartheta_1, & \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_3)\vartheta_3 = \vartheta_2, \\ \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_3)\vartheta_2 = -\vartheta_3, & \overset{\star}{\mathscr{R}}(\vartheta_1, \vartheta_3)\vartheta_3 = \vartheta_1, & \overset{\star}{\mathscr{R}}(\vartheta_3, \vartheta_1)\vartheta_1 = \vartheta_3. \end{array} \right.$$

We conclude that κ=1 and μ=0. Consequently ϑ_1=$\hat{\zeta} \in N(1,0)$-nullity distribution. Also the value of $\overset{\star}{\mathscr{S}}$ as below:

$$\overset{\star}{\mathscr{S}}(\vartheta_1, \vartheta_1) = \overset{\star}{\mathscr{S}}(\vartheta_2, \vartheta_2) = \overset{\star}{\mathscr{S}}(\vartheta_3, \vartheta_3) = 2. \tag{9.1}$$

In this case, equation (8.6) reduces to

$$\overset{\star}{\mathscr{S}}(\vartheta_1, \vartheta_1) = \overset{\star}{\mathscr{S}}(\vartheta_2, \vartheta_2) = \overset{\star}{\mathscr{S}}(\vartheta_3, \vartheta_3) = -(\lambda + \dot{\varphi}). \tag{9.2}$$

It is clear that from (9.1) and (9.2) that λ=$-(2+\dot{\varphi})$ and hence k=1, for n=1. Therefore, the Theorem 8.4 is verified.

10. Conclusion

The exploration of the $\mathscr{Z}^*$-tensor in pseudo-Riemannian manifolds and space-times delves into their geometric characteristics, curvature patterns, and overall behavior using mathematical methods like differential forms. This research into such manifolds not only enhances our comprehension of geometric structures with limited symmetries but also has practical implications in various fields, including physics. For instance, Mantica and Molinari defined the $\mathscr{Z}^*$-tensor [29] in 2012 and introduced many interesting results and applications in physics. Thereafter many authors study various properties of these tensors ([49]- [51]). Inspired by these works we study some geometric properties of $N(k)-(CMM)_{2n+1}$, whose metrics are the $\mathscr{Z}^*$-soliton and deduce some interesting results.

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