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Dynamics of the rational difference equations

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ABSTRACT

Discrete-time systems are sometimes used to explain natural phenomena that happen in non-linear sciences. We study the periodicity, boundedness, oscillation, stability, and certain exact solutions of nonlinear difference equations of generalized order in this paper. Using the standard iteration method, exact solutions are obtained. Some well-known theorems are used to test the stability of the equilibrium points. Some numerical examples are also provided to confirm the theoretical work's validity. The numerical component is implemented with Wolfram Mathematica. The method presented may be simply applied to other rational recursive issues.
In this research, we examine the qualitative behavior of rational recursive sequences provided that the initial conditions are arbitrary real numbers. We examine the behavior of solutions on graphs according to the state of their initial value

$$x_{n+1} = \frac{x_n x_{n-8}}{\pm x_{n-7} \pm x_n x_{n-7} x_{n-8}}, \quad n \in \mathbb{N}_0.$$

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1. Introduction

Differential equations are often used to describe some natural phenomena when the time is continuous. However, some real life problems can be simply investigated using discrete-time equations. Differential equations occur naturally in many nonlinear sciences, including ecology and economics. In such cases, the state of a phenomenon at a specific point in time completely predicts its state after a year. Dynamical systems theory is useful in discussing the behavior of some models without solving them. Most natural phenomena are studied using difference equations. For instance, recursive equations have been well used in modeling some natural phenomena such as the size of a population, the Fibonacci sequence, the drug in the blood system, the transmission of information, the pricing of a certain commodity, the propagation of annual plants, and others [12]. In addition, some scholars have used difference equations to find the numerical solutions of some differential equations. More specifically, discretizing a given differential equation gives a difference equation. For example, Runge-Kutta scheme is obtained from discretizing a first order differential equation. This raises the question of the convergence of the difference scheme to the solution of a differential equation. Or, in a broader sense, the question of the correspondence between the properties of solutions of differential equations and their difference approximations. The work [17] is devoted to questions of conservation of a solution bounded on the entire axis in the transition from differential to difference equations and vice versa. In [18], similar questions were considered to preserve the oscillatory property of solutions to second-order equations. The development of technology has motivated the use of recurrence equations as approximations to partial differential equations.

It is worth mentioning that fractional order difference equations are often utilized to investigate some real life phenomena emerging in nonlinear sciences.

Alayachi et al. [5] analyzed the local and global attractivity, periodicity and the solutions of a sixth order difference equation. Some numerical examples have been also presented in [5]. In [28], Sanbo and Elsayed presented the periodicity, stability and some solutions of a fifth order recursive equation. Almatrafi and Alzubaidi [8] discussed the dynamical behaviors of an eighth order difference relation and showed some 2D figures for the obtained results. Moreover, Ahmed et al. [2], found new solutions and investigated the dynamical analysis for some nonlinear difference relations of fifteenth order. More discussions about nonlinear recursive problems can be seen in refs. [1–30].

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions

$$x_{-k}, U_{-k+1}, ..., x_0 \in I$$

the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad (1)$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [23]. A point $\overline{x} \in I$ is called an equilibrium point of equation (1) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of equation (1), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.1. *(Stability).*

(i) *The equilibrium point $\overline{x}$ of equation (1) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all*

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I,$$

with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq k.$$

(ii) *The equilibrium point $\overline{x}$ of equation (1) is called locally asymptotically stable if $\overline{x}$ is a locally stable solution of equation (1) and there exists $\gamma > 0$, such that for all*

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I,$$

with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) *The equilibrium point $\overline{x}$ of equation (1) is called a global attractor if for all*

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) *The equilibrium point $\overline{x}$ of equation (1) is called a global asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of equation (1).*

(v) *The equilibrium point $\overline{x}$ of equation (1) is called unstable if $\overline{x}$ is not locally stable. The linearized equation of equation (1) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{U}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 1. *(see[21]). Assume that $p, q \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + p x_n + q x_{n-k} = 0, \quad n \in \mathbb{N}_0.$$

