NEU PRESS

FEN ve MÜHENDİSLİK BİLİMLERİ DERGİSİ

Cilt/Vol: 6 Sayı/No: 1 Yıl/Year: 2024 Araştırma Makalesi/Research Article e-ISSN: 2667-7989

https://doi.org/10.47112/neufmbd.2024.42

NLS Yüzeyinin Spacelike Eğrisinin Pseudo Null Darboux Çatısı ile Geometrik Analizi

Melek ERDOĞDU [1] **Ayşe YAVUZ** [2]

[1] Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computational Science, Konya, Türkiye

[2] Necmettin Erbakan University, Ahmet Keleşoğlu Faculty of Education, Department of Mathematics and Science Teaching, Konya, Türkiye

Makale Bilgisi

Makale Geçmişi

Geliş Tarihi: 16.01.2024

Kabul Tarihi: 17.03.2024

Yayın Tarihi: 30.04.2024

Anahtar kelimeler:
Darboux çatısı,
Doğrusal olmayan Schrödinger denklemi,
Hasimoto yüzeyi.

ÖZET

Bu çalışmada, Minkowski uzayında birim hızlı spacelike eğri ile ilişkilendirilmiş dinamik bir sistem olarak ele alarak duman halkası denklemi olarak bilinen girdap filaman denkleminin araştırılması verilmiştir. Darboux çatısı kullanılarak, doğrusal olmayan Schrödinger (NLS) denklemine karşılık gelen soliton yüzeyinin diferensiyel geometrik özellikleri incelenmiştir. Bu yüzey genellikle NLS yüzeyi veya Hasimoto yüzeyi olarak adlandırılır. Ayrıca, NLS yüzeyinin Gauss ve ortalama eğrilikleri, Darboux eğrilikleri kullanılarak belirlenmiştir. Bu analiz, NLS yüzeyinin geometrik özelliklerine derinlemesine bir bakış sunarak, bu alandaki bilgiyi artırmaya ve potansiyel uygulamalara ışık tutmaya yönelik önemli bir katkı sağlayacaktır.

Geometric Analysis of the NLS Surface with the Pseodu Null Darboux Frame of Spacelike Curve

Article Info

Article History

Received: 16.01.2024

Accepted: 17.03.2024

Published: 30.04.2024

Keywords:
Darboux frame,
Hasimoto surface,
Nonlinear Schrödinger equation,

ABSTRACT

In this study, examination of the vortex filament equation, known as the smoke ring equation, is given by considering it as a dynamic system associated with a unit speed spacelike curve in Minkowski space. Using the Darboux framework, the differential geometric features of the soliton surface that relate to the nonlinear Schrödinger (NLS) equation were investigated. This surface is often called the NLS surface or Hasimoto surface. Additionally, Gaussian and the mean curvatures of the NLS surface were determined using the Darboux curvatures. By providing an in-depth look at the geometric properties of the NLS surface, this analysis will make a significant contribution to increasing knowledge in this field and shedding light on potential applications.

To cite this article:

Erdoğdu, M. & Yavuz, A. (2024). Geometric analysis of the NLS surface with the Pseudo null Darboux frame of spacelike curve, *Necmettin Erbakan University Journal of Science and Engineering*, *6*(1), 188-198. https://doi.org/10.47112/neufmbd.2024.42

***Sorumlu Yazar/Corresponding Author:** Melek Erdoğdu, *melekrdgd@erbakan.edu.tr*

INTRODUCTION

The exploration of curves on surfaces within the realm of differential geometry unveils intricate mathematical relationships and geometric phenomena. Using the Darboux frame to better understand a curve located on the Hasimoto surface is an interesting point of exploration in the field of geometry. The intertwining of differential geometry and the study of Hasimoto surfaces, which emerge from solitonic solutions of certain physical models like the nonlinear Schrödinger equation, offers a rich terrain for exploration.

