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# RICCI CURVATURES ON HYPERSURFACES OF ALMOST PRODUCT-LIKE STATISTICAL MANIFOLDS[1]

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**Abstract**

Riemannian curvature invariants on hypersurfaces of an almost product-like manifold with constant curvature $\upsilon$ are computed. Various relationships involving sectional curvatures and Ricci curvatures have been obtained. Using the Chen-Ricci inequality, some characterizations are presented.

**Keywords:** Riemannian curvature invariants, hypersurfaces, product manifolds, Chen-Ricci inequality

## 1. Introduction

Riemannian invariants are crucial in the manifold theory because they determine the intrinsic properties of Riemannian manifolds. The most popularly studied Riemannian invariants are the curvature invariants. Among the Riemann curvature invariants, the classically known are sectional, Ricci, and scalar curvatures. In the literature, we expose various studies related to fundamental inequalities, including Ricci curvature and squared mean curvature, for dissimilar types of submanifolds of real space forms. The first studies in this direction were made by B. Y. Chen in [1-4], etc. This kind of inequalities was studied in [5-10], etc.

The theory of statistical manifolds has recently been studied intensively by authors who have conducted research on differential geometry and its applications. Some applications can be seen in several fields such as image processing, physic, computer science and machine learning [11-14], etc. The notion of statistical manifolds was firstly introduced by C. R. Rao in [15]. The theory of hypersurfaces or submanifolds of statistical manifolds was examined in [16-24], etc.

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In the literature, there exist remarkable applications of Riemannian product manifolds [25-34], etc. The concept of almost product-like manifolds was introduced as follows [19]:

Let $(\widetilde{M}, \tilde{h})$ be a Riemannian manifold with two almost product structures $F$ and $F^*$ providing the condition

$$\tilde{h}(FY_1, Y_2) = \tilde{h}(Y_1, F^* Y_2) \tag{1}$$

for each $Y_1, Y_2 \in \Gamma(TM)$. Then, $(\widetilde{M}, \tilde{h}, F)$ is entitled to an almost product-like manifold. We remark that an almost product-like manifold is an almost product-like Riemannian manifold if $F = F^*$. If an almost product manifold admitting a statistical structure is entitled to an almost product-like statistical manifold.

In light of the above-mentioned situations, we derive some relations involving the Riemannian curvature invariants on hypersurfaces of almost product-like statistical manifolds and locally product-like statistical manifolds. With the help of these equalities and inequalities, we obtain some characterizations of these hypersurfaces.

## 2. Preliminaries

Let $(\widetilde{M}, \tilde{h})$ be an $(n+1)$ −dimensional Riemannian manifold furnished with a Riemannian metric $\tilde{h}$ and $\{Z_1, Z_2, \ldots, Z_{n+1}\}$ be an orthonormal basis on $(\widetilde{M}, \tilde{h})$. The Ricci curvature at $Z_i$, $i \in \{1,2, \ldots, n+1\}$ is formulated as

$$\widetilde{Ric}^0(Z_i) = \sum_{i=1}^{n+1} \tilde{h}(\tilde{R}^0(Z_i, Z_j)Z_j, Z_i), \tag{2}$$

where $\tilde{R}^0$ is the Riemannian curvature tensor field. Note that Ricci curvature can be given by

$$\widetilde{Ric}^0(Z_i) = \sum_{i=1}^{n+1} \widetilde{K}^0_{ij}\ , \tag{3}$$

where $\widetilde{K}^0_{ij}$ indicates the sectional curvature of a plane section spanned by $Z_i$ and $Z_j$ for $i, j \in \{1,2, \ldots, n+1\}$. The scalar curvature at $p \in \widetilde{M}$ is formulated as

$$\tilde{\tau}^0(p) = \sum_{i=1}^{n+1} \widetilde{Ric}^0(Z_i). \tag{4}$$

Let $(M, h)$ be an orientable hypersurface of $(\widetilde{M}, \tilde{h})$ with the induced metric $h$ of $\tilde{h}$. Suppose that $N$ is a local unit normal field of $M$. The Gauss and Weingarten formulas are indicated by

$$\widetilde{\nabla}^0_{Y_a} Y_b = \nabla^0_{Y_a} + \mathrm{g}(A^0_N Y_a, Y_b) \qquad \text{and} \qquad \widetilde{\nabla}_{Y_a} N = -A^0_N Y_a \tag{5}$$

respectively. The Gauss equation is indicated by

$$R^0(Y_a, Y_b)Y_c = \tilde{R}^0(Y_a, Y_b)Y_c + \mathrm{h}(A^0_N Y_b, Y_c)A^0_N Y_a - \mathrm{h}(A^0_N Y_a, Y_c)A^0_N Y_b \tag{6}$$

for each $Y_a, Y_b, Y_c \in \Gamma(TM)$.

**Theorem 2.1 (Chen-Ricci inequality)** Let $M$ be an $n-$dimensional submanifold. Then, we have the following situations:

- **i.** For any unit vector field $Y$, we have

$$Ric^0(Y) \leq \frac{1}{4}n^2\|H\|^2 + \widetilde{Ric}^0_{T_pM}(Y), \tag{7}$$

where $\widetilde{Ric}^0_{T_pM}$ is the $n-$Ricci curvature of $T_pM$.
- **ii.** The equality case of (7) is satisfied for each $Y \in T_pM$ if and only if $M$ is totally geodesic or $n = 2$ and $M$ is totally umbilical.