Remark 2. *Theorem 1 can be easily extended to general linear equations of the form*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n \in \mathbb{N}_0, \quad (2)$$

where, $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \mathbb{N}$. Then (2) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Definition 1.2. *(Periodicity). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = U_n$ for all $n \geq -k$.*

Definition 1.3. *The equilibrium point $\overline{x}$ is said to be hyperbolic if $|f(\overline{x})| \neq 1$. If $|f(\overline{x})| = 1$, x is non hyperbolic.*

2. The Difference Equation $x_{n+1} = \frac{x_n x_{n-8}}{x_{n-7}+x_n x_{n-7} x_{n-8}}$

In this part we give the solutions of

$$x_{n+1} = \frac{x_n x_{n-8}}{x_{n-7} + x_n x_{n-7} x_{n-8}}, \quad n \in \mathbb{N}_0, \tag{3}$$

where the initials are arbitrary real numbers.

Theorem 3. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Eq. 3. Then for $n \in \mathbb{N}_0$*

$$\begin{aligned}
x_{16n+1} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+9)aj)}{h\prod_{i=0}^{n}(1+(16i+1)aj)},\\
x_{16n+2} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+10)aj)}{g\prod_{i=0}^{n}(1+(16i+2)aj)},\\
x_{16n+3} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+11)aj)}{f\prod_{i=0}^{n}(1+(16i+3)aj)},\\
x_{16n+4} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+12)aj)}{e\prod_{i=0}^{n}(1+(16i+4)aj)},\\
x_{16n+5} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+13)aj)}{d\prod_{i=0}^{n}(1+(16i+5)aj)},\\
x_{16n+6} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+14)aj)}{c\prod_{i=0}^{n}(1+(16i+6)aj)},\\
x_{16n+7} &= \frac{aj\prod_{i=0}^{n-1}(1+(16i+15)aj)}{b\prod_{i=0}^{n}(1+(16i+7)aj)},\\
x_{16n+8} &= \frac{j\prod_{i=0}^{n-1}(1+(16i+16)aj)}{\prod_{i=0}^{n}(1+(16i+8)aj)},\\
x_{16n+9} &= \frac{h\prod_{i=0}^{n}(1+(16i+1)aj)}{\prod_{i=0}^{n}(1+(16i+9)aj)},\\
x_{16n+10} &= \frac{g\prod_{i=0}^{n}(1+(16i+2)aj)}{\prod_{i=0}^{n}(1+(16i+10)aj)},\\
x_{16n+11} &= \frac{f\prod_{i=0}^{n}(1+(16i+3)aj)}{\prod_{i=0}^{n}(1+(16i+11)aj)},\\
x_{16n+12} &= \frac{e\prod_{i=0}^{n}(1+(16i+4)aj)}{\prod_{i=0}^{n}(1+(16i+12)aj)},\\
x_{16n+13} &= \frac{d\prod_{i=0}^{n}(1+(16i+5)aj)}{\prod_{i=0}^{n}(1+(16i+13)aj)},\\
x_{16n+14} &= \frac{c\prod_{i=0}^{n}(1+(16i+6)aj)}{\prod_{i=0}^{n}(1+(16i+14)aj)},\\
x_{16n+15} &= \frac{b\prod_{i=0}^{n}(1+(16i+7)aj)}{\prod_{i=0}^{n}(1+(16i+15)aj)},\\
x_{16n+16} &= \frac{a\prod_{i=0}^{n}(1+(16i+8)aj)}{\prod_{i=0}^{n}(1+(16i+16)aj)}.
\end{aligned}$$

where,

$$\begin{aligned}
&x_{-8} = j, \quad x_{-7} = h, \quad x_{-6} = g, \quad x_{-5} = f,\\
&x_{-4} = e, \quad x_{-3} = d, \quad x_{-2} = c, \quad x_{-1} = b,\\
&x_0 = a.
\end{aligned} \tag{4}$$

