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Research Article

Polynomially accretive operators

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Abstract

In this paper, we introduce a new class of operators on a complex Hilbert space $\mathcal{H}$ which is called *polynomially accretive operators*, and thereby extending the notion of accretive and n–real power positive operators. We give several properties of the newly introduced class, and generalize some results for accretive operators. We also prove that every 2–normal and $(2k+1)$–real power positive operator, for some $k \in \mathbb{N}$, must be n–normal for all $n \geq 2$. Finally, we give sufficient conditions for the normality in the preceding implication.

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1. Introduction

Let $\mathcal{H}$ denote the complex Hilbert space and let $\mathfrak{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$. For $T \in \mathfrak{B}(\mathcal{H})$, we denote by T^* the adjoint operator of T and by $|T| = (T^*T)^{1/2}$ the absolute value of T. Furthermore, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ will represent the range and the null space of operator T, respectively. We say that operator T is normal if $TT^* = T^*T$, i.e., if T commutes with T^*, and quasinormal if T commutes with T^*T. The set of all normal operators includes the Hermitian ($T = T^*$), unitary ($TT^* = T^*T = I$), and positive operators ($T \geq 0$). Clearly, every normal operator is quasinormal.

In [15], the author introduced another generalization of normal operators, called n–power normal operators. Namely, the operator T is n–power normal operator for some $n \in \mathbb{N}$ if T^n commutes with T^*, i.e., $T^nT^* = T^*T^n$. For more information on n–power normal operators, see [1], [7], [8].

More recently, in [10], the authors further generalized the notion of n–power normal operator to the class of polynomially normal operators. An operator T is said to be polynomially normal if there exists a non-trivial polynomial p such that $p(T)$ is normal. We also have to mention that the idea of considering this class of operators is not new, and can be traced back to the work of Kittaneh [16].

The class of accretive operators recently followed the similar path as the class of normal operators. First recall that for $T \in \mathfrak{B}(\mathcal{H})$, we can write

$$T = \mathrm{Re}\,(T) + i\,\mathrm{Im}\,(T),$$

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where Re (T) and Im (T) are Hermitian. Such a decomposition is unique, and

$$\mathrm{Re}\,(T) = \frac{T + T^*}{2}, \quad \mathrm{Im}\,(T) = \frac{T - T^*}{2i}.$$

Operators Re (T) and Im (T) are called the real and imaginary part of T, respectively. The class of accretive operators is a subset of $\mathfrak{B}(\mathcal{H})$ consisting of all operators which have the positive real part. In other words, operator T is accretive if and only if Re $(T) \geq 0$. Throughout the literature, accretive operators are also known as real positive operators in the case of general Hilbert spaces, and *Re–nnd* (Re-nonnegative definite) matrices, in a finite–dimensional case (cf. [6], [9], [26], [27], [28]).

In [13], the authors introduced and studied the operator T satisfying $T^2 \geq -T^{*2}$, and in [3], the author further generalized the notion of accretive operators by introducing the n–real power positive operator. Namely, for $n \in \mathbb{N}$, an operator T is said to be n–real power positive operator if

$$T^n + T^{*n} \geq 0,$$

or, equivalently, Re $(T^n) \geq 0$. The author in [3] also gave several properties regarding this notion. Inspired by this results, as well as the development and the path taken in generalizing the class of normal operators, it is natural to extend the notion of n–real power positive operators to an even wider class related to polynomials.

In the sequel, $\mathbb{C}[z]$ will denote the set of all non–trivial complex polynomials in one variable. Note that if $p \in \mathbb{C}[z]$, then $\overline{p} \in \mathbb{C}[z]$, as well, where $\overline{p}(z) = \overline{p(\overline{z})}$, $z \in \mathbb{C}$.

Definition 1.1. Let $T \in \mathfrak{B}(\mathcal{H})$ and $p \in \mathbb{C}[z]$. If T satisfies the inequality

$$p(T) + \overline{p}(T^*) \geq 0, \tag{1.1}$$

then T is called p–accretive operator.

Operator $T \in \mathfrak{B}(\mathcal{H})$ is polynomially accretive, if T is q–accretive for some polynomial $q \in \mathbb{C}[z]$.

Remark 1.2. Note that if $T \in \mathfrak{B}(\mathcal{H})$ and $p(t) = t^n$, for some $n \in \mathbb{N}$, then T is a n–real power positive operator. Also, if T is p–accretive for $p(t) = t$, then T is accretive. Thus, the set of all polynomially accretive operators contains all accretive and all n–real power positive operator.

Remark 1.3. In the sequel, real positive and n–power real positive operators will be called accretive and n–accretive operators, respectively. Also, n–power normal operators will be simply called n–normal operators.

The paper is organized as follows. In Section 2, we give certain well known theoretical results which will be used throughout the paper. In Section 3, we give some elementary properties and characterizations of the class of polynomially accretive operators, as well as generalizing results from [3] regarding n–accretive operators. In Section 4, we show that under certain conditions, we are able to explicitly present the stucture of polynomially accretive operators. As a consequence of our results, among other things, we prove that every 2–normal and $(2k+1)$–accretive operator, for some $k \in \mathbb{N}$, must be n–normal for all $n \geq 2$. Finally, we give sufficient conditions in order to include $n = 1$ in the previous implication.