Now we recall some basic facts related to statistical manifolds:

Let $\widetilde{D}$ indicates a torsion-free connection on $(\widetilde{M}, \widetilde{h}, F)$. If $\widetilde{D}\widetilde{h}$ is symmetric, $(\widetilde{M}, \widetilde{h}, \widetilde{D}, F)$ is said to be an almost product-like statistical manifold. For any $(\widetilde{M}, \widetilde{h}, \widetilde{D}, F)$, we have

$$\widetilde{h}\left(\widetilde{D}_{Y_c}Y_a, Y_b\right) = Y_c\widetilde{h}\,(Y_a, Y_b) - \widetilde{h}\left(\widetilde{D}^*_{Y_c}Y_b, Y_a\right), \tag{8}$$

where

$$\widetilde{D}^0_{Y_a}Y_b = \frac{1}{2}\left(\widetilde{D}_{Y_a}Y_b + \widetilde{D}^*_{Y_a}Y_b\right) \tag{9}$$

for any $Y_a, Y_b, Y_c \in \Gamma\big(T\widetilde{M}\big)$. An almost product-like manifold is entitled to a locally product-like statistical manifold if $\widetilde{D}F = 0$. We note that $\widetilde{D}^*F = 0$ holds for any locally product-like statistical manifold [19].

Denote the Riemannian curvature tensors with regard to $\widetilde{D}$ and $\widetilde{D}^*$ by $\tilde{R}$ and $\tilde{R}^*$. It is known that

$$\widetilde{h}\left(\tilde{R}(Y_a, Y_b)Y_d, Y_c\right) = -\widetilde{h}\left(\tilde{R}^*(Y_a, Y_b)Y_c, Y_d\right) \tag{10}$$

is satisfied for each $Y_a, Y_b, Y_c, Y_d \in \Gamma\big(T\widetilde{M}\big)$. From (10), it is clear that $R$ and $R^*$ are not symmetric. The manifold $(\widetilde{M}, \widetilde{h}, \widetilde{D}, F)$ is said to have constant curvature $\upsilon$ if

$$\tilde{R}(Y_a, Y_b)Y_c = \upsilon\left\{\widetilde{h}\,(Y_b, Y_c)Y_a - \widetilde{h}\,(Y_a, Y_c)Y_b\right\} \tag{11}$$

holds. In view of (10) and (11), it follows that $(\widetilde{M}, \widetilde{h}, \widetilde{D}, F)$ is also of constant curvature with regard to $\tilde{R}^*$. We note that Riemannian curvatures are not symmetric.

An almost product manifold of constant curvature $\upsilon = 0$ is indicated by $\widetilde{M}(\upsilon)$. If $\upsilon = 0$, $\widetilde{M}(\upsilon)$ is entitled to a Hessian manifold [20].

For simplicity, we indite

$$\widetilde{Ric}(Z_i) = \sum_{j=1}^{n+1} \tilde{h}(\tilde{R}(Z_i, Z_j)Z_j, Z_i) = \sum_{j=1}^{n+1} \tilde{K}_{ij}, \tag{12}$$

$$\widetilde{Ric}^*(Z_i) = \sum_{j=1}^{n+1} \tilde{h}(\tilde{R}^*(Z_i, Z_j)Z_j, Z_i) = \sum_{j=1}^{n+1} \tilde{K}^*_{ij}, \tag{13}$$

$$\tilde{\tau}(p) = \sum_{i=1}^{n+1} \widetilde{Ric}(Z_i), \quad \tilde{\tau}^*(p) = \sum_{i=1}^{n+1} \widetilde{Ric}^*(Z_i). \tag{14}$$

## 3. Hypersurfaces of almost product-like statistical manifolds

Let $(\widetilde{M}, \widetilde{h}, \widetilde{D}, F)$ be an almost product-like statistical manifold and $(M, h)$ be a hypersurface of $(\widetilde{M}, \widetilde{h}, \widetilde{D}, F)$. Denote the unit normal vector field of $(M, h)$ by $N$. If $FN$ and $F^*N$ belong to $\Gamma(TM)$, then $(M, h)$ is entitled to a tangential hypersurface. For any tangential hypersurface, we write $FN = \xi$, $F^*N = \xi^*$ and

$$FY = \varphi Y + \mu^*(Y)N, \tag{15}$$
$$F^*Y = \varphi^* Y + \mu(Y)N \tag{16}$$

for any $Y \in \Gamma(TM)$, where $\mu(Y) = h(Y, \xi)$ and $\mu^*(Y) = h(Y, \xi^*)$. Using (1), (15) and (16), we find the following relations:

$$h(\varphi Y_a, Y_b) = h(Y_a, \varphi^* Y_b), \tag{17}$$
$$h(\varphi Y_a, \varphi Y_b) = h(Y_a, Y_b) - \mu^*(Y_a)\mu(Y_b), \tag{18}$$
$$\varphi^2 Y_a = X - \mu^*(Y_a)\xi, \tag{19}$$
$$\eta^*(\varphi Y_a) = \mu(\varphi^* Y_a) = 0, \tag{20}$$
$$(\varphi^*)^2 Y_a = Y_a - \mu(Y_a)\xi^*. \tag{21}$$