Proof Suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{16n-15} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+9)aj)}{h\prod_{i=0}^{n-1}(1+(16i+1)aj)},\\
x_{16n-14} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+10)aj)}{g\prod_{i=0}^{n-1}(1+(16i+2)aj)},\\
x_{16n-13} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+11)aj)}{f\prod_{i=0}^{n-1}(1+(16i+3)aj)},\\
x_{16n-12} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+12)aj)}{e\prod_{i=0}^{n-1}(1+(16i+4)aj)},\\
x_{16n-11} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+13)aj)}{d\prod_{i=0}^{n-1}(1+(16i+5)aj)},\\
x_{16n-10} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+14)aj)}{c\prod_{i=0}^{n-1}(1+(16i+6)aj)},\\
x_{16n-9} &= \frac{aj\prod_{i=0}^{n-2}(1+(16i+15)aj)}{b\prod_{i=0}^{n-1}(1+(16i+7)aj)},\\
x_{16n-8} &= \frac{j\prod_{i=0}^{n-2}(1+(16i+16)aj)}{\prod_{i=0}^{n-1}(1+(16i+8)aj)},\\
x_{16n-7} &= \frac{h\prod_{i=0}^{n-1}(1+(16i+1)aj)}{\prod_{i=0}^{n-1}(1+(16i+9)aj)},\\
x_{16n-6} &= \frac{g\prod_{i=0}^{n-1}(1+(16i+2)aj)}{\prod_{i=0}^{n-1}(1+(16i+10)aj)},\\
x_{16n-5} &= \frac{f\prod_{i=0}^{n-1}(1+(16i+3)aj)}{\prod_{i=0}^{n-1}(1+(16i+11)aj)},\\
x_{16n-4} &= \frac{e\prod_{i=0}^{n-1}(1+(16i+4)aj)}{\prod_{i=0}^{n-1}(1+(16i+12)aj)},\\
x_{16n-3} &= \frac{d\prod_{i=0}^{n-1}(1+(16i+5)aj)}{\prod_{i=0}^{n-1}(1+(16i+13)aj)},\\
x_{16n-2} &= \frac{c\prod_{i=0}^{n-1}(1+(16i+6)aj)}{\prod_{i=0}^{n-1}(1+(16i+14)aj)},\\
x_{16n-1} &= \frac{b\prod_{i=0}^{n-1}(1+(16i+7)aj)}{\prod_{i=0}^{n-1}(1+(16i+15)aj)},\\
x_{16n} &= \frac{a\prod_{i=0}^{n-1}(1+(16i+8)aj)}{\prod_{i=0}^{n-1}(1+(16i+16)aj)}.
\end{aligned}$$

where, $x_{-8}, \ldots, x_0$ defines as in 4 Now, it follows from Equation 3 that

$$x_{16n+1} = \frac{x_{16n}x_{16n-8}}{x_{16n-7} + x_{16n}x_{16n-7}x_{16n-8}}. \quad (5)$$

If the found values are substituted in the equation5, we have

$$x_{16n+1} = \frac{aj\prod_{i=0}^{n-1}(1+(16i+9)aj)}{h\prod_{i=0}^{n}(1+(16i+1)aj)}.$$

Other relations can also be obtained in a similar way, and thus the proof is complete.

Theorem 4. *Equation 3 has a unique equilibrium $\overline{x} = 0$ and it is not locally asymptotically stable.*

Proof

We have

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(1+\overline{x}^2)}.$$

Then

$$1 + \overline{x}^2 = 1, \quad \overline{x}^2 = 0.$$

Thus the equilibrium of Equation 3 is $\overline{x} = 0$.
Define the function F by

$$F(\alpha,\beta,\gamma) = \frac{\alpha\gamma}{\beta(1+\alpha\gamma)}.$$

Then it follows that,

$$F_\alpha(\alpha,\beta,\gamma) = \frac{\gamma}{\beta(1+\alpha\gamma)^2}; \quad F_\beta(\alpha,\beta,\gamma) = -\frac{\alpha\gamma}{\beta^2(1+\alpha\gamma)};$$
$$F_\gamma(\alpha,\beta,\gamma) = \frac{\alpha}{\beta(1+\alpha\gamma)^2};$$

we see that,

$$F_\alpha(\overline{x},\overline{x},\overline{x}) = 1; F_\beta(\overline{x},\overline{x},\overline{x}) \quad = -1; F_\gamma(\overline{x},\overline{x},\overline{x}) = 1$$

By using Theorem 1 , the proof is completed.