2. Preliminaries

In this section, we give some well–known results which will be used (implicitly or explicitly) throughout the paper, and especially in Section 4. In other words, we'll effectively combine the theory of normal operators, theory of operator matrices and theory of generalized inverses in order to obtain our main results.

Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ will denote the orthogonal projection onto $\mathcal{S}$. For any $T \in \mathfrak{B}(\mathcal{H})$, the operator matrix decomposition of T induced by S is given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{2.1}$$

where $T_{11} = P_S T P_S|_S$, $T_{12} = P_S T(I - P_S)|_{S^\perp}$, $T_{21} = (I - P_S)TP_S|_S$ and $T_{22} = (I - P_S)T(I - P_S)|_{S^\perp}$.

Also, for $T \in \mathfrak{B}(\mathcal{H})$ there exists a linear operator $T' : \mathcal{D}(T') \subseteq \mathcal{H} \mapsto \mathcal{H}$ such that $\mathcal{R}(T) \subseteq \mathcal{D}(T')$ and

$$TT'T = T.$$

Operator T' is called the inner inverse of T. In general, note that T' may not be bounded, i.e., $T' \notin \mathfrak{B}(\mathcal{H})$. Moreover, for $T \in \mathfrak{B}(\mathcal{H})$ there exists an inner inverse of T, T', such that $T' \in \mathfrak{B}(\mathcal{H})$ if and only if T has closed range [21]. Additionaly, if T' also satisfies

$$T'TT' = T',$$

then T' is called a reflexive inverse of T. Furthermore, there exists a unique reflexive inverse X of operator T which satisfies the system of equations

$$XT = P_{\overline{R(T^*)}} \quad \text{and} \quad TX = P_{\overline{R(T)}} \restriction_{\mathcal{R}(T) \oplus \mathcal{R}(T)^\perp},$$

Such an operator is called the Moore-Penrose (generalized) inverse of T and will be denoted by $T^\dagger$. For more details, see ([18], [4], [5], [22], [23]).

We also state the celebrated Douglas' Lemma, we can say freely, an irreplaceable tool when dealing with operator range inclusions.

Theorem 2.1 (Douglas' Lemma [11])**.** Let A and B be bounded operators on Hilbert space $\mathcal{H}$. The following statements are equivalent:

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;
(ii) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$;
(iii) there exists a bounded operator C on $\mathcal{H}$ such that $A = BC$.

Moreover, if any of the previous conditions holds, then there exists a unique operator C so that

(1) $\|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$;
(2) $\mathcal{N}(A) = \mathcal{N}(C)$;
(3) $\mathcal{R}(C) \subseteq \overline{\mathcal{R}(B^*)}$.

As an operator T is p–accretive for some $p \in \mathbb{C}[z]$ if and only if $\operatorname{Re}(p(T))$ is positive (see Theorem 3.1 below), the following operator matrix representation will be crucial in our work.

Theorem 2.2. [2] Let S be a closed subspace of $\mathcal{H}$ and $T \in \mathfrak{B}(\mathcal{H})$ have the matrix operator decomposition induced by S and given by (2.1). Then, T is positive if and only if

(i) $T_{11} \geq 0$;
(ii) $T_{21} = T_{12}^*$;
(iii) $\mathcal{R}(T_{12}) \subseteq \mathcal{R}(T_{11}^{1/2})$;
(iv) $T_{22} = \left((T_{11}^{1/2})^\dagger T_{12}\right)^* (T_{11}^{1/2})^\dagger T_{12} + F$, where $F \geq 0$.

Finally, we give several standard results concerning normal and positive operators.

Theorem 2.3 (Fuglede Theorem [12])**.** If M and N are commuting normal operators, then MN is also normal.

Theorem 2.4. [25, Theorem 12.12] If $n \in \mathbb{N}$, then the commutants of a positive operator and its n-th root coincide.

Theorem 2.5. [19, Corollary 5.1.36] If $A, B \in \mathfrak{B}(\mathcal{H})$ are two commuting and positive operators, then

$$\sqrt[n]{AB} = \sqrt[n]{A}\,\sqrt[n]{B},$$

for all $n \in \mathbb{N}$.

Theorem 2.6 (Löwner-Heinz inequality [14, 17])**.** If $A, B \in \mathfrak{B}(\mathcal{H})$ are positive operators such that $B \leq A$ and $p \in [0, 1]$, then $B^p \leq A^p$.

Remark 2.7. In general, the previous theorem does not hold for $p > 1$ (see, for example, [20, page 55]). However, if A and B commute, and $p \in \mathbb{N}$, then $B \leq A$ implies $B^p \leq A^p$. Indeed, since A and B commute, we may write

$$A^p - B^p = (A - B)(A^{p-1} + A^{p-2}B + \ldots + B^{p-1}).$$

Since A and B commute and $B \leq A$, we have that $A - B$ and $A^