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New Conformable P-Type $(3+1)$-Dimensional Evolution Equation and its Analytical and Numerical Solutions

Mehmet Şenol[1] , **Meliha Özlem Erol**[2]

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Abstract — The paper examines the conformable nonlinear evolution equation in $(3+1)$-dimensions. First, basic definitions and characteristics for the conformable derivative are given. Then, the modified extended tanh-function and $\exp(-\phi(\xi))$-expansion techniques are utilized to determine the exact solutions to this problem. The consequences of some of the acquired data's physical 3D and 2D contour surfaces are used to demonstrate the findings, providing insight into how geometric patterns are physically interpreted. These solutions help illustrate how the studied model and other nonlinear representations in physical sciences might be used in real-world scenarios. It is clear that these methods have the capacity to solve a large number of fractional differential equations with beneficial outcomes.

Keywords $(3+1)$*-dimensional evolution equation, modified extended tanh-function method,* $\exp(-\phi(\xi))$*-expansion method, residual power series method, conformable derivative*

Mathematics Subject Classification (2020) 35R11, 65J15

1. Introduction

In several fields of the social and fundamental sciences, as well as engineering, fractional differential equations are encountered. Their significance in several disciplines requiring complex physical processes, from electrical circuits and control theory to wave propagation, has earned more attention in recent years. Many engineering issues are modeled and designed using them. Solutions to these equations have been helpful since they highlight nonlinear physical properties more clearly and provide a path for further study. In mathematical physics, nonlinear wave equations play a role in several fields, notably chemical kinetics, solid-state physics, optical fibers, fluid mechanics, and plasma physics.

A particular kind of partial differential equation that depicts how a system changes over time is called an evolution partial differential equation(PDE). PDEs are equations involving functions and their partial derivatives concerning several independent variables in mathematics and science. Time is one of these factors that evolution PDEs particularly include, so they represent how a system changes or evolves. Dynamic processes are frequently modeled using evolution PDEs in physics, engineering, biology, and economics, among other disciplines. The wave equation, the heat equation, and the Schrödinger equation in quantum mechanics are a few examples of the evolution of PDEs. These equations, essential to comprehend physical systems' behavior, explain how variables like temperature, displacement, or wave function change over time and space.

[1]msenol@nevsehir.edu.tr(Corresponding Author); [2]m.ozlemeroll@gmail.com
[1,2]Department of Mathematics, Faculty of Arts and Sciences, Nevşehir Hacı Bektaş Veli University, Nevşehir, Türkiye

Much focus has been placed on the nonlinear evolution equations recently. It is becoming more and more attractive to look for PDE solutions directly. In applied research and sciences, the mathematical modeling of physical occurrences is an essential tool for analysis. Various mathematical methods have been used to search for solutions and an advanced knowledge of these equations. Many analytical and numerical methods have been used to solve these equations and gain an excellent grasp of them. A few of these analytical methods are: Unified Ansätze Method [1] for the optical solitons and traveling wave solutions to Kudryashov's equation, Sub-equation Method [2] for the generalized Benjamin, modified generalized multidimensional Kadomtsev–Petviashvili, modified generalized multidimensional Kadomtsev–Petviashvili–Benjamin–Bona–Mahony, and the variant Boussinesq system of equations, Sardar Sub-equation Method [3] for the Korteweg–de Vries–Zakharov–Kuznetsov equation, Jacobi Elliptic Function Expansion Method [4] for the Korteweg–de Vries, Boussinesq, Klein–Gordon, and variant Boussinesq equations, Exp-function Method [5] for the generalized shallow water-like equation, Boiti–Leon–Manna–Pempinelli, generalized variable-coefficient B-type Kadomtsev–Petviashvili, and Caudrey-Dodd-Gibbon-Kotera-Sawada equations, Extended sinh-Gordon Equation Expansion Method [6] for the Kundu–Eckhaus equation, Modified Kudryashov Method [7] for the Kuramoto–Sivashinsky and seventh-order Sawada–Kotera equations, Modified Exponential Function Method [8] for the modified Benjamin-Bona-Mahony and Sharma-Tasso-Olver equations, (G'/G)-expansion Method [9] for the the higher order Broer–Kaup, breaking soliton, and asymmetric Nizhnik–Novikov–Vesselov equations, Modified Simple Equation Method [10] for the Kaup-Newell equation, the Extended Trial Equation Method [11] for the $B(n+1,1,n)$ equation, and the Variational Direct Method [12] for the complex Ginzburg–Landau equation.

Scientists became deeply interested in inventing fractional models and discovering approximations to the generated problems. Scientists also place extensive attention on the creation and use of different methods to get these solutions. Multiple methods, frequently discovered in literature, are used to find numerical solutions for FDEs. These include the Residual Power Series Method(RPSM) [13], Homotopy Analysis Method [14], Homotopy-Perturbation, and Variational Iteration Methods [15]. It has become clear that no single method can strictly and universally solve every nonlinear problem. Many techniques were created as the result of this insight, such as Modified Extended tanh-function Method [16] and $\exp(-\phi(\xi))$-expansion Method [17].

For fractional differential equations, multiple definitions of derivative have been put upward, such as the Riemann-Liouville [18], Caputo [19] and conformable derivatives [20]. The Riemann-Liouville and Caputo fractional derivatives are notable for often used in modern mathematical discourse. Similarly, the conformable fractional derivative technique is prominent due to its dependability and ease of use.

Recently, Mohan et al. [21] has presented a new $(3+1)$-dimensional P-type evolution equation as

$$u_{xxxy} + \alpha_1 u_{yt} + \alpha_2 (uux)_y + \alpha_3 u_{xx} + \alpha_4 u_{zz} = 0 \tag{1.1}$$

The authors in this work present the Painlevé integrability analysis of the model. Using Cole-Hopf transformation and symbolic computation, they obtain the rogue waves up to the third order. Finally, they introduce dispersive-soliton solutions to this equation.

In this paper, we address some new analytical and numerical solutions of the model that do not exist in the literature. The structure of the paper is as follows: Section 2 provides some basic definitions to be needed for the following sections. Section 3 details the modified, extended tanh-function method. Section 4 describes $\exp(-\phi(\xi))$-expansion method in detail. Section 5 presents the approximation approach known as the residual power series method(RPSM). Section 6 contains analytical and numerical solutions to the underlying equation. Section 7 discusses the need for further research.

2. Preliminaries

This section provides some basic notions to be needed for the following sections.