Example 1. *Assume that*

$x_{-8} = 6.5, \quad x_{-7} = 5.5, \quad x_{-6} = 24, \quad x_{-5} = 23, \quad x_{-4} = 22,$
$x_{-3} = 21, \quad x_{-2} = 5, \quad x_{-1} = 4, \quad x_0 = 3.$

See figure 1.

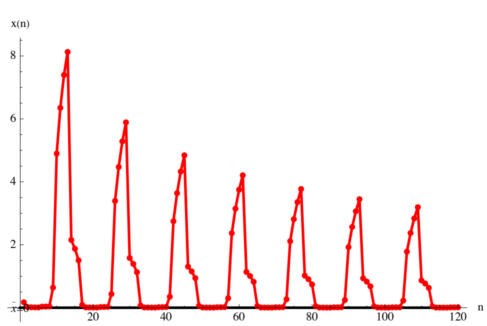

Figure 1

Example 2. *Assume that,*

$x_{-8} = 7.5, \quad x_{-7} = 3.5, \quad x_{-6} = 20, \quad x_{-5} = 21, \quad x_{-4} = 19,$
$x_{-3} = 18, \quad x_{-2} = 6, \quad x_{-1} = 5, \quad x_0 = 2.5$

See figure 2.

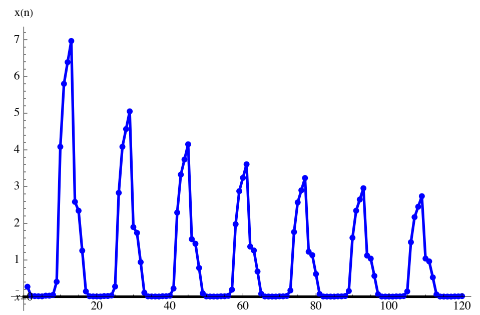

Figure 2

3. The Equation $x_{n+1} = \frac{x_n x_{n-8}}{x_{n-7} - x_n x_{n-7} x_{n-8}}$

We deal with the difference equation

$$x_{n+1} = \frac{x_n x_{n-8}}{x_{n-7} - x_n x_{n-7} x_{n-8}}, \quad n \in \mathbb{N}_0. \quad (6)$$

where the initials are arbitrary real numbers.

Theorem 5. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Equation 6 Then for $n \in \mathbb{N}_0$*

$$x_{16n+1} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+9)aj)}{h\prod_{i=0}^{n}(1-(16i+1)aj)},$$
$$x_{16n+2} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+10)aj)}{g\prod_{i=0}^{n}(1-(16i+2)aj)},$$
$$x_{16n+3} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+11)aj)}{f\prod_{i=0}^{n}(1-(16i+3)aj)},$$
$$x_{16n+4} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+12)aj)}{e\prod_{i=0}^{n}(1-(16i+4)aj)},$$
$$x_{16n+5} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+13)aj)}{d\prod_{i=0}^{n}(1-(16i+5)aj)},$$
$$x_{16n+6} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+14)aj)}{c\prod_{i=0}^{n}(1-(16i+6)aj)},$$
$$x_{16n+7} = \frac{aj\prod_{i=0}^{n-1}(1-(16i+15)aj)}{b\prod_{i=0}^{n}(1-(16i+7)aj)},$$
$$x_{16n+8} = \frac{j\prod_{i=0}^{n-1}(1-(16i+16)aj)}{\prod_{i=0}^{n}(1-(16i+8)aj)},$$