The Darboux frame, a set of vectors typically encompassing the tangent, normal, and binormal vectors associated with a curve, serves as a fundamental tool in characterizing the geometric properties of curves in three-dimensional space. Examining the Darboux frame within the context of Hasimoto surfaces adds an extra layer of complexity and intrigue, as these surfaces are intimately connected to solitonic behavior a phenomenon known for its localized, stable, and particle-like nature.

The main aim of this work is to uncover insights that not only contribute to the theoretical foundations of differential geometry, but also offer practical applications in understanding the behavior of curves in the unique context of Hasimoto surfaces. Through a comprehensive review and analysis, this study endeavors to broaden our understanding of the interplay between geometric structures, providing valuable contributions to both mathematical theory and potential applications in physical modeling.

Examining the movement of a vortex filament reveals the key differential geometry and mathematical physics difficulties. One of the most significant studies that guides us in resolving those critical problems is Hasimoto's study in 1972. It focused on an approximate representation of the self-induced motion of an isolated filament of a vortex moving in an a fluid that is incompressible without stretching. The equation holds

$$\varphi_t = \varphi_s \times \varphi_{ss}$$

if the vortex filament's position vector is represented by $\varphi = \varphi(s,t)$. These vortex motions can be considered to be travelling wave solutions with respect to the Nonlinear Schrödinger (NLS) equation, as they involve no change in shape [1].

The NLS surfaces or Hasimoto surfaces have connections with the NLS equation. Hasimoto established the complex function $q = \kappa e^{i\int \tau ds}$ of a curvature functions κ, τ and discovered that if the curve expands according to the vortex filament equation, it yields the focusing on the cubic NLS equation [1].

The curves on the Hasimoto surface have been the subject of several research in the literature, some of which have made use of the Darboux frame, also referred to as the frame of the curve-surface pair. A adaptable frame characterized by three invariant functions, the Darboux frame is defined on a curve which occurs on a surface [2]. The study of the smoke ring equation, which involves investigating the the soliton surface soliton surface connected to the NLS formula, also known as the Hasimoto surface, using the Darboux frame [3]. Additionally, a study has been conducted to examine the Darboux frame of a pseudo null curve situated on a lightlike surface, emphasizing its significance in Minkowski space [4].

Hasimoto surfaces for several types of curve establishment that correspond to a Frenet frame in Minkowski space were investigated, further emphasizing the significance of the Darboux frame in understanding these surfaces [5]. Additionally, it has been investigated what the Darboux frame is defined for lightlike surface, providing insights into its application in physics and linguistics [6]. The study of the Hasimoto surfaces using the Bishop frame also contributes to the understanding of the Darboux frame's role in characterizing surfaces in different spaces [7].

In summary, the literature review reveals that a curve on a Hasimoto surface has been extensively studied in various geometric contexts, including different space geometries, mathematical physics, and surface characterizations. These studies demonstrate the broad applicability and significance of the Darboux frame in understanding the geometric properties of curves on Hasimoto surfaces. Overall, the literature provides a comprehensive understanding of the Darboux frame, offering insights into its geometric properties, invariants, and its connection to various mathematical and physical concepts.

Binormal motion of curves having constant curvature gives rise to integrable expansions of the Dym equations, as the research [8] explains. Moreover, It can also be attributed to integrable variations of the classical sine-Gordon equations and was previously proven that it comes within the binormal motion of curves with constant torsion. Simultaneously, a Backlund transformation is carried out, helping us to create the related soliton surfaces. The research paper demonstrates analogies across Backlund's and Bianchi's standard transformations regarding the extended sine-Gordon system in [9]. Initially, the NLS equation was examined in kinematic examinations that focused on specific hydrodynamic motions, and later, its general intrinsic geometric characteristics were investigated [10], and then utilized in magneto-hydrodynamics [11]. Recent developments indicated a requirement for research into non-Euclidean geometries. A normal congruence in three-dimensional Minkowski space is the main topic of the study [12], and it investigates the NLS equation of repulsive type and nonlinear heat systems. In [13] Hasimoto surfaces are examined in Minkowski 3-space and identified their distinctive geometric characteristics.