Definition 2.1. [24] The following defines the conformable derivative of a function of order ω, $j : [0, \infty) \to \mathbb{R}$, $t > 0$, $\omega \in (0, 1)$,

$$\mathscr{D}_t^\omega(j)(t) = \lim_{\delta \to 0} \frac{j\left(t + \delta t^{1-\omega}\right) - j(t)}{\delta}$$

In addition, if $\lim_{t \to 0^+} \mathscr{D}_t^\omega(j)(t)$ exists and j is ω-differentiable in the range $(0, k)$ for $k > 0$, the definition becomes

$$\mathscr{D}_t^\omega(j)(0) = \lim_{t \to 0^+} \mathscr{D}_t^\omega(j)(t)$$

Lemma 2.2. [22–24] For $0 < \omega \leq 1$, let j_1 and j_2 be ω-differentiable at $t > 0$. Then,

i. $\mathscr{D}_t^\omega(t^{p_1}) = p_1 t^{p_1 - \omega}$, $p_1 \in \mathbb{R}$

ii. $\mathscr{D}_t^\omega(p_1 j_1 + p_2 j_2) = p_1 \mathscr{D}_t^\omega(j_1) + p_2 \mathscr{D}_t^\omega(j_2)$, $p_1, p_2 \in \mathbb{R}$

iii. $\mathscr{D}_t^\omega(\frac{j_1}{j_2}) = \frac{j_2 . \mathscr{D}_t^\omega(j_1) - j_1 \mathscr{T}_t^\omega(j_2)}{j_2^2}$

iv. $\mathscr{D}_t^\omega(j_1 . j_2) = j_1 \mathscr{D}_t^\omega(j_2) + j_2 \mathscr{D}_t^\omega(j_1)$

v. $\mathscr{D}_t^\omega(j_1)(t) = t^{1-\omega} \frac{dj_1(t)}{dt}$

vi. $\mathscr{D}_t^\omega(S) = 0$, if S is a constant

Definition 2.3. [25] Let $j(y_1, y_2, \cdots, y_n)$ be the function with n variables. Following is the partial derivatives of j in y_i of order $\omega \in (0, 1]$.

$$\frac{d^\omega}{dy_i^\omega} j(y_1, y_2, \cdots, y_n) = \lim_{\delta \to 0} \frac{j(y_1, y_2, \cdots, y_{i-1}, y_i + \delta y_i^{1-\omega}, y_n) - j(y_1, y_2, \cdots, y_n)}{\delta}$$

The following sections will introduce modified extended tanh-function, $\exp(-\phi(\xi))$-expansion, and RPS methods.

3. Modified Extented tanh-function Method

The primary stages of the modified extended tanh-function method [25–27] are explained in this section as follows. Suppose we have a nonlinear evolution equation of the type

$$\beta\left(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \cdots\right) = 0 \tag{3.1}$$

where β is a polynomial in $u(x, y, \cdots, t)$ and nonlinear components are found in its partial derivatives. Utilizing the transformation,

$$u(x, y, \cdots, t) = u(\xi),\ \xi = kx + wy + \cdots + \frac{ct^\omega}{\omega}$$

will turn (3.1) to an ODE as

$$\beta\left(u(\xi), u'(\xi), u''(\xi), \cdots\right) = 0 \tag{3.2}$$

Suppose that the form of the solution of (3.1),

$$u(\xi) = A_0 + \sum_{m=1}^{N} \left(A_m \phi^m(\xi) + B_m \phi^{-m}(\xi)\right),\ m \in \{0, 1, 2, \cdots, N\} \tag{3.3}$$

where $A_N \neq 0, B_N \neq 0$, and A_m and B_m are constants that have to be found and $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi) = \sigma + \phi(\xi)^2 \tag{3.4}$$

In this case, σ is an unknown parameter. Numerous solutions can be found for (3.4), as illustrated below

i. If $\sigma < 0$, then

$$\phi(\xi) = -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right) \text{ or } \phi(\xi) = -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right)$$

ii. If $\sigma > 0$, then

$$\phi(\xi) = \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right) \text{ or } \phi(\xi) = -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right)$$

iii. If $\sigma = 0$, then

$$\phi(\xi) = -\frac{1}{\xi}$$

In (3.3), the positive integer N is obtained by balancing the biggest nonlinear variable and the highest-order derivatives.

By replacing (3.3), its derivative, and (3.4) into (3.2), as well as collecting all the terms of the same power ϕ^m, $(m \in \{0, 1, 2, \cdots, N\})$ and equating them to zero, one can use a symbolic computation tool to determine the values of A_m and B_m. By entering these values and the solutions to (3.4) into (3.3), we can obtain the exact solutions to (3.1).

4. exp$(-\phi(\xi))$-expansion Method

Examine the nonlinear evolution equation presented in the following manner

$$\mathscr{D}(u, \mathscr{D}_t^\omega, \mathscr{D}_x u, \mathscr{D}_y u, \mathscr{D}_x^2 u, \mathscr{D}_y^2 u, \cdots) = 0 \tag{4.1}$$

The arbitrary order conformable derivative operator is represented by $\mathscr{D}_t^\omega$ in this case. $u = u(x, y, \cdots, t)$ is an unknown function, and the subscripts stand for partial derivatives. When using $\exp(-\phi(\xi))$-expansion method [28–30] in order to obtain wave solutions for (4.1), the following steps must be carried out.

i. The real variables $x, y, z, \cdots, t$ are combined using a compound variable named ξ as

$$\xi = kx + wy + \cdots + \frac{ct^\omega}{\omega}, \quad u(x, y, z, \cdots, t) = u(\xi)$$

ii. The next ordinary differential equation may be obtained by reducing (4.1)

$$\mathscr{G}(u(\xi), u'(\xi), u''(\xi), \cdots) = 0 \tag{4.2}$$

iii. As the following finite series, the exact solutions may be constructed:

$$u(\xi) = \sum_{r=0}^{N} B_r(\exp\left(\xi(-\phi)\right))^r, \quad B_N \neq 0, 0 \leq r \leq N \tag{4.3}$$

iv. The following ordinary differential equation can be solved for $\phi = \phi(\xi)$

$$\phi'(\xi) = \exp(-\phi(\xi)) + \eta\exp(\phi(\xi)) + \lambda \tag{4.4}$$

v. The following are the possible solutions to (4.4) for $\lambda^2 - 4\eta > 0$ and $\eta \neq 0$, depending on the pertinent parameters.