The Gauss and Weingarten formulas are indicated by

$$\widetilde{D}_{Y_a} Y_b = D_{Y_a} Y_b + h(A^*_N Y_a, Y_b)N, \tag{22}$$
$$\widetilde{D}_{Y_a} N = -A_N Y_a + \kappa(Y_a)N, \tag{23}$$
$$\widetilde{D}^*_{Y_a} Y_b = D^*_{Y_a} Y_b + h(A_N Y_a, Y_b)N, \tag{24}$$
$$\widetilde{D}^*_{Y_a} N = -A^*_N Y_a - \kappa(Y_a)N, \tag{25}$$

where $D_{Y_a} Y_b, D^*_{Y_a} Y_b \in \Gamma(TM)$, $A_N, A^*_N$ are the shape operators with regard to $\widetilde{D}, \widetilde{D}^*$, respectively and $\kappa$ is $1-$form. A tangential hypersurface is entitled to totally geodesic with regard to $\widetilde{D}$ (resp. $\widetilde{D}^*$) if $A_N = 0$ (resp. $A^*_N = 0$), totally umbilical with regard to $\widetilde{D}$ (resp. $\widetilde{D}^*$) if there exists a smooth function $\lambda$ such that $A_N Y_a = \lambda Y_a$ (resp. $A^*_N Y_a = \lambda Y_a$) holds [19].

The Gauss formulae of a statistical manifold is given by

$$R(Y_a, Y_b)Y_c = \tilde{R}(Y_a, Y_b)Y_c - h(A^*_N Y_a, Y_c)A_N Y_b + h(A^*_N Y_b, Y_c)A_N Y_a. \tag{26}$$

An almost product-like statistical manifold is said to have constant curvature $\upsilon$ if

$$\tilde{R}(Y_a,Y_b)Y_c = \upsilon\{\widetilde{h}\,(Y_b,Y_c)Y_a - \widetilde{h}\,(Y_a,Y_c)Y_b + \widetilde{h}\,(Y_b,FY_c)Y_a - \widetilde{h}\,(Y_a,FY_c)Y_2 + \widetilde{h}\,(FY_a,Y_b)FY_c - \widetilde{h}\,(Y_a,FY_b)FY_c\} \tag{27}$$

and

$$\tilde{R}^*(Y_a,Y_b)Y_c = \upsilon\{\widetilde{h}\,(Y_b,Y_c)Y_a - \widetilde{h}\,(Y_a,Y_c)Y_b + \widetilde{h}\,(Y_b,F^*Y_c)Y_a - \widetilde{h}\,(Y_a,F^*Y_c)FY_b + \widetilde{h}\,(F^*Y_a,Y_b)F^*Y_c - \widetilde{h}\,(Y_a,FY_b)F^*Y_c\} \tag{28}$$

are satisfied [19].

## 4. Sectional curvatures

Let $\widetilde{M}(\upsilon)$ be an almost product-like statistical manifold of constant curvature $\upsilon$ and let $(M,h)$ be an $n-$dimensional tangential hypersurface of $\widetilde{M}(\upsilon)$. Consider an orthonormal basis $\{Z_1,Z_2,\dots,Z_{n-1},X\}$ such that we write $X = a(X)\xi + b(X)\xi^*$ for some function $a,b$ defined on $\widetilde{M}$. In this case, we obtain the following orthogonal decomposition:

$$TM = \mathbb{D}_0 \oplus \mathbb{D}_1, \tag{29}$$

where $\mathbb{D}_0 = span\{Z_1,Z_2,\dots,Z_{n-1}\}$ and $\mathbb{D}_1 = span\{X\}$. Thus, we have

$$[a(X)]^2\mu(\xi) + 2a(X)b(X) + [b(X)]^2\mu^*(\xi^*) = 1, \tag{30}$$

$$\mu^*(X) = a(X) + b(X)\mu^*(X)\xi^*, \tag{31}$$

$$\mu(X) = a(X)\mu(\xi) + b(X). \tag{32}$$

**Lemma 4.1** For any tangential hypersurface of $\widetilde{M}(\upsilon)$, we have the following relations for each orthonormal vector fields $Y_1,Y_2 \in \Gamma(\mathbb{D}_0)$:

$$\widetilde{h}\left(\tilde{R}(Y_a,Y_b)Y_b,Y_a\right) = \upsilon\{1 + h(\varphi Y_a,Y_a)h(\varphi Y_b,Y_b) - 2h^2(\varphi^*Y_a,Y_b) + h(\varphi Y_a,Y_b)h(\varphi^*Y_a,Y_b)\}, \tag{33}$$

$$\widetilde{h}\left(\tilde{R}^*(Y_a,Y_b)Y_b,Y_a\right) = \upsilon\{1 + h(\varphi^*Y_a,Y_a)h(\varphi^*Y_b,Y_b) - 2h^2(\varphi Y_a,Y_b) + h(\varphi Y_a,Y_b)h(\varphi^*Y_a,Y_b)\}, \tag{34}$$

$$\widetilde{h}\left(\tilde{R}(Y_a,\xi)\xi,Y_a\right) = \upsilon\,\mu(\xi), \tag{35}$$