$$x_{16n+9} = \frac{h\prod_{i=0}^{n}(1-(16i+1)aj)}{\prod_{i=0}^{n}(1-(16i+9)aj)},$$
$$x_{16n+10} = \frac{g\prod_{i=0}^{n}(1-(16i+2)aj)}{\prod_{i=0}^{n}(1-(16i+10)aj)},$$
$$x_{16n+11} = \frac{f\prod_{i=0}^{n}(1-(16i+3)aj)}{\prod_{i=0}^{n}(1-(16i+11)aj)},$$
$$x_{16n+12} = \frac{e\prod_{i=0}^{n}(1-(16i+4)aj)}{\prod_{i=0}^{n}(1-(16i+12)aj)},$$
$$x_{16n+13} = \frac{d\prod_{i=0}^{n}(1-(16i+5)aj)}{\prod_{i=0}^{n}(1-(16i+13)aj)},$$
$$x_{16n+14} = \frac{c\prod_{i=0}^{n}(1-(16i+6)aj)}{\prod_{i=0}^{n}(1-(16i+14)aj)},$$
$$x_{16n+15} = \frac{b\prod_{i=0}^{n}(1-(16i+7)aj)}{\prod_{i=0}^{n}(1-(16i+15)aj)},$$
$$x_{16n+16} = \frac{a\prod_{i=0}^{n}(1-(16i+8)aj)}{\prod_{i=0}^{n}(1-(16i+16)aj)}.$$

holds.

Proof The proof is similar to the proof of Theorem 3 and therefore it will be omitted.

Theorem 6. *Equation 6 has a unique equilibrium $\overline{x} = 0$, and it is not locally asymptotically stable.*

Proof The proof is similar to the proof Theorem 4 and there it will be omitted.

For confirming the outcomes of this section, we take into consideration mathematical instances which stand for various kind of solutions to (3).

Example 3. *The solution in given by Figure 3 when,*

$x_{-8} = 6,\quad x_{-7} = 6.5,\quad x_{-6} = 11,\quad x_{-5} = 19,\quad x_{-4} = 13,$
$x_{-3} = 10,\quad x_{-2} = 11,\quad x_{-1} = 8,\quad x_0 = 9.5$

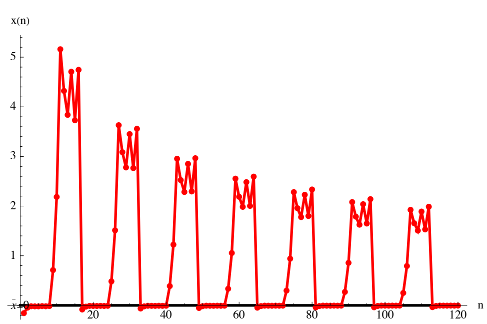

Figure 3

Example 4. *The solution is given by Figure 4 when,*

$x_{-8} = 6.8,\quad x_{-7} = 6,\quad x_{-6} = 13,\quad x_{-5} = 17,\quad x_{-4} = 16,$
$x_{-3} = 21,\quad x_{-2} = 19,\quad x_{-1} = 17,\quad x_0 = 10.5$

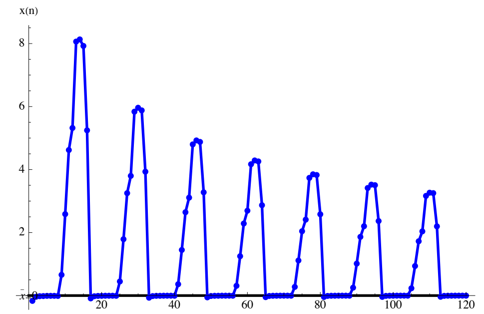

Figure 4

4. The Equation $x_{n+1} = \frac{x_n x_{n-8}}{-x_{n-7}+x_n x_{n-7} x_{n-8}}$

We deal with the difference equation

$$x_{n+1} = \frac{x_n x_{n-8}}{-x_{n-7}+x_n x_{n-7} x_{n-8}}, \quad n \in \mathbb{N}_0. \qquad (7)$$

where the initials are arbitrary real numbers.