$$u_1(\xi) = \frac{\ln\left(-\sqrt{(\lambda^2 - 4\eta)}\tanh\left(\frac{\sqrt{(\lambda^2-4\eta)}}{2}(\xi + h)\right) - \lambda\right)}{2\eta}$$

when $\lambda^2-4\eta<0$ and $\eta\neq 0$ are present,

$$u_2(\xi)=\frac{\ln\left(\sqrt{(4\eta-\lambda^2)}\tanh\left(\frac{\sqrt{(4\eta-\lambda^2)}}{2}(\xi+h)\right)-\lambda\right)}{2\eta}$$

when $\lambda^2-4\eta>0$, $\lambda\neq 0$, and $\eta=0$ are present,

$$u_3(\xi)=-\ln\left(\frac{\lambda}{\sinh(\lambda(\xi+h))+\cosh(\lambda(\xi+h))-1}\right)$$

when $\lambda^2-4\eta=0$, $\lambda\neq 0$, and $\eta\neq 0$ are present,

$$u_4(\xi)=\ln\left(-\frac{2(\lambda(\xi+h)+2)}{\lambda^2(h+\xi)}\right)$$

when $\lambda^2-4\eta=0$, $\lambda=0$, and $\eta=0$ are present,

$$u_5(\xi)=\ln(\xi+h)$$

in which h serves as the integration constant.

vi. The positive integer N is determined by considering the homogeneous balance between the highest order derivatives of $u(\xi)$ as given in (4.2) and the biggest nonlinear term. When (4.2) is replaced by (4.3) along with (4.4), and terms with the same powers of $\exp(-\phi)$ are combined, the left side of (4.2) becomes a polynomial. A series of algebraic equations in terms of $B_r(r\in\{0,1,2,3,\cdots,N\})$, c, λ, and η are produced. We get solutions for (4.2) by equating all of this polynomial's coefficients to zero, solving the ensuing system of algebraic equations, and then substituting the solutions back into (4.3).

5. Residual Power Series Method(RPSM)

Examine the following nonlinear fractional differential equation to illustrate the basis of the RPS method [31, 32].

$$h(x,y,z,t)=\mathscr{D}_\omega u(x,y,z,t)+R[x,y,z]u(x,y,z,t)+N[x,y,z]u(x,y,z,t) \tag{5.1}$$

The initial condition is

$$u(x,y,z,0)=f_0(x,y,z)=f(x,y,z) \tag{5.2}$$

$R[x,y,z]$ is a linear operators and $N[x,y,z]$ is a nonlinear operators. The RPS method requires expanding the unknown function to a fractional series at $t=0$ to find the approximate solutions to (5.1), subject to (5.2).

Thus, the solution may be represented as follows using a series expansion

$$u(x,y,z,t)=\sum_{n=0}^{\infty}f_n(x,y,z)\frac{t^{n\omega}}{\omega^n n!}$$

Consequently, for $0\leq t<\mathbb{R}^{\frac{1}{v}}$ and $0<\omega\leq 1$, the k-th series of $u(x,y,z,t)$, or $u_k(x,y,z,t)$, is determined to be as follows

$$u_k(x,y,z,t)=f(x,y,z)+\sum_{n=1}^{k}f_n(x,y,z)\frac{t^{n\omega}}{\omega^n n!},\quad k\in\{1,2,3,\cdots\} \tag{5.3}$$

Then, we express the residual function and the coefficient k-th residual function as

$$\text{Res}u_k(x,y,z,t)=\mathscr{D}_\omega u_k(x,y,z,t)+R[x,y,z]u_k(x,y,z,t)+N[x,y,z]u_k(x,y,z,t)-h(x,y,z,t) \tag{5.4}$$

where $k\in\{1,2,3,\cdots\}$. For $\text{Res}u(x,y,z,t)=0$ and $\lim_{k\to\infty}\text{Res}u_k(x,y,z,t)=\text{Res}u(x,y,z,t)$, it is obvious that $t\geq 0$.

Calculating out $\text{Res}u_1(x, y, z, 0) = 0$, yields the first unknown function, $f_1(x, y, z)$. The fractional derivative of a constant is 0 in the conformable sense, hence $\mathscr{D}_t^{(n-1)\omega}\text{Res}u_k(x, y, z, t) = 0$ relative to $n \in \{1, 2, 3, \cdots, k\}$. The desired $f_n(x, y, z)$ coefficients are obtained by solving this equation for $t = 0$. Thus, $u_n(x, y, z, t)$ solutions may be determined, respectively.

6. Solutions for the Equation

Examine the conformable version of (1.1) in the specific situation provided as follows for the next two analytical methods

$$u_{xxxy} + \mathscr{D}_t^{\omega}\alpha_1 u_y + \alpha_2(uux)_y + \alpha_3 u_{xx} + \alpha_4 u_{zz} = 0 \tag{6.1}$$

A conformable fractional derivative is a mathematical concept that extends the notion of classical derivatives to non-integer orders in a more flexible and generalized manner. It is a relatively recent development in the field of fractional calculus [24]. Compared to classical fractional derivatives, conformable fractional derivative has two essential advantages. First, most of the properties of the classical derivative, including linearity, quotient rule, product rule, power rule, and chain rule, are satisfied by the conformable fractional derivative. Second, differential equations with a conformable fractional derivative are more straightforward to solve numerically than those involving the Riemann-Liouville or Caputo fractional derivatives, making it very convenient to model many physical problems.

After doing the transformation as $u(x, y, z, t) = u(\xi)$ with $\xi = kx + wy + sz + \frac{ct^{\omega}}{\omega}$, the following ODE is obtained

$$u(\xi) = \left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + k^3 w u''(\xi) + \frac{1}{2}\alpha_2 kwu(\xi)^2$$

By balancing, $u^2 = 2N$, $u'' = N + 2$, and $N = 2$ is calculated. The exact solutions are obtained by substituting them into (3.3) and (4.3).