$$\widetilde{h}\left(\tilde{R}(Y_a,\xi)\xi^*,Y_a\right) = \widetilde{h}\left(\tilde{R}(Y_a,\xi^*)\xi,Y_a\right) = \upsilon, \tag{36}$$

$$\widetilde{h}\left(\tilde{R}(Y_a,\xi^*)\xi^*,Y_a\right) = \upsilon\mu^*(\xi^*), \tag{37}$$

$$\widetilde{h}\left(\tilde{R}^*(Y_a,\xi)\xi,Y_a\right) = \upsilon\mu(\xi), \tag{38}$$

$$\widetilde{h}\left(\tilde{R}^*(Y_a,\xi^*)\xi,Y_a\right) = \widetilde{h}\left(\tilde{R}^*(Y_1,\xi)\xi^*,Y_a\right) = \upsilon, \tag{39}$$

$$\widetilde{h}\left(\tilde{R}^*(Y_a,\xi^*)\xi^*,Y_a\right) = \upsilon\mu^*(\xi^*). \tag{40}$$

**Proof.** Since $\widetilde{M}(v)$ is an almost product-like statistical manifold of constant curvature $v$, we write

$$\begin{aligned}\widetilde{h}\left(\tilde{R}(Y_a,Y_b)Y_c,Y_d\right) = v\{&\widetilde{h}\,(Y_b,Y_c)\widetilde{h}\,(Y_a,Y_d) - \widetilde{h}\,(Y_a,Y_c)\widetilde{h}\,(Y_b,Y_d) + \widetilde{h}\,(Y_b,FY_c)\widetilde{h}\,(FY_a,Y_d)\\ &-\widetilde{h}\,(Y_a,F\,Y_c)\widetilde{h}\,(FY_b,Y_d) + \widetilde{h}\,(FY_a,Y_b)\widetilde{h}\,(FY_c,Y_d)\\ &-\widetilde{h}(Y_a,FY_b)\widetilde{h}\,(FY_c,Y_d)\}\end{aligned} \tag{41}$$

and

$$\begin{aligned}\widetilde{h}\left(\tilde{R}^*(Y_a,Y_b)Y_c,Y_d\right) = v\{&\widetilde{h}\,(Y_b,Y_c)\widetilde{h}\,(Y_a,Y_d) - \widetilde{h}\,(Y_a,Y_c)\widetilde{h}\,(Y_b,Y_d) + \widetilde{h}\left(Y_b,\tilde{F}^*Y_c\right)\widetilde{h}\left(\tilde{F}^*Y_a,Y_d\right)\\ &-\widetilde{h}\left(Y_a,\tilde{F}^*\,Y_c\right)\widetilde{h}\left(\tilde{F}^*Y_b,Y_d\right) + \widetilde{h}\left(\tilde{F}^*Y_a,Y_b\right)\widetilde{h}\left(\tilde{F}^*Y_c,Y_d\right)\\ &-\widetilde{h}\left(Y_a,\tilde{F}^*Y_b\right)\widetilde{h}\left(\tilde{F}^*Y_c,Y_d\right)\}.\end{aligned} \tag{42}$$

By a straightforward computation, the proofs of (33)-(40) are easy to follow. ■

**Theorem 4.2** For any tangential hypersurface,

$$\widetilde{K}(\pi) = \widetilde{K}^*(\pi) = 0$$

is satisfied for any plane section $\pi$ spanned by a vector field $\Gamma(\mathbb{D}_0)$ and $\xi$ or $\xi^*$ if and only if $v = 0$.

**Lemma 4.3** For any tangential hypersurface of $\widetilde{M}(v)$, we have the following relations for each unit vector fields $X \in \Gamma(\mathbb{D}_1)$ and $Y \in \Gamma(\mathbb{D}_0)$:

$$\widetilde{h}\left(\tilde{R}(Y,X)X,Y\right) = v, \tag{44}$$

$$\widetilde{h}\left(\tilde{R}^*(Y,X)X,Y\right) = v. \tag{45}$$

**Proof.** Putting $Y$ instead of $Y_1, Y_4$ and $X$ instead of $Y_2, Y_3$ in (41), we have

$$\begin{aligned}\widetilde{h}\left(\tilde{R}(Y,X)X,Y\right) &= \widetilde{h}\left(\tilde{R}(Y,a(X)\xi + b(X)\xi^*)a(X)\xi + b(X)\xi^*,Y\right)\\ &= [a(X)]^2\widetilde{h}\left(\tilde{R}(Y,X)X,Y\right) + a(X)b(X)\widetilde{h}\left(\tilde{R}(Y,\xi)\xi^*,Y\right)\\ &\quad +a(X)\,b(X)\widetilde{h}\left(\tilde{R}(Y,\xi^*)\xi,Y\right) + [b(X)]^2\widetilde{h}\left(\tilde{R}(Y,\xi^*)\xi^*,Y\right).\end{aligned} \tag{46}$$