The purpose of this research is to contribute to the body of knowledge in both differential geometry and mathematical physics by offering a focused exploration of the Darboux frame in the specific context of curves on the NLS surfaces. Through this investigation, we seek to uncover new insights, establish connections between geometric and physical phenomena, and potentially open avenues for further research and applications. In this study, we explore the differential geometric features of the soliton surface associated with the Nonlinear Schrödinger equation, also known as the NLS surface or Hasimoto surface. A quick overview is presented to provide background details and context beforehand we get started. The geometric characteristics of the NLS surface is then discussed. We determine the Gaussian and mean curvature of the NLS surface. In addition, we obtain new information as well as the essential and sufficient requirements for the NLS surfaces to be flat or minimal.

MATERIALS AND METHODS

In this part, we provide the background information about spacelike curves on lightlike surfaces that is required to grasp the primary subject of the study. Let $\alpha: I \rightarrow E_1^3$ be a regular unit speed spacelike curve. At any point along the curve α, we may construct a Frenet frame $\{T, N, B\}$ satisfying the followings:

$$\langle T, T\rangle = 1,$$

$$\langle N, N\rangle = -\langle B, B\rangle = \varepsilon_1 = \pm 1,$$

$$\langle T, N\rangle = \langle N, B\rangle = \langle T, B\rangle = 0,$$

$$T \times N = -\varepsilon_1 B,$$

$$N \times B = T,$$

$$B \times T = \varepsilon_1 N.$$

The Serret-Frenet formulas of α are given by

$$T'(s) = \varepsilon_1 \kappa(s) N(s),$$

$$N'(s) = -\kappa(s) T(s) - \varepsilon_1 \tau(s) B(s),$$

$$B'(s) = -\varepsilon_1 \tau(s) N(s).$$

However, since the unit speed spacelike curve α lies on the lightlike surface, we can additionally establish alternative pseudo null frame fields on the curve α, which is referred to as the Darboux frame [4].

The unit tangent vector field T of the unit speed spacelike curve α and the null normal vector field n of the surface on the curve α comprise the Darboux frame. There is only one way to select the last frame field g, which is null vector field in order to characterize a pseudo null frame that includes these vector fields. This suggests that we have the following relationships

$$\langle T, T\rangle = 1,$$

$$\langle n, n\rangle = \langle g, g\rangle = \langle T, g\rangle = \langle T, n\rangle = 0,$$

$$\langle g, n\rangle = \varepsilon_1 = \pm 1$$

$$T \times g = \varepsilon_1 g,$$

$$g \times n = T,$$

$$n \times T = \varepsilon_1 n.$$

The derivative formulas of the Darboux frame fields $\{T, g, n\}$ are given by

$$T'(s) = \varepsilon_1 k_n(s) g(s) + \varepsilon_1 k_g(s) n(s),$$

$$g'(s) = -k_g(s) T(s) + \varepsilon_1 \tau_g(s) g(s),$$

$$n'(s) = -k_n(s) T(s) - \varepsilon_1 \tau_g(s) n(s)$$

where k_n is the normal curvature, k_g is the geodesic curvature, and τ_g is the geodesic torsion of the curve α [4].

The following situations are satisfied for a curve lying on a surface:

i) $k_g = 0$ if and only if the curve α is a geodesic curve.

ii) $k_n = 0$ if and only if the curve α is an asymptotic curve.

iii) $\tau_g = 0$ if and only if the curve α is a principal line curve.

RESULTS

This section will provide an overview and look at the geometric features of the soliton surface related to the NLS equation. For this investigation, s-parameter curves on the soliton surface $\varphi = \varphi(s,t)$ will be used. Derivative formulas will be obtained depending on the time parameter of the Darboux vector fields T, g, n of the spacelike s parameter curve $\varphi = \varphi(s,t)$ for all t.