6.1. Modified Extended tanh-function Method Solutions

For $N = 2$, (3.3) takes the following form,

$$u = A_0 + A_1\phi(\xi) + B_1\phi(\xi)^{-1} + A_2\phi(\xi)^2 + B_2\phi(\xi)^{-2}$$

When combined with (3.4), the following algebraic equation system is created.

$$\alpha_2 A_1 B_1 kw + \alpha_2 A_2 B_2 kw + A_0\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 2A_2 k^3\sigma^2 w + \frac{1}{2}\alpha_2 A_0^2 kw + 2B_2 k^3 w = 0$$

$$A_2\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 8A_2 k^3\sigma w + \frac{1}{2}\alpha_2 A_1^2 kw + \alpha_2 A_0 A_2 kw = 0$$

$$2A_1 k^3 w + \alpha_2 A_1 A_2 kw = 0$$

$$6A_2 k^3 w + \frac{1}{2}\alpha_2 A_2^2 kw = 0$$

$$\alpha_2 A_2 B_1 kw + A_1\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 2A_1 k^3\sigma w + \alpha_2 A_0 A_1 kw = 0$$

$$\alpha_2 A_0 B_2 kw + B_2\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 8B_2 k^3\sigma w + \frac{1}{2}\alpha_2 B_1^2 kw = 0$$

$$\alpha_2 A_0 B_1 kw + \alpha_2 A_1 B_2 kw + B_1\left(\alpha_1 cw + \alpha_3 k^2 + \alpha_4 s^2\right) + 2B_1 k^3\sigma w = 0$$

$$2B_1 k^3\sigma^2 w + \alpha_2 B_1 B_2 kw = 0$$

and

$$6B_2 k^3\sigma^2 w + \frac{1}{2}\alpha_2 B_2^2 kw = 0$$

For A_0, A_1, A_2, B_1, B_2, and c, we have six cases and six sets of solutions

Case 1.

$$A_0{=}-\frac{24k^2\sigma}{\alpha_2},\quad A_1{=}0,\quad A_2{=}-\frac{12k^2}{\alpha_2},\quad B_1{=}0,\quad B_2{=}-\frac{12k^2\sigma^2}{\alpha_2},\quad \text{and}\quad c{=}-\frac{k^2\left(\alpha_3-16k\sigma w\right)+\alpha_4 s^2}{\alpha_1 w}$$

Set 1.

For $\sigma<0$,

$$\begin{aligned} u_1(x,y,z,t) \;=\; & -\frac{24k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\ & +\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} \end{aligned} \tag{6.2}$$

or

$$\begin{aligned} u_2(x,y,z,t) \;=\; & -\frac{24k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\ & +\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} \end{aligned}$$

For $\sigma>0$,

$$\begin{aligned} u_3(x,y,z,t) \;=\; & -\frac{24k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\ & -\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} \end{aligned}$$

or

$$\begin{aligned} u_4(x,y,z,t) \;=\; & -\frac{24k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}\\ & -\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} \end{aligned}$$

For $\sigma=0$,

$$u_5(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(-\frac{t^\omega(\alpha_3 k^2+\alpha_4 s^2)}{\alpha_1 w\omega}+kx+sz+wy\right)^2}$$

Case 2.

$$A_0{=}-\frac{12k^2\sigma}{\alpha_2},\quad A_1{=}0,\quad A_2{=}0,\quad B_1{=}0,\quad B_2{=}-\frac{12k^2\sigma^2}{\alpha_2},\quad \text{and}\quad c{=}-\frac{k^2\left(\alpha_3-4k\sigma w\right)+\alpha_4 s^2}{\alpha_1 w}$$

Set 2.

For $\sigma<0$,

$$u_6(x,y,z,t)=\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}-\frac{12k^2\sigma}{\alpha_2}$$

or

$$u_7(x,y,z,t) = \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{12k^2\sigma}{\alpha_2} \tag{6.3}$$

For $\sigma > 0$,

$$u_8(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \cot^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

or

$$u_9(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \tan^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{10}(x,y,z,t) = 0$$

which is a trivial solution.

Case 3.

$$A_0 = -\frac{12k^2\sigma}{\alpha_2}, \quad A_1 = 0, \quad A_2 = -\frac{12k^2}{\alpha_2}, \quad B_1 = 0, \quad B_2 = 0, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 - 4k\sigma w\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 3.

For $\sigma < 0$,

$$u_{11}(x,y,z,t) = \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{12k^2\sigma}{\alpha_2}$$

or

$$u_{12}(x,y,z,t) = \frac{12k^2\sigma \coth^2\left(\sqrt{-\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2} - \frac{12k^2\sigma}{\alpha_2}$$

For $\sigma > 0$,

$$u_{13}(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \tan^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

or

$$u_{14}(x,y,z,t) = -\frac{12k^2\sigma}{\alpha_2} - \frac{12k^2\sigma \cot^2\left(\sqrt{\sigma}\left(-\frac{t^{\omega}\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega} + kx + sz + wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{15}(x,y,z,t) = -\frac{12k^2}{\alpha_2\left(-\frac{t^{\omega}(\alpha_3 k^2+\alpha_4 s^2)}{\alpha_1 w\omega} + kx + sz + wy\right)^2}$$

Case 4.

$$A_0 = -\frac{4k^2\sigma}{\alpha_2}, \quad A_1 = 0, \quad A_2 = 0, \quad B_1 = 0, \quad B_2 = -\frac{12k^2\sigma^2}{\alpha_2}, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 + 4k\sigma w\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 4.

For $\sigma < 0$,

$$u_{16}(x,y,z,t) = \frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

or

$$u_{17}(x,y,z,t) = \frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

For $\sigma > 0$,

$$u_{18}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

or

$$u_{19}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{20}(x,y,z,t) = 0$$

which is a trivial solution.

Case 5.

$$A_0 = -\frac{4k^2\sigma}{\alpha_2}, \quad A_1=0, \quad A_2 = -\frac{12k^2}{\alpha_2}, \quad B_1=0, \quad B_2=0, \quad \text{and} \quad c = -\frac{k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2}{\alpha_1 w}$$

Set 5.

For $\sigma < 0$,

$$u_{21}(x,y,z,t) = \frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

or

$$u_{22}(x,y,z,t) = \frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2} - \frac{4k^2\sigma}{\alpha_2}$$

For $\sigma > 0$,

$$u_{23}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

or

$$u_{24}(x,y,z,t) = -\frac{4k^2\sigma}{\alpha_2} - \frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

For $\sigma = 0$,

$$u_{25}(x,y,z,t) = -\frac{12k^2}{\alpha_2\left(-\frac{t^\omega(\alpha_3 k^2+\alpha_4 s^2)}{\alpha_1 w\omega}+kx+sz+wy\right)^2}$$

Case 6.

$$A_0=\frac{8k^2\sigma}{\alpha_2},\quad A_1=0,\quad A_2=-\frac{12k^2}{\alpha_2},\quad B_1=0,\quad B_2=-\frac{12k^2\sigma^2}{\alpha_2},\quad \text{and}\quad c=-\frac{k^2\left(\alpha_3+16k\sigma w\right)+\alpha_4 s^2}{\alpha_1 w}$$

Set 6.