Substituting (33), (35), (36) and (37) into (46), we obtain (44). Writing $Y$ instead of $Y_a, Y_d$ and $X$ instead of $Y_b, Y_c$ in (42), we have

$$\begin{aligned}\widetilde{h}\left(\tilde{R}^*(Y,X)X,Y\right) &= \widetilde{h}\left(\tilde{R}^*(Y,a(X)\xi + b(X)\xi^*)a(X)\xi + b(X)\xi^*,Y\right)\\ &= [a(X)]^2\widetilde{h}\left(\tilde{R}^*(Y,X)X,Y\right) + a(X)b(X)\widetilde{h}\left(\tilde{R}^*(Y,\xi)\xi^*,Y\right)\\ &\quad +a(X)\,b(X)\widetilde{h}\left(\tilde{R}^*(Y,\xi^*)\xi,Y\right) + [b(X)]^2\widetilde{h}\left(\tilde{R}^*(Y,\xi^*)\xi^*,Y\right).\end{aligned} \tag{47}$$

From (30) and substituting (4), (38), (39) and (40) into (47), we obtain (45). ■

From Lemma 4.3, we find

**Theorem 4.4** For any tangential hypersurface, we have

$$\widetilde{K}(\pi)=\widetilde{K}^{*}(\pi) \tag{48}$$

for any plane section $\pi$ spanned by unit vector fields $X \in \Gamma(\mathbb{D}_1)$ and $Y \in \Gamma(\mathbb{D}_0)$.

Let $\pi = span\{Z_1, Z_2\}$ be a two-dimensional orthonormal plane section of $\Gamma(\mathrm{TM})$. The $K-$ sectional curvature is defined by [35]

$$\widetilde{K}^{s}(\pi)=\frac{1}{2}\{\tilde{h}\left(\tilde{R}(Z_1,Z_2)Z_2,Z_1\right)+\tilde{h}\left(\tilde{R}^{*}(Z_1,Z_2)Z_2,Z_1\right)-\tilde{h}\left(\tilde{R}^{0}(Z_1,Z_2)Z_2,Z_1\right)\}. \tag{49}$$

We note that $\widetilde{K}^{s}$ is symmetric and independent of choosing linearly independent vector fields on plane section $\pi$.

**Theorem 4.5** For any tangential hypersurface $(M,h)$ of $\widetilde{M}(\upsilon)$, the following relations are satisfied:

$$\begin{aligned}\widetilde{K}^{s}(\pi_0)+\widetilde{K}^{0}(\pi_0)=\upsilon\{&1+h(\varphi Y_a,Y_a)h(\varphi Y_b,Y_b)-h^2(\varphi Y_a,Y_b)-h^2(\varphi^{*}Y_a,Y_b)\\ &h(\varphi Y_a,Y_b)+h(\varphi^{*}Y_a,Y_b)\},\end{aligned} \tag{50}$$

where $\pi_0 = span\{Y_a, Y_b\}$ for any $Y_a, Y_b \in \Gamma(\mathbb{D}_0)$ and $\widetilde{K}^{0}(\pi_0)$ indicates the sectional curvature of $\pi_0$.

**Proof.** In view of (33), (34) and (49), it follows that

$$\begin{aligned}\widetilde{K}^{s}(\pi_0)=&\frac{\upsilon}{2}\{1+h(\varphi Y_a,Y_a)h(\varphi Y_b,Y_b)-2\,h^2(\varphi^{*}Y_a,Y_b)+h(\varphi Y_a,Y_b)h(\varphi^{*}Y_a,Y_b)\}\\ &+\frac{\upsilon}{2}\{1+h(\varphi^{*}Y_a,Y_a)h(\varphi^{*}Y_b,Y_b)-2\,h^2(\varphi Y_a,Y_b)+h(\varphi Y_a,Y_b)h(\varphi^{*}Y_a,Y_b)\}\\ &-\tilde{h}\left(\tilde{R}^{0}(Z_1,Z_2)Z_2,Z_1\right),\end{aligned}$$

which is in equivalent to (50). ■

**Theorem 4.6** For any tangential hypersurface $(M,h)$ of $\widetilde{M}(\upsilon)$, the following relations are satisfied:

$$\widetilde{K}^{s}(\pi_1)+\widetilde{K}^{0}(\pi_1)=\upsilon, \tag{51}$$

where $\pi_1 = span\{X, \xi\}$ for any $X \in \Gamma(\mathbb{D}_1)$ and $\widetilde{K}^{0}(\pi_1)$ indicates the sectional curvature of $\pi_1$.

**Proof.** The proof of (51) is straightforward based on (44), (45) and (49). ■

**Theorem 4.7** Let $(M,h)$ be a tangential hypersurface of $\widetilde{M}(\upsilon)$ and let $\pi$ be a plane section spanned by $\xi$ and $\xi^{*}$ .Then, we have the following situations:

- **i.** $\widetilde{K}^{s}(\pi)+\widetilde{K}^{0}(\pi)>0$ if and only if $\upsilon>0$.
- **ii.** $\widetilde{K}^{s}(\pi)+\widetilde{K}^{0}(\pi)=0$ if and only if $\upsilon=0$.
- **iii.** $\widetilde{K}^{s}(\pi)+\widetilde{K}^{0}(\pi)<0$ if and only if $\upsilon<0$.

**Proof.** Substituting $a=0$ and $b=1$ into (51), we find

$\widetilde{K}^s(\pi) + \widetilde{K}^0(\pi) = \upsilon\mu^*(\xi^*).$

The proof is easy to obtain because $\mu^*(\xi^*) > 0$. ■

It is known that a space form with constant curvature $\widetilde{K}^0$ is isometric to

- **i.** the Euclidean space if $\widetilde{K}^0 = 0$,
- **ii.** a sphere if $\widetilde{K}^0 > 0$,
- **iii.** the hyperbolic space if $\widetilde{K}^0 < 0$.