The metric coefficient matrix is obtained corresponding to the Darboux frame fields as follows:

$$I^* = \begin{bmatrix} \langle T(s), T(s)\rangle & \langle T(s), g(s)\rangle & \langle T(s), n(s)\rangle \\ \langle g(s), T(s)\rangle & \langle g(s), g(s)\rangle & \langle g(s), n(s)\rangle \\ \langle n(s), T(s)\rangle & \langle n(s), g(s)\rangle & \langle n(s), n(s)\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \varepsilon_1 \\ 0 & \varepsilon_1 & 0 \end{bmatrix}.$$

If we denote semi skew symmetric matrix with respect to the above metric coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$$

then we get the following matrix equation

$$-\begin{bmatrix} a_{11} & a_{21} & a_{32} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & \varepsilon_1 a_{13} & \varepsilon_1 a_{12} \\ \varepsilon_1 a_{31} & a_{33} & a_{32} \\ \varepsilon_1 a_{21} & a_{23} & a_{22} \end{bmatrix}$$

by

$$I^* A I^* = -A^T.$$

This means that

$$a_{11} = 0,$$

$$a_{21} + \varepsilon_1 a_{13} = 0,$$

$$a_{31} + \varepsilon_1 a_{12} = 0,$$

$$a_{12} + \varepsilon_1 a_{13} = 0,$$

$$a_{22} + a_{33} = 0,$$

$$a_{13} + \varepsilon_1 a_{12} = 0,$$

$$a_{32} = a_{23} = 0.$$

Thus, semi skew symmetric matrix A can be written as follows;

$$A = \begin{bmatrix} 0 & \delta & \beta \\ -\varepsilon_1 \beta & \gamma & 0 \\ -\varepsilon_1 \delta & 0 & -\gamma \end{bmatrix}$$

for some real numbers δ, β and γ.

Theorem. Consider a NLS surface $\varphi = \varphi(s,t)$ such that, for any t, $\varphi = \varphi(s,t)$ is a unit speed spacelike curve. Partial derivative formulas according to the s and t parameters of the Darboux frame defined on the soliton surface are obtained as follows:

i) $$\frac{\partial}{\partial s}\begin{bmatrix}T\\g\\n\end{bmatrix}=\begin{bmatrix}0 & \varepsilon_1 k_n & \varepsilon_1 k_g\\ -k_g & \varepsilon_1\tau_g & 0\\ -k_n & 0 & -\varepsilon_1\tau_g\end{bmatrix}\begin{bmatrix}T\\g\\n\end{bmatrix},$$

ii) $$\frac{\partial}{\partial t}\begin{bmatrix}T\\g\\n\end{bmatrix}=\begin{bmatrix}0 & \delta & \beta\\ -\varepsilon_1\beta & \gamma & 0\\ -\varepsilon_1\delta & 0 & -\gamma\end{bmatrix}\begin{bmatrix}T\\g\\n\end{bmatrix},$$

where

$$\delta=\varepsilon_1 k_n\tau_g+\frac{\partial}{\partial s}k_n,$$

$$\beta=\varepsilon_1 k_g\tau_g-\frac{\partial}{\partial s}k_g,$$

$$\gamma=\frac{1}{k_n}\left(\varepsilon_1\frac{\partial}{\partial s}(\varepsilon_1 k_n\tau_g+\frac{\partial}{\partial s}k_n)+\varepsilon_1 k_n\tau_g^2+\frac{\partial}{\partial s}k_n\tau_g-\frac{\partial}{\partial t}k_n\right).$$

Here k_n is nonzero normal, k_g is the geodesic curvatures and τ_g is the geodesic torsion of spacelike curve $\varphi=\varphi(s,t)$ for all t.