For $\sigma<0$,

$$u_{26}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$+\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

or

$$u_{27}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}+\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$+\frac{12k^2\sigma\coth^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

For $\sigma>0$,

$$u_{28}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$-\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

or

$$u_{29}(x,y,z,t)=\frac{8k^2\sigma}{\alpha_2}-\frac{12k^2\sigma\tan^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$
$$-\frac{12k^2\sigma\cot^2\left(\sqrt{\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3+16k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}$$

For $\sigma=0$,

$$u_{30}(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(-\frac{t^\omega(\alpha_3k^2+\alpha_4s^2)}{\alpha_1 w\omega}+kx+sz+wy\right)^2}$$

6.2. exp $(-\phi(\xi))$-expansion Method Solutions

Considering that $N=2$, (4.3) is as follows:

$$u=B_0+B_1\exp(-\phi(\xi))+B_2\exp(-\phi(\xi))^2$$

The algebraic system of equations next develops when (4.4) is included

$$\alpha_1B_0cw+2B_2\eta^2k^3w+B_1\eta\lambda k^3w+\alpha_3B_0k^2+\frac{1}{2}\alpha_2B_0^2kw+\alpha_4B_0s^2=0$$

$$\alpha_1 B_1 cw + 6B_2\eta\lambda k^3 w + 2B_1\eta k^3 w + B_1\lambda^2 k^3 w + \alpha_3 B_1 k^2 + \alpha_2 B_0 B_1 kw + \alpha_4 B_1 s^2 = 0$$

$$\alpha_1 B_2 cw + 8B_2\eta k^3 w + 4B_2\lambda^2 k^3 w + 3B_1\lambda k^3 w + \alpha_3 B_2 k^2 + \frac{1}{2}\alpha_2 B_1^2 kw + \alpha_2 B_0 B_2 kw + \alpha_4 B_2 s^2 = 0$$

$$10B_2\lambda k^3 w + 2B_1 k^3 w + \alpha_2 B_1 B_2 kw = 0$$

and

$$6B_2 k^3 w + \frac{1}{2}\alpha_2 B_2^2 kw = 0$$

Case 7.

$$B_0 = -\frac{12\eta k^2}{\alpha_2}, \quad B_1 = -\frac{12k^2\lambda}{\alpha_2}, \quad B_2 = -\frac{12k^2}{\alpha_2}, \quad \text{and} \quad c = -\frac{k^2\left(\alpha_3 + kw\left(\lambda^2 - 4\eta\right)\right) + \alpha_4 s^2}{\alpha_1 w}$$

Set 7.

For $\lambda^2 - 4\eta > 0$, $\eta \neq 0$,

$$\begin{aligned} v_1(x,y,z,t) = & -\frac{48\eta^2 k^2}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+kw\left(\lambda^2-4\eta\right)\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)^2} \\ & -\frac{24\eta k^2\lambda}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+kw\left(\lambda^2-4\eta\right)\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)} \\ & -\frac{12\eta k^2}{\alpha_2} \end{aligned} \tag{6.4}$$

For $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$,

$$\begin{aligned} v_2(x,y,z,t) = & -\frac{48\eta^2 k^2}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+kw\left(\lambda^2-4\eta\right)\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)^2} \\ & -\frac{24\eta k^2\lambda}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+kw\left(\lambda^2-4\eta\right)\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-\lambda\right)} \\ & -\frac{12\eta k^2}{\alpha_2} \end{aligned}$$

For $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$\begin{aligned} v_3(x,y,t,z) = & -\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+k\lambda^2 w\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)+\cosh\left(\lambda\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+k\lambda^2 w\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-1\right)} \\ & -\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+k\lambda^2 w\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)+\cosh\left(\lambda\left(G - \frac{t^\omega\left(k^2\left(\alpha_3+k\lambda^2 w\right)+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)\right)-1\right)^2} \end{aligned}$$

For $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$ and $\eta \neq 0$,

$$v_4(x,y,z,t) = -\frac{3k^2\lambda^4\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)^2}{\alpha_2\left(\lambda\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)+2\right)^2} + \frac{6k^2\lambda^3\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)}{\alpha_2\left(\lambda\left(G - \frac{t^\omega\left(\alpha_3 k^2+\alpha_4 s^2\right)}{\alpha_1 w\omega}\right)+2\right)} - \frac{12\eta k^2}{\alpha_2}$$

For $\lambda^2-4\eta=0,\ \lambda=0$ and $\eta=0$,

$$v_5(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(G-\frac{t^\omega(k^2(\alpha_3+4\eta kw)+\alpha_4s^2)}{\alpha_1w\omega}\right)^2}$$

Case 8.

$$B_0{=}-\frac{2k^2\left(2\eta+\lambda^2\right)}{\alpha_2},\quad B_1{=}-\frac{12k^2\lambda}{\alpha_2},\quad B_2{=}-\frac{12k^2}{\alpha_2},\quad \text{and}\quad c{=}\frac{k^3w\left(\lambda^2-4\eta\right)-\alpha_3k^2-\alpha_4s^2}{\alpha_1w}$$

Set 8.

For $\lambda^2-4\eta>0$, $\eta\neq0$,

$$\begin{aligned}v_6(x,y,z,t)=&-\frac{48\eta^2k^2}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)^2}\\&-\frac{24\eta k^2\lambda}{\alpha_2\left(-\sqrt{\lambda^2-4\eta}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\eta}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)}\\&-\frac{2k^2\left(2\eta+\lambda^2\right)}{\alpha_2}\end{aligned}$$

For $\lambda^2-4\eta<0$ and $\eta\neq0$,

$$\begin{aligned}v_7(x,y,z,t)=&-\frac{48\eta^2k^2}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)^2}\\&-\frac{24\eta\lambda k^2}{\alpha_2\left(\sqrt{4\eta-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\eta-\lambda^2}\left(G+\frac{t^\omega(k^3w(\lambda^2-4\eta)-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-\lambda\right)}\\&-\frac{2k^2\left(2\eta+\lambda^2\right)}{\alpha_2}\end{aligned}$$

For $\lambda^2-4\eta>0,\ \lambda\neq0$ and $\eta=0$,

$$\begin{aligned}v_8(x,y,z,t)=&-\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)+\cosh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-1\right)}\\&-\frac{12k^2\lambda^2}{\alpha_2\left(\sinh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)+\cosh\left(\lambda\left(G+\frac{t^\omega(\lambda^2k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)\right)-1\right)^2}\\&-\frac{2k^2\lambda^2}{\alpha_2}\end{aligned}$$