As a result of Theorem 4.7, we get

**Corollary 4.8** Let $(M, h)$ be a tangential hypersurface of a Hessian manifold. Then, we have

- **i.** $\widetilde{K}^s(\pi) > 0$ if and only if $(M, h)$ is a hypersurface of the hyperbolic space.
- **ii.** $(M, h)$ is a hypersurface of the Euclidean space.
- **iii.** $(M, h)$ is a hypersurface of the sphere.

## 5. Ricci curvatures of tangential hypersurfaces

We begin this section with recalling sub-plane sections of Ricci curvatures of $\left(\widetilde{M}, \widetilde{h}, \widetilde{D}, F\right)$.

Let $\dim \widetilde{M} = n + 1$ and $\pi_l$ be an $l-$ dimensional subsection of $\Gamma(T\widetilde{M})$. Suppose that $\{Z_1, Z_2, \ldots, Z_l\}$ is an orthonormal basis of $\pi_l$ . Then, the Ricci curvature of $\pi_l$ is defined as [2]

$$\widetilde{Ric}^0_{\pi_l}(Z_m) = \sum_{j \neq m}^{l} \widetilde{K}^0_{mj}. \tag{52}$$

Inspired by this definition, we write

$$\widetilde{Ric}_{\pi_l}(Z_m) = \sum_{j=1}^{l} \widetilde{h}\left(\tilde{R}(Z_m, Z_j)Z_j, Z_m\right), \tag{53}$$

$$\widetilde{Ric}^*_{\pi_l}(Z_m) = \sum_{j=1}^{l} \widetilde{h}\left(\tilde{R}^*(Z_m, Z_j)Z_j, Z_m\right) \tag{54}$$

and

$$\widetilde{Ric}^s_{\pi_l}(Z_m) = \sum_{j=1}^{l} \widetilde{K}^s_{mj}. \tag{55}$$

We call $\widetilde{Ric}^s_{\pi_l}$ the $s-$Ricci curvature of $\pi_l$. For $l = n,\ \pi_l = T_pM$ for $p \in M$ and

$$\widetilde{Ric}^0_{T_pM}(Z_m) = \sum_{j\neq m}^{n} \widetilde{K}^0_{mj}, \tag{56}$$

$$\widetilde{Ric}_{T_pM}(Z_m) = \sum_{j=1}^{n} \widetilde{h}\left(\widetilde{R}\big(Z_m, Z_j\big)Z_j, Z_m\right), \tag{57}$$

$$\widetilde{Ric}^*_{T_pM}(Z_m) = \sum_{j=1}^{n} \widetilde{h}\left(\widetilde{R}^*\big(Z_m, Z_j\big)Z_j, Z_m\right) \tag{58}$$

and

$$\widetilde{Ric}^s_{T_pM}(Z_m) = \sum_{j=1}^{n} \widetilde{K}^s_{mj}. \tag{59}$$

We call $\widetilde{Ric}^s_{T_pM}$ the $s-$Ricci curvature of $T_pM$.

**Lemma 5.1** For any tangential hypersurface of $\widetilde{M}(\upsilon)$, we have the following relations:

$$\begin{aligned}\widetilde{Ric}_{T_pM}(Y_a) = \upsilon\{&n + 1 + h(\varphi Y_a, Y_a)[trace\ \varphi - a(X)b(X)h(\varphi\xi^*, \xi)] - 2\|\varphi^* Y_a\|^4 \\ &-2[a(X)]^2 h^2(\varphi^* Y_a, \xi^*) - a(X)b(X)h(\varphi Y_a, \xi)h(\varphi^* Y_{1a}, \xi^*)\},\end{aligned} \tag{60}$$

$$\begin{aligned}\widetilde{Ric}^*_{T_pM}(Y_a) = \upsilon\{&n + 1 + h(\varphi^* Y_a, Y_a)[trace\ \varphi^* - a(X)b(X)h(\varphi^*\xi, \xi^*)] - 2\|\varphi Y_a\|^4 \\ &-2[a(X)]^2 h^2(\varphi Y_a, \xi) - a(X)b(X)h(\varphi Y_a, \xi)h(\varphi^* Y_a, \xi^*)\},\end{aligned} \tag{61}$$

$$\begin{aligned}\widetilde{Ric}_{T_pM}(X) = \upsilon\Bigg\{&n - 1 + \sum_{j=1}^{n-1} h\big(\varphi Z_j, Z_j\big)h(\varphi\xi^*, \xi) + \sum_{j=1}^{n-1} h\big(\varphi^* Z_j, \xi^*\big)h\big(\varphi^*\xi, Z_j\big) \\ &+[a(X)]^2 \sum_{j=1}^{n-1} h\big(\varphi^* Z_j, \xi\big)h\big(\varphi Z_j, \xi\big) - 2[a(X)]^2 \sum_{j=1}^{n-1} h^2\big(\varphi Z_j, \xi\big)\Bigg\},\end{aligned} \tag{62}$$

$$\begin{aligned}\widetilde{Ric}^*_{T_pM}(X) = \upsilon\Bigg\{&n - 1 + \sum_{j=1}^{n-1} h\big(\varphi^* Z_j, Z_j\big)h(\varphi^*\xi, \xi) + \sum_{j=1}^{n-1} h\big(\varphi Z_j, \xi\big)h\big(\varphi\xi^*, Z_j\big) \\ &+[a(X)]^2 \sum_{j=1}^{n-1} h\big(\varphi^* Z_j, \xi\big)h\big(\varphi Z_j, \xi\big) - 2[a(X)]^2 \sum_{j=1}^{n-1} h^2\big(\varphi^* Z_j, \xi\big)\Bigg\}\end{aligned} \tag{63}$$

for each unit vector fields $Y_1 \in \Gamma(\mathbb{D}_0)$ and $X \in \Gamma(\mathbb{D}_1)$.