Theorem 7. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of difference equation 7. Then solutions are 16 period.*

$$\left\{ x_{16n+1} = \frac{aj}{h(-1+aj)}, x_{16n+2} = \frac{aj}{g}, x_{16n+3} = \frac{aj}{f(-1+aj)}, x_{16n+4} = \frac{aj}{e}, x_{16n+5} = \frac{aj}{d(-1+aj)}, x_{16n+6} = \frac{aj}{c}, x_{16n+7} = \frac{aj}{b(-1+aj)}, x_{16n+8} = j, x_{16n+9} = h, x_{16n+10} = g, x_{16n+11} = f, x_{16n+12} = e, x_{16n+13} = d, x_{16n+14} = c, x_{16n+15} = b, x_{16n+16} = a \right\}.$$

Proof Suppose that $n > 0$ and that our assumption holds for $n-1$. Then

$$x_{16n-15} = \frac{aj}{h(-1+aj)}, x_{16n-14} = \frac{aj}{g}, x_{16n-13} = \frac{aj}{f(-1+aj)},$$
$$x_{16n-12} = \frac{aj}{e}, x_{16n-11} = \frac{aj}{d(-1+aj)}, x_{16n-10} = \frac{aj}{c},$$
$$x_{16n-9} = \frac{aj}{b(-1+aj)}, x_{16n-8} = j, x_{16n-7} = h, x_{16n-6} = g,$$
$$x_{16n-5} = f, x_{16n-4} = e, x_{16n-3} = d, x_{16n-2} = c,$$
$$x_{16n-1} = b, x_{16n} = a.$$

Now, it follows from 7 that

$$x_{16n+1} = \frac{x_{16n} x_{16n-8}}{-x_{16n-7}+x_{16n} x_{16n-7} x_{16n-8}} \qquad (8)$$

If the found values are substituted in the Eq. 8, we have

$$x_{16n+1} = \frac{aj}{h(-1+aj)}.$$

We can prove other relations similarly.

Theorem 8. *Equation 7 has a unique equilibrium points which are* $0, \pm\sqrt{2}$*, and these equilibrium points are not locally asymptotically stable.*

Proof The proof is similar to the proof Theorem 4 and there it will be omitted.

Example 5. *The solution in given by Figure 5 when,*

$x_{-8} = 7.5, \quad x_{-7} = 9, \quad x_{-6} = 13, \quad x_{-5} = 11, \quad x_{-4} = 15,$
$x_{-3} = 13.5, \quad x_{-2} = 19, \quad x_{-1} = 12, \quad x_0 = 14.$

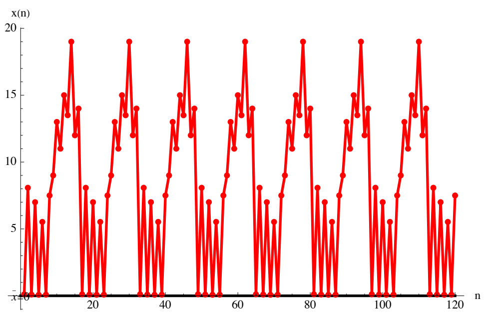

Figure 5

Example 6. *The solution in given by Figure 6 when,*

$x_{-8} = 1.2, \quad x_{-7} = 1.5, \quad x_{-6} = 1.3, \quad x_{-5} = 1.6, \quad x_{-4} = 1.35,$
$x_{-3} = 1.25, \quad x_{-2} = 1.45, \quad x_{-1} = 1.465, \quad x_0 = 1.245$