Proof. i) This is accomplished using derivative formulae of vector fields T,g,n.

ii) It is known that there are smooth functions δ , β , γ such that

$$\frac{\partial}{\partial t}\begin{bmatrix}T\\g\\n\end{bmatrix}=\begin{bmatrix}0 & \delta & \beta\\ -\varepsilon_1\beta & \gamma & 0\\ -\varepsilon_1\delta & 0 & -\gamma\end{bmatrix}\begin{bmatrix}T\\g\\n\end{bmatrix}.$$

We obtain

$$\begin{aligned}\frac{\partial}{\partial s}(T_t)&=\frac{\partial}{\partial s}(\delta g+\beta n)\\ &=\left(-\delta k_g-\beta k_n\right)T\\ &\quad+\left(\frac{\partial\delta}{\partial s}+\delta\varepsilon_1\tau_g\right)g+\left(\frac{\partial\beta}{\partial s}-\beta\varepsilon_1\tau_g\right)n\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(T_s)&=\frac{\partial}{\partial t}(\varepsilon_1 k_n g+\varepsilon_1 k_g n)\\ &=\varepsilon_1\left(\varepsilon_1\beta k_n-\varepsilon_1\delta k_g\right)T\\ &\quad+\left(\frac{\partial k_n}{\partial t}+k_n\gamma\right)g+\left(\frac{\partial k_g}{\partial t}-k_g\gamma\right)n.\end{aligned}$$

The following equation is obtained from the equality of the coefficients of the Darboux frame fields in the compatibility $\frac{\partial}{\partial s}(T_t)=\frac{\partial}{\partial t}(T_s)$.

$$\frac{\partial\delta}{\partial s}=\varepsilon_1\frac{\partial k_n}{\partial t}+\varepsilon_1 k_n\gamma-\delta\varepsilon_1\tau_g \qquad (1)$$

and

$$\frac{\partial \beta}{\partial s} = \beta \varepsilon_1 \tau_g + \frac{\partial k_g}{\partial t} - k_g \gamma$$

By using the compatibility conditions of the vector field g and making the necessary adjustments, the following equations are obtained.

$$\frac{\partial}{\partial s}(g_t) = \frac{\partial}{\partial s}(-\varepsilon_1 \beta T + \gamma g)$$

$$= \left(\varepsilon_1 \frac{\partial \beta}{\partial s} - k_g \gamma\right) T$$

$$+ \left(-\beta k_n + \frac{\partial}{\partial s}\gamma + \varepsilon_1 \gamma \tau_g\right) g + (-\beta k_g) n$$

and

$$\frac{\partial}{\partial t}(g_s) = \frac{\partial}{\partial t}(-k_g T + \varepsilon_1 \tau_g g)$$

$$= \left(\frac{\partial}{\partial t} k_g - \beta \tau_g\right) T$$

$$+ \left(-\delta k_g + \varepsilon_1 \frac{\partial}{\partial t}\tau_g + \varepsilon_1 \gamma \tau_g\right) g + (-\beta k_g) n.$$

Then the following equation is obtained

$$\frac{\partial}{\partial s}\gamma = \beta k_n - \delta k_g + \varepsilon_1 \frac{\partial}{\partial t}\tau_g.$$

We have

$$\varphi_t = \varphi_s \times \varphi_{ss}$$

because $\varphi = \varphi(s,t)$ is a NLS surface in which $\varphi = \varphi(s,t)$ is a unit spacelike curve for every t.