**Proof.** Let $\{Z_1, Z_2, \dots, Z_{n-1}\}$ be an orthonormal frame field of $\Gamma(TM)$. From (33), (44) and (57), we have

$$\begin{aligned}\widetilde{Ric}_{T_pM}(Z_1) &= \sum_{j=1}^{n-1} \tilde{h}\left(\tilde{R}(Z_1, Z_j)Z_j, Z_1\right) + \tilde{h}\left(\tilde{R}(Z_1, X)X, Z_1\right) \\ &= \sum_{j=1}^{n-1} v\{2 + h(\varphi Z_1, Z_1)h(\varphi Z_j, Z_j) - 2h^2(\varphi^* Z_1, Z_j) + h(\varphi Z_1, Z_j)h(\varphi^* Z_1, Z_j)\} \\ &= v\left\{2(n-1) + \sum_{j=1}^{n-1} h(\varphi Z_1, Z_1)h(\varphi Z_j, Z_j) - 2\sum_{j=1}^{n-1} h^2(\varphi^* Z_1, Z_j)\right. \\ &\quad \left. + \sum_{j=1}^{n-1} h(\varphi Z_1, Z_j)\, h(\varphi^* Z_1, Z_j)\right\}. \end{aligned} \tag{64}$$

Since

$$\sum_{j=1}^{n-1} h(\varphi Z_1, Z_j) = \sum_{j=1}^{n-1} h(\varphi Z_j, Z_j) + h(\varphi X, X) - h(\varphi X, X)$$

$$\begin{aligned} h(\varphi X, X) &= h(\varphi(a(X)\xi + b(X)\xi^*), a(X)\xi + b(X)\xi^*) \\ &= a(X)b(X)h(\varphi\xi^*, \xi), \end{aligned}$$

we have

$$\sum_{j=1}^{n-1} h(\varphi Z_j, Z_j) = trace\ \varphi - a(X)b(X)h(\varphi\xi^*, \xi). \tag{65}$$

Furthermore, we have

$$\sum_{j=1}^{n-1} h(\varphi^* Z_1, Z_j) = \sum_{j=1}^{n-1} h(\varphi^* Z_1, Z_j) + h(\varphi^* Z_1, X) - h(\varphi^* Z_1, X),$$

$$h(\varphi^* Z_1, X) = h(\varphi^* Z_1, a(X)\xi + b(X)\xi^*) = b(X)h(\varphi^* Z_1, \xi^*).$$

From the above equations, we get

$$\sum_{j=1}^{n-1} h^2(\varphi^* Z_1, Z_j) = \|\varphi^* Z_1\|^4 - b^2(X)h^2(\varphi^* Z_1, \xi^*), \tag{66}$$

$$\begin{aligned}\sum_{j=1}^{n-1} h(\varphi Z_1, Z_j)\, h(\varphi^* Z_1, Z_j) = &\sum_{j=1}^{n-1} h(\varphi Z_1, Z_j)\, h(\varphi^* Z_1, Z_j) + h(\varphi Z_1, X)h(\varphi^* Z_1, X) \\ &- h(\varphi Z_1, X)h(\varphi^* Z_1, X),\end{aligned}$$

$$\begin{aligned} h(\varphi Z_1, X)h(\varphi^* Z_1, X) &= h(\varphi Z_1, a(X)\xi + b(X)\xi^*)h(\varphi^* Z_1, a(X)\xi + b(X)\xi^*) \\ &= a(X)b(X)h(\varphi Z_1, \xi)\, h(\varphi^* Z_1, \xi^*), \end{aligned}$$

$$\sum_{j=1}^{n-1} h(\varphi Z_1, Z_j)\, h(\varphi^* Z_1, Z_j) = 1 - a(X)b(X)h(\varphi Z_1, \xi)h(\varphi^* Z_1, \xi^*). \tag{67}$$

By considering (65), (66) and (67) in (64), we find (60). The proofs of (61), (62) and (63) can be obtained using a similar proof technique of (60). ■

**Lemma 5.2** For any tangential hypersurface of $\widetilde{M}(\upsilon)$, we have

$$\begin{aligned}\widetilde{Ric}^s_{T_pM}(Z_1) = \upsilon\{&n + 1 + h(\varphi Z_1, Z_1)[trace\ \varphi - a(X)b(X)h(\varphi\xi^*, \xi)] + \|\varphi Z_1\|^4 \\ &-[b(X)]^2 h^2(\varphi Z_1, \xi) + \|\varphi^* Z_1\|^4 - [b(X)]^2 h^2(\varphi^* Z_1, \xi) \\ &-a(X)b(X)h(\varphi Z_1, \xi)h(\varphi^* Z_1, \xi^*)\}.\end{aligned} \tag{68}$$