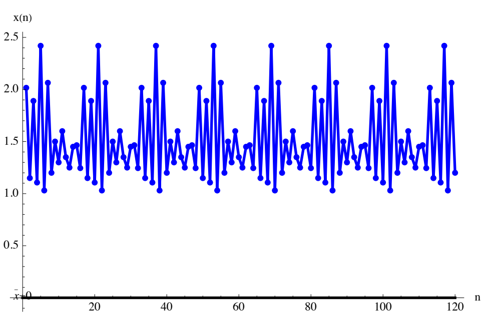

Figure 6

5. The Equation $x_{n+1} = \frac{x_n x_{n-8}}{-x_{n-7} - x_n x_{n-7} x_{n-8}}$

$$x_{n+1} = \frac{x_n x_{n-8}}{-x_{n-7} - x_n x_{n-7} x_{n-8}}, \quad n \in \mathbb{N}_0, \qquad (9)$$

where the initials are arbitrary real numbers. $x_0.x_{-8} \neq -1$.

Theorem 9. *Let* $\{x_n\}_{n=-8}^{\infty}$ *be a solution of difference equation 9 .*

$$\left\{ \frac{-aj}{h(1+aj)}, \frac{aj}{g}, \frac{aj}{-f(1+aj)}, \frac{aj}{e}, \frac{-aj}{d(1+aj)}, \frac{aj}{c}, \frac{-aj}{b(1+aj)}, j, h, g, f, e, d, c, b, a \right\}.$$

where equilibriums $x_0.x_{-8} \neq -1$*. The solutions are obtained with 16 periods.*

Proof The proof is the same as the proof of Theorem 7 and hence is omitted.

Theorem 10. *Equation 6 has a unique equilibrium points which are* $0, \pm\sqrt{-2}$*, and these equilibrium points are not locally asymptotically stable.*

Proof The proof is similar to the proof Theorem 4 and there it will be omitted.

Example 7. *See Fig. 7 for the initials*

$x_{-8} = 6.1, \quad x_{-7} = 9.3, \quad x_{-6} = 13, \quad x_{-5} = 11, \quad x_{-4} = 15,$
$x_{-3} = 13.5, \quad x_{-2} = 19, \quad x_{-1} = 12, \quad x_0 = 14.$

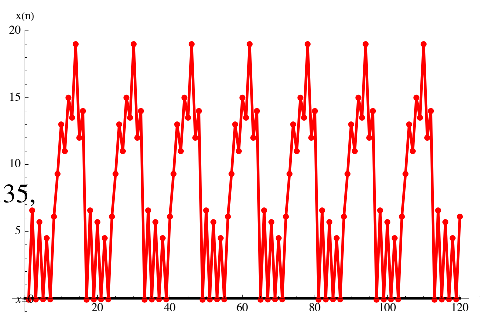

Figure 7

Example 8. *We consider*

$x_{-8} = 6.1, \quad x_{-7} = 9.3, \quad x_{-6} = 13, \quad x_{-5} = 11,$
$x_{-4} = 15.2, \quad x_{-3} = 13.5, \quad x_{-2} = 19.2, \quad x_{-1} = 12.2,$
$x_0 = 16.$

See figure 8

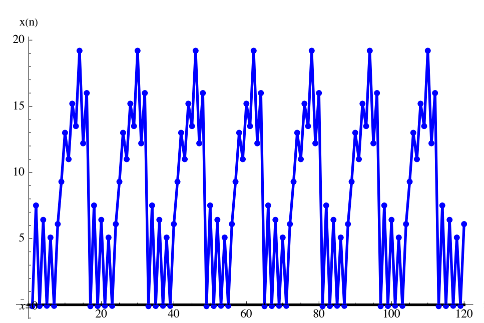

Figure 8

6. Conclusion

We investigate the dynamics of the difference equation expressed as

$$x_{n+1} = \frac{x_n x_{n-8}}{\pm x_{n-7} \pm x_n x_{n-7} x_{n-8}}$$

where the initial values are positive real numbers. Our analysis includes a discussion on local stability. Additionally, we obtain solutions for specific cases and provide numerical examples to further illustrate our findings.

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