This implies

$$\varphi_t = T \times (\varepsilon_1 k_n g + \varepsilon_1 k_g n)$$

$$= \varepsilon_1 \varepsilon_1 k_n g + (-\varepsilon_1)\varepsilon_1 k_g n$$

$$= k_n g - k_g n.$$

We have found

$$\frac{\partial}{\partial s}(\varphi_t) = \frac{\partial}{\partial s}(k_n g - k_g n)$$

$$= \left(\varepsilon_1 k_n \tau_g + \frac{\partial}{\partial s} k_n\right) g + \left(\varepsilon_1 k_g \tau_g - \frac{\partial}{\partial s} k_g\right) n$$

and

$$\frac{\partial}{\partial t}(\varphi_s) = \frac{\partial}{\partial s}(T) = \delta g + \beta n.$$

Compatibility condition $\frac{\partial}{\partial s}(\varphi_t) = \frac{\partial}{\partial t}(\varphi_s)$ implies that

$$\delta = \varepsilon_1 k_n \tau_g + \frac{\partial}{\partial s} k_n,$$

$$\beta = \varepsilon_1 k_g \tau_g - \frac{\partial}{\partial s} k_g.$$

Substituting above equations to Equation 1, we find

$$\gamma = \frac{1}{k_n}\left(\varepsilon_1 \frac{\partial}{\partial s}(\varepsilon_1 k_n \tau_g + \frac{\partial}{\partial s} k_n) + \varepsilon_1 k_n \tau_g^2 + \frac{\partial}{\partial s} k_n \tau_g - \frac{\partial}{\partial t} k_n\right).$$

Theorem. Consider a NLS surface $\varphi = \varphi(s,t)$ such that, for any t, $\varphi = \varphi(s,t)$ is a unit speed spacelike curve. The Gaussian and mean curvatures of the NLS surface are

$$K = \varepsilon\left(\left(-2k_n k_g \gamma - \frac{\partial}{\partial t}(k_g - k_n)\right) + \frac{\left(\frac{\partial}{\partial s}(k_n k_g)\right)^2}{2\varepsilon_1 k_n k_g}\right)$$

and

$$H = \varepsilon \frac{-4{k_n}^2{k_g}^2 + 2k_n k_g \gamma + \frac{\partial}{\partial t}(k_g - k_n)}{-4\varepsilon_1 k_n k_g},$$

where $\varepsilon = \langle Z, Z\rangle$, respectively.

Proof. The Lorentzian product of a surface in three-dimensional Minkowski space on its tangent space is the first fundamental form. We need to find the first fundamental form of the NLS surface $\varphi = \varphi(s,t)$. We may write

$$I = \langle \frac{\partial \varphi}{\partial s} ds + \frac{\partial \varphi}{\partial t} dt, \frac{\partial \varphi}{\partial s} ds + \frac{\partial \varphi}{\partial t} dt\rangle$$

$$= \langle T ds + (k_n g - k_g n) dt, T ds + (k_n g - k_g n) dt\rangle$$
$$= ds^2 - 2\varepsilon_1 k_n k_g dt^2.$$

Thus, coefficients of first fundamental form are obtained as

$$\mathbf{E} = 1,$$

$$\mathbf{F} = 0,$$

$$\mathbf{G} = -2\varepsilon_1 k_n k_g.$$

So, we have

$$\mathbf{EG} - \mathbf{F}^2 = -2\varepsilon_1 k_n k_g.$$

Suppose that normal vector field of the surface is written as follows

$$Z = \varpi T + \rho g + \varsigma n$$

for some differentiable functions ϖ, ρ and ς. We know that

$$\langle Z, \varphi_s \rangle = 0.$$

This implies

$$\varpi = 0.$$

Similarly, we have

$$\langle Z, \varphi_t \rangle = 0,$$

Then we get

$$\langle \rho g + \varsigma n, k_n g - k_g n \rangle = 0.$$

So, we find

$$-k_g \rho + \varsigma k_n = 0.$$

We obtain

$$\rho = k_n, \varsigma = k_g.$$

Thus, the normal vector field of the Hasimoto surface $\varphi = \varphi(s,t)$ is obtained as follows