**Proof.** From (56) and (59), it follows that

$$\begin{aligned}\widetilde{Ric}^s_{T_pM}(Z_1) &= \sum_{j=1}^{n-1} \widetilde{K}^s_{1j} + \widetilde{K}(Z_1, X) \\ &= \upsilon\left\{n + \sum_{j=1}^{n-1} h(\varphi Z_1, Z_1)\, h(\varphi Z_j, Z_j) + \sum_{j=1}^{n-1} h^2(\varphi Z_1, Z_j) + \sum_{j=1}^{n-1} h^2(\varphi^* Z_1, Z_j)\right. \\ &= \left.\sum_{j=1}^{n-1} h(\varphi Z_1, Z_j)\, h(\varphi^* Z_1, Z_j)\right\} - \widetilde{Ric}^0_{T_pM}(Z_1).\end{aligned} \tag{69}$$

Using a technique similar to that of Lemma 5.1, the proof of (68) is straightforward. ■

**Theorem 5.3** Let $(M, h)$ be an $n$ $-$dimensional hypersurface of $\widetilde{M}(\upsilon)$, we have

$$\begin{aligned}Ric^0(Y_a) \le \frac{1}{4}n^2\|H\|^2 + \upsilon\{&n + 1 + h(\varphi Y_a, Y_a)[trace\ \varphi - a(X)b(X)h(\varphi\xi^*, \xi)] + \|\varphi Y_a\|^4 \\ &-[b(X)]^2 h^2(\varphi Y_a, \xi) + \|\varphi^* Y_a\|^4 - [b(X)]^2 h^2(\varphi^* Y_a, \xi) \\ &-a(X)b(X)h(\varphi Y_a, \xi)h(\varphi^* Y_a, \xi^*)\}\end{aligned} \tag{70}$$

for each unit vector field $Y_a \in \Gamma(\mathbb{D}_0)$. The equality case (70) is satisfied for each unit vector field $Y_a \in \Gamma(\mathbb{D}_0)$ if and only if $M$ is $\mathbb{D}_0$ $-$geodesic or $n = 2$ and $M$ is $\mathbb{D}_0$ $-$totally umbilical.

**Proof.** Using (68) in (7), (70) is satisfied for any $Y_a \in \Gamma(\mathbb{D}_0)$. In the equality case, we obtain from Theorem 2.1 (Chen-Ricci inequality) that $A^0_N = 0$ on $\mathbb{D}_0$ or there is a smooth function $\lambda$ such that $A^0_N Y_a = \lambda Y_a$. This completes the proof. ■

**Lemma 5.4** For any tangential hypersurface of $\widetilde{M}(\upsilon)$, we have for each unit vector field $X \in \Gamma(\mathbb{D}_1)$ that

$$\widetilde{Ric}^s_{T_pM}(X) + \widetilde{Ric}^0_{T_pM}(X) = (n-1)\upsilon. \tag{71}$$

**Theorem 5.5** Let $(M, g)$ be an $n$ $-$dimensional hypersurface of $\widetilde{M}(\upsilon)$, we have

$$Ric^0(X) \leq \frac{1}{4}n^2\|H\|^2 + (n-1)\upsilon - \widetilde{Ric}^s_{T_pM}(X) \tag{72}$$

for each unit vector field $X \in \Gamma(\mathbb{D}_1)$. The equality case of (72) is satisfied for each unit vector field $X \in \Gamma(\mathbb{D}_1) \Leftrightarrow M$ is $\mathbb{D}_1$ −geodesic or $n = 2$ and $M$ is $\mathbb{D}_1$ −totally umbilical.

**Proof.** From (71) in (7), we find (72) is satisfied for any $X \in \Gamma(\mathbb{D}_1)$. From Theorem 2.1 (Chen-Ricci inequality), we obtain $A_N^0 = 0$ on $\mathbb{D}_1$ or $n = 2$ and there is a smooth function $\lambda$ such that $A_N^0 X = \lambda X$. This completes the proof. ■

**Corollary 5.6** Let $(M, g)$ be an $n$ −dimensional hypersurface of a Hessian manifold. Then, we have

$$Ric^0(X) \leq \frac{1}{4}n^2\|H\|^2 - \widetilde{Ric}^s_{T_pM}(X) \tag{73}$$

for each unit vector field $X \in \Gamma(\mathbb{D}_1)$. The equality case of (73) is satisfied for each unit vector field $X \in \Gamma(\mathbb{D}_1) \Leftrightarrow M$ is $\mathbb{D}_1$ −geodesic or $n = 2$ and $M$ is $\mathbb{D}_1$ −totally umbilical.

**Corollary 5.7** Let $(M, g)$ be an $n$ −dimensional minimal hypersurface of a Hessian manifold. Then, we have

$$Ric^0(X) \geq \widetilde{Ric}^s_{T_pM}(X) \tag{74}$$

for each unit vector field $X \in \Gamma(\mathbb{D}_1)$. The equality case of (74) is satisfied for each unit vector field $X \in \Gamma(\mathbb{D}_1) \Leftrightarrow M$ is $\mathbb{D}_1$ −geodesic or $n = 2$ and $M$ is $\mathbb{D}_1$ −totally umbilical.

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