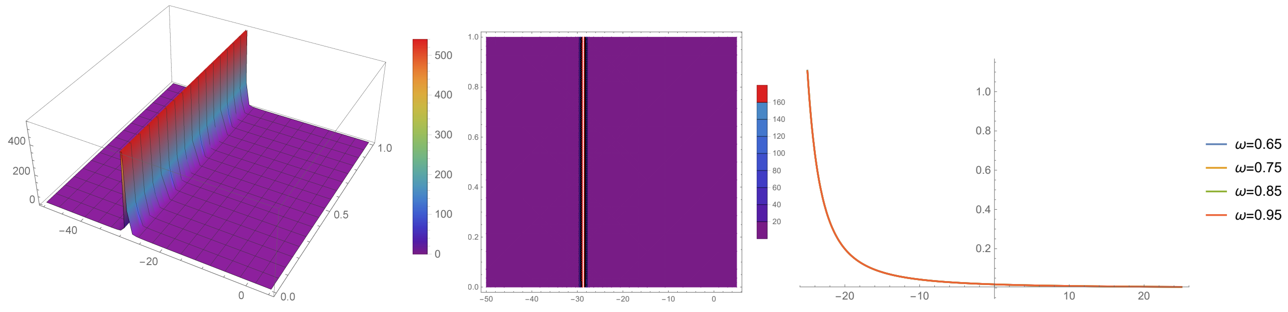
For $\lambda^2-4\eta=0,\ \lambda\neq0$ and $\eta\neq0$,

$$v_9(x,y,z,t)=-\frac{3k^2\lambda^4\left(G+\frac{t^\omega\left(\alpha_3\left(-k^2\right)-\alpha_4s^2\right)}{\alpha_1w\omega}\right)^2}{\alpha_2\left(\lambda\left(G+\frac{t^\omega(\alpha_3(-k^2)-\alpha_4s^2)}{\alpha_1w\omega}\right)+2\right)^2}+\frac{6k^2\lambda^3\left(G+\frac{t^\omega\left(\alpha_3\left(-k^2\right)-\alpha_4s^2\right)}{\alpha_1w\omega}\right)}{\alpha_2\left(\lambda\left(G+\frac{t^\omega(\alpha_3(-k^2)-\alpha_4s^2)}{\alpha_1w\omega}\right)+2\right)}-\frac{12\eta k^2}{\alpha_2}$$

For $\lambda^2-4\eta=0,\ \lambda=0$, and $\eta=0$,

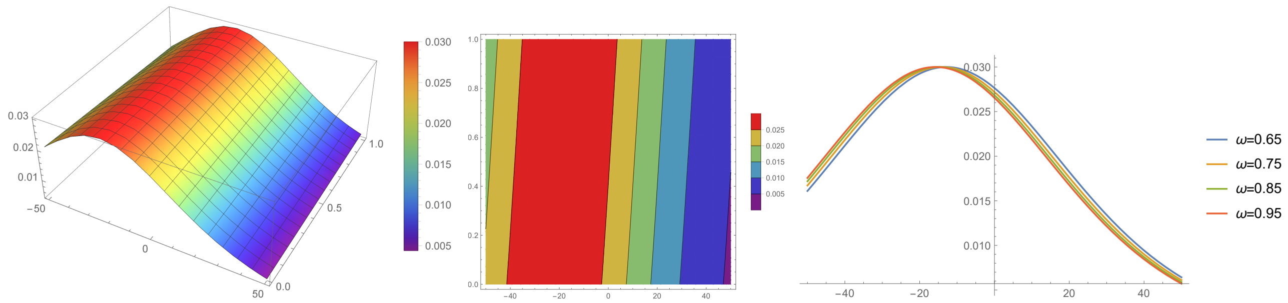
$$v_{10}(x,y,z,t)=-\frac{12k^2}{\alpha_2\left(G+\frac{t^\omega(4\eta k^3w-\alpha_3k^2-\alpha_4s^2)}{\alpha_1w\omega}\right)^2}-\frac{8\eta k^2}{\alpha_2}$$

where, $G=h+kx+sz+wy$.



(a) **(b)** **(c)**

Figure 1. Tanh-function method solution $u_1(x,y,z,t)$ of (6.2) in three dimensions(a), contour(b), and two dimensions(c)



(a) **(b)** **(c)**

Figure 2. (a) 3D,(b) contour and (c) 2D plots of $\exp(-\phi(\xi))$-expansion method solution $v_1(x,y,z,t)$ of (6.4)

6.3. RPSM Solutions

First, consider an initial condition for $t=0$, using any of the previously obtained exact solutions. If (6.3) is taken as the exact solution, the initial condition is becomes

$$u_7(x,y,z,0)=\frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}\left(-\frac{t^\omega\left(k^2(\alpha_3-4k\sigma w)+\alpha_4 s^2\right)}{\alpha_1 w\omega}+kx+sz+wy\right)\right)}{\alpha_2}-\frac{12k^2\sigma}{\alpha_2}$$

The RPSM solution takes the form of (5.3) for the approximate solutions to the $(3+1)$-dimensional P-type evolution (6.1), where $u=u(x,y,z,t)$ and $t\geq 0,\quad 0<\omega\leq 1$ the generic form of the $k-th$ residual function of the time-fractional equation may be shown using (5.4) as follows:

$$\text{Res}u_k(x,y,z,t)=u_{xxxy}+\mathscr{D}_t^\omega\alpha_1 u_y+\alpha_2(uux)_y+\alpha_3 u_{xx}+\alpha_4 u_{zz}=0$$

It is required to determine $f_1(x,y,z)$ for a known $f(x,y,z)$ function in order to establish $\text{Res}u_1(x,y,z,t)$. In considering it, $\text{Res}u_1(x,y,z,t)$ is obtained as

$$\begin{aligned}\text{Res}u_1(x,y,z,t) \;=\; &\alpha_1\left(f_1\right)_y+\alpha_2\left(\left(\left(\frac{(f_1)_x t^\omega}{\omega}+(f)_x\right)\left(\frac{(f_1)_y t^\omega}{\omega}+(f)_y\right)+\left(\frac{(f_1)\, t^\omega}{\omega}+f\right)\left(\frac{(f_1)_{xy} t^\omega}{\omega}+(f)_{xy}\right)\right)\right.\\ &+\alpha_3\left(\frac{(f_1)_{xx} t^\omega}{\omega}+(f)_{xx}\right)+\left(\frac{(f_1)_{xxxy} t^\omega}{\omega}\right)+\alpha_4\left(\frac{(f_1)_{zz} t^\omega}{\omega}+(f)_{zz}\right)+(f)_{xxxy}\end{aligned}$$

when $f_1=f_1(x,y,z)$ and $f=f(x,y,z)$ occur. Thus, the first unknown coefficient is calculated when $t=0$.