$$Z = k_n g + k_g n.$$

We also see that

$$\langle Z, Z \rangle = 2\varepsilon_1 k_n k_g$$

The character of the normal vector Z is determined by the signs of $\varepsilon_1,\ k_n,\ k_g$ variables. We know that

$$\varphi_{ss} = \varepsilon_1 k_n g + \varepsilon_1 k_g n,$$

$$\varphi_{st} = \left(\varepsilon_1 k_n \tau_g + \frac{\partial}{\partial s} k_n\right) g + \left(\varepsilon_1 k_g \tau_g - \frac{\partial}{\partial s} k_g\right) n,$$

$$\varphi_{tt} = (-\varepsilon_1 k_n \beta + \varepsilon_1 k_g \delta) T + \left(k_n \gamma + \frac{\partial}{\partial t}(k_n)\right) g + \left(k_g \gamma - \frac{\partial}{\partial t}(k_g)\right) n.$$

Coefficients of the second fundamental form of the NLS surface $\varphi = \varphi(s,t)$ are determined as

$$\mathbf{e} = \langle Z, \varphi_{ss} \rangle = 2\varepsilon_1 k_n k_g,$$

$$\mathbf{f} = \langle Z, \varphi_{st} \rangle = \frac{\partial}{\partial s}(k_n k_g),$$

$$\mathbf{g} = \langle Z, \varphi_{tt} \rangle = 2k_n k_g \gamma + \frac{\partial}{\partial t}(k_g - k_n).$$

If the necessary operations are carried out, we obtain

$$H = \varepsilon \frac{-4{k_n}^2{k_g}^2 + 2k_n k_g \gamma + \frac{\partial}{\partial t}(k_g - k_n)}{-4\varepsilon_1 k_n k_g},$$

$$K = \varepsilon \frac{2\varepsilon_1 k_n k_g\left(2k_n k_g \gamma + \frac{\partial}{\partial t}(k_g - k_n)\right) - \left(\frac{\partial}{\partial s}(k_n k_g)\right)^2}{-2\varepsilon_1 k_n k_g},$$

respectively.

DISCUSSION AND CONCLUSIONS

It is known that the relationship between geometric analyzes in Minkowski space and differential equations sheds light on studies in the field of physics. For example, the solution to the optical fiber differential equations involving the position vector field is identified. This normal congruence of surfaces has intrinsic geometrical characteristics that are described in the context of vectors of electromagnetic waves. We examine situations in which particular electric charge and current densities allow electromagnetic and magnetic vectors to meet the requirements of Maxwell's equations [14]. Moreover, sufficient as well as necessary conditions for the geometric phase are provided to allow for EM wave vectors to be executed in parallel with the pseudo null frame[15]. There are also studies on some engineering topics that are known to be related to differential equations and geometric analysis. Engineering studies addressing some of these issues can be found at [16-18].

The study examines the spacelike curve in Minkowski space as a dynamic system and analyzes the smoke ring equation, additionally referred to as the vortex filament equation. Using the Darboux frame, the geometric properties of the soliton surface corresponding to NLS equation have been studied in detail; this surface is often referred to as the NLS surface or Hasimoto surface.

The results obtained provide an in-depth look at the mathematical properties of the NLS surface, increasing knowledge in this field and opening new perspectives on potential applications. In agreement with relevant literature, our analyzes helped us understand the dynamic properties of spacelike curves on the NLS surface.

The results of this study provide several recommendations for future researchers and those working in related fields. Further experimental and theoretical studies on the dynamic behavior of spacelike curves on the NLS surface can deepen the knowledge in this field. Additionally, the results obtained may inspire researchers interested in mathematical physics, differential geometry and nonlinear equations to develop new methods and approaches. In this context, in future studies, further research can be recommended to explore the potential applications of curves on the NLS surface in various physical systems.

Conflict of interest

The authors have no conflicts of interest to disclose for this study.