$$f_1=-\frac{24k^2\sigma^2\left(4k^3\sigma w-\alpha_3 k^2-\alpha_4 s^2\right)\tanh\left(\sqrt{-\sigma}(kx+sz+wy)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w}$$

Hence, the first approximate RPSM solution $u_1 = u_1(x, y, z, t)$ is subsequently obtained as

$$u_1 = -\frac{12k^2\sigma}{\alpha_2} + \frac{12k^2\sigma \tanh^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_2} - \frac{24k^2\sigma^2 t^\omega \left(4k^3\sigma w - \alpha_3 k^2 - \alpha_4 s^2\right)\tanh\left(\sqrt{-\sigma}(kx+sz+wy)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega}$$

Likewise, to get the second unknown parameter, the second residual function is established as

$$\begin{aligned}\mathrm{Res}u_2 =\ & \frac{(f_2)_{xxxy}\, t^{2\omega}}{2\omega^2} + \alpha_1 t^{1-\omega}\left((f_1)_y\, t^{\omega-1} + \frac{(f_2)_y\, t^{2\omega-1}}{\omega}\right) + \frac{(f_1)_{xxxy}\, t^\omega}{\omega} \\ & +\alpha_2\left(\left(\frac{(f_2)_x\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_x\, t^\omega}{\omega} + (f)_x\right)\left(\frac{(f_2)_y\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_y\, t^\omega}{\omega} + (f)_y\right)\right) \\ & +\alpha_3\left(\frac{(f_2)_{xx}\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_{xx}\, t^\omega}{\omega} + (f)_{xx}\right) \\ & +\alpha_2\left(\frac{(f_2)\, t^{2\omega}}{2\omega^2} + \frac{(f_1)\, t^\omega}{\omega} + f\right)\left(\frac{(f_2)_{xy}\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_{xy}\, t^\omega}{\omega} + (f)_{xy}\right) \\ & +\alpha_4\left(\frac{(f_2)_{zz}\, t^{2\omega}}{2\omega^2} + \frac{(f_1)_{zz}\, t^\omega}{\omega} + (f)_{zz}\right) + (f)_{xxxy}\end{aligned}$$

where $f_2(x, y, z) = f_2$. Taking the first order derivative, we can get the second unknown parameter for $t = 0$ as follows:

$$f_2 = \frac{24\sigma^2\left(-4k^4\sigma w + \alpha_3 k^3 + \alpha_4 k s^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+sz+wy)\right) - 2\right)\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1^2\alpha_2 w^2}$$

As a result, the second approximation of $u_2 = u_2(x, y, z, t)$ becomes

$$\begin{aligned}u_2 =\ & -\frac{12k^2\sigma}{\alpha_2} + \frac{12k^2\sigma\tanh^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_2} \\ & + \frac{12\sigma^2 t^{2\omega}\left(-4k^4\sigma w + \alpha_3 k^3 + \alpha_4 k s^2\right)^2\left(\cosh\left(2\sqrt{-\sigma}(kx+sz+wy)\right) - 2\right)\operatorname{sech}^4\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1^2\alpha_2 w^2\omega^2} \\ & - \frac{24k^2\sigma^2 t^\omega\left(4k^3\sigma w - \alpha_3 k^2 - \alpha_4 s^2\right)\tanh\left(\sqrt{-\sigma}(kx+sz+wy)\right)\operatorname{sech}^2\left(\sqrt{-\sigma}(kx+sz+wy)\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega}\end{aligned}$$

Likewise, the following approximate solutions appear

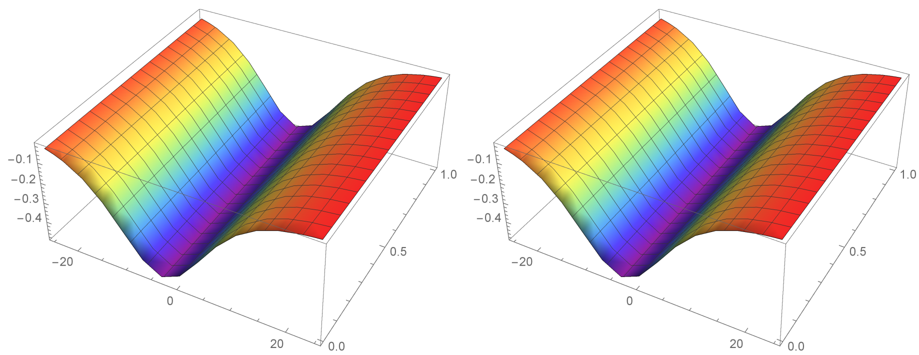
$$\begin{aligned}u_3 =\ & -\frac{12k^2\sigma}{\alpha_2} + \frac{12k^2\sigma\tanh^2((A))}{\alpha_2} \\ & + \frac{12\sigma^2 t^{2\omega}\left(-4k^4\sigma w + \alpha_3 k^3 + \alpha_4 k s^2\right)^2\left(\cosh(2(A)) - 2\right)\operatorname{sech}^4((A))}{\alpha_1^2\alpha_2 w^2\omega^2} \\ & - \frac{8k^2\sigma^3 t^{3\omega}\left(-4k^3\sigma w + \alpha_3 k^2 + \alpha_4 s^2\right)^3\left(\cosh(2(A)) - 5\right)\tanh((A))\operatorname{sech}^4((A))}{\alpha_1^3\alpha_2\sqrt{-\sigma}w^3\omega^3} \\ & - \frac{24k^2\sigma^2 t^\omega\left(4k^3\sigma w - \alpha_3 k^2 - \alpha_4 s^2\right)\tanh((A))\operatorname{sech}^2((A))}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega}\end{aligned}$$

$$
\begin{aligned}
u_4 = \; & \frac{12k^2\sigma\tanh^2(A)}{\alpha_2} + \frac{12\sigma^2(\cosh(2A)-2)\text{sech}^4(A)t^{2\omega}\left(-4k^4\sigma w+\alpha_3k^3+\alpha_4ks^2\right)^2}{\alpha_1^2\alpha_2w^2\omega^2} \\
& + \frac{k^2\sigma^3(-26\cosh(2A)+\cosh(4A)+33)\text{sech}^6(A)t^{4\omega}\left(4k^3\sigma w-\alpha_3k^2-2\alpha_4s^2\right)\left(-4k^3\sigma w+\alpha_3k^2+\alpha_4s^2\right)^3}{2\alpha_1^4\alpha_2w^4\omega^4} \\
& - \frac{8k^2\sigma^3(\cosh(2A)-5)\tanh(A)\text{sech}^4(A)t^{3\omega}\left(-4k^3\sigma w+\alpha_3k^2+\alpha_4s^2\right)^3}{\alpha_1^3\alpha_2\sqrt{-\sigma}w^3\omega^3} \\
& - \frac{24k^2\sigma^2\tanh(A)\text{sech}^2(A)t^{\omega}\left(4k^3\sigma w-\alpha_3k^2-\alpha_4s^2\right)}{\alpha_1\alpha_2\sqrt{-\sigma}w\omega} - \frac{12k^2\sigma}{\alpha_2}
\end{aligned} \tag{6.5}
$$

where, $A=\sqrt{-\sigma}(kx+sz+wy)$.

Table 1. Comparison of specific numerical values of RPSM approximate solution u_4 of (6.5) and Modified Extended tanh-function Method exact solution u_7 of (6.3)

	$\omega=0.55$			$\omega=0.85$			$\omega=0.95$		
t	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*
0.0	−0.418977	−0.418977	0.00000	−0.418977	−0.418977	0.00000	−0.418977	−0.418977	0.00000
0.1	−0.421380	−0.421380	4.4640×10^{-11}	−0.419759	−0.419759	4.5158×10^{-13}	−0.419533	−0.419533	1.1368×10^{-13}
0.2	−0.422485	−0.422485	2.1718×10^{-10}	−0.420385	−0.420385	4.9379×10^{-12}	−0.420050	−0.420050	1.6314×10^{-12}
0.3	−0.423351	−0.423351	5.5319×10^{-10}	−0.420961	−0.420961	2.0229×10^{-11}	−0.420553	−0.420553	7.8329×10^{-12}
0.4	−0.424090	−0.424090	1.0792×10^{-9}	−0.421508	−0.421508	5.5392×10^{-11}	−0.421045	−0.421045	2.4008×10^{-11}
0.5	−0.424747	−0.424747	1.8182×10^{-9}	−0.422032	−0.422032	1.2156×10^{-10}	−0.421531	−0.421531	5.7522×10^{-11}
0.6	−0.425345	−0.425345	2.7903×10^{-9}	−0.422539	−0.422539	2.3187×10^{-10}	−0.42201	−0.42201	1.1791×10^{-10}
0.7	−0.425898	−0.425898	4.0145×10^{-9}	−0.423033	−0.423033	4.0137×10^{-10}	−0.422484	−0.422484	2.1700×10^{-10}
0.8	−0.426415	−0.426415	5.5082×10^{-9}	−0.423514	−0.423514	6.4705×10^{-10}	−0.422953	−0.422953	3.6900×10^{-10}
0.9	−0.426901	−0.426901	7.2881×10^{-9}	−0.423986	−0.423986	9.8777×10^{-10}	−0.423418	−0.423418	5.9063×10^{-10}
1.0	−0.427363	−0.427363	9.3700×10^{-9}	−0.424448	−0.424448	1.4443×10^{-9}	−0.423880	−0.423880	9.0121×10^{-10}



(a) RPSM solution **(b)** Exact solution

Figure 3. Comparison of surface plots of RPSM approximate solution u_4 of (6.5) and Modified Extended tanh-function Method exact solution u_7 of (6.3)

The surface plots show some novel solutions to the present equation that might be useful for other types of differential equations of arbitrary order. Figures 1 and 2 display some of the physical characteristics of the acquired analytical solutions in 3D, 2D, and contour representations. Besides, Figure 3 compares the surface graphics of the approximate and exact solutions obtained in 3D. Concurrently, for the given Table 1 and the mentioned figures, the following values and ranges are used for the exact and approximate solutions.

i. Figure 1: $k=0.01$, $s=0.01$, $\alpha_1=0.9$, $\alpha_2=-0.8$, $\alpha_3=0.7$, $\alpha_4=0.4$, $\sigma=-0.04$, $y=0.3$, $z=0.2$, $w=0.95$, and $\omega=0.95$, $-50\le x\le 5$, for (a) and (b); $t=0.99$, for (c).

ii. Figure 2: $k = 0.2$, $w = 1$, $s = 0.01$, $y = 0.1$, $z = 0.5$, $h = 0.1$, $\eta = 0.05$, $\lambda = 0.5$, $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.1$, $\alpha_4 = 0.1$, $\omega = 0.95$, $-50 \leq x \leq 50$, for (a) and (b); $t = 0.99$, for (c).

iii. Table 1: $\alpha_1 = 0.3$, $\alpha_2 = -0.2$, $\alpha_3 = 0.01$, $\alpha_4 = 0.4$, $k = 0.3$, $s = 0.1$, $\sigma = -0.09$, $w = 0.9$, $x = 1$, $\omega = 0.95$, $y = 1$, $z = 1$, and $0 \leq t \leq 1$.

iv. Figure 3: $\alpha_1 = 0.3$, $\alpha_2 = -0.2$, $\alpha_3 = 0.01$, $\alpha_4 = 0.4$, $k = 0.3$, $s = 0.1$, $\sigma = -0.09$, $w = 0.9$, $x = 1$, $\omega = 0.95$, $y = 1$, $z = 1$ and $\omega = 0.95$, $-25 \leq x \leq 25$, for (a) and (b); $0 \leq t \leq 1$.

7. Conclusion

In the main study [21], the authors presented the Painlevé integrability analysis of the model. Additionally, they can acquire the rogue waves up to the third order by using symbolic computation and the Cole-Hopf transformation. Dispersive-soliton solutions to this equation are finally introduced. Next, very recently, multi-wave, breather, and other localized wave solutions via the Hirota bilinear method have been presented in [33]. In this paper, using modified extended tanh-function and the $\exp(-\phi(\xi))$-expansion methods, solutions to the $(3+1)$-dimensional P-type evolution equation with conformable derivative were explored in this study. The residual power series method(RPSM) was also employed to get approximate solutions. Modified extended tanh-function and $\exp(-\phi(\xi))$-expansion methods produced several accurate exact solutions with low processing complexity. Furthermore, there is no requirement for discretization, translation, or perturbation when applying the RPSM to the governing equation. 3D, 2D, and contour plots were illustrated to visually present the solutions discovered. Besides, a comparison table is presented to compare the approximate solutions with the exact solutions. These solutions have important physical characteristics that have not been previously reported in the literature and are unique. According to some interpretations of the figures, the exact solutions' physical behavior appears for particular values. Comprehending these applications is essential for their possible practical uses. Thus, analytical and numerical solutions are essential to understanding real-world scenarios. As a result, further fractional order differential equations may be handled and solved using the suggested methods in later research.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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