Authorship contribution statement

M.E.: Conceptualization, Methodology, Writing-Original Draft, Investigation, Writing- Review & Editing **A.Y.:** Conceptualization, Methodology, Writing-Original Draft, Formal Analysis, Writing-Review & Editing.

KAYNAKLAR (REFERENCES)

[1] H. Hasimoto, A soliton on a vortex filament, *Journal of Fluid Mechanics*. 51(3) (1972), 477-485.

[2] S. Izumiya, S. Otani, Flat approximations of surfaces along curves, *Demonstratio Mathematica*. 48(2) (2015), 217-241.

[3] M. Erdoğdu, A. Yavuz, Differential geometric aspects of nonlinear Schrödinger equation. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*. 70(1) (2021), 510-521.

[4] J. Djordjević, E. Nešović, On generalized Darboux frame of a pseudo null curve lying on a lightlike surface in Minkowski 3-Space. *International Electronic Journal of Geometry*. 16(1) (2023), 81-94.

[5] N. Gürbüz, W. Y. Dae, Hasimoto surfaces for two classes of curve evolution in Minkowski 3-space, *Demonstratio Mathematica*. 53(1) (2020), 277-284.

[6] E. S. Y. Topbaş, İ. Gök, N. Ekmekçi, Y. Yaylı, Darboux frame of a curve lying on a lightlike surface, *Mathematical Sciences and Applications E-Notes*. 4(2) (2016), 121–130.

[7] E. Hamouda, O. Moaaz, C. Cesarano, S. Askar, A. Elsharkawy, Geometry of solutions of the quasi vortex filament equation in Euclidean 3-Space E^3, *Mathematics*. 10(6) (2022), 891.

[8] W. K. Schief, C. Rogers, Binormal motion of curves of constant curvature and torsion. Generation of soliton surfaces. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*. 455 (1988), 3163-3188.

[9] C. Rogers, W. K. Schief, Intrinsic geometry of the NLS equation and its auto-Bäcklund transformation. *Studies in Applied Mathematics*. 101(3) (1998), 267-287.

[10] A. W. Marris, S.L. Passman, Vector fields and flows on developable surfaces, *Archive for Rational Mechanics and Analysis*. 32 (1969), 29-86.

[11] C. Rogers, J. G. Kingston, Nondissipative magneto-hydrodynamic flows with magnetic and velocity field lines orthogonal geodesics, *SIAM Journal on Applied Mathematics*. 26(1) (1974), 183-195.

[12] N. Gürbüz, Intrinsic geometry of the NLS equation and heat system in 3-dimensional Minkowski space, *Advanced Studies in Theoretical Physics*. 4 (2010), 557-564.

[13] M. Erdoğdu, M. Özdemir, Geometry of Hasimoto surfaces in Minkowski 3-space. *Mathematical Physics, Analysis and Geometry*. 17 (2014), 169-181.

[14] A. Yavuz, On normal congruence of surfaces and position vector of optical fiber by electromagnetic wave vectors. *Optik*. 260 (2022), 168994.

[15] M. Erdoğdu, Electromagnetic waves along pseudo null curves in Minkowski space. *Heliyon*. 8(8) (2022).

[16] M. Uyaner, A. Yar, Nano elyaf takviyeli nanokompozit üretimi ve karakterizasyonu. *Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi*. 1(1) (2019), 10-19.

[17] S. Ata, M.E. Boyacıoğlu, R. Şahin, A. Kahraman, ORÇ ile düşük sıcaklıklı ısı kaynaklarından elektrik üretilmesinde ıslak ve yeni nesil akışkanların çevresel ve termodinamik performanslarının karşılaştırılması. *Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi*. 3(1) (2021), 13-23.

[18] B. Gökçe, O. B. Özden, Kaynaklı bağlantıya sahip karmaşık bir yapıda goldak modeli kullanılarak distorsiyonların ve kalıntı gerilmelerin nümerik analizler ile belirlenmesi. *Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi*. 5(2) (2023), 16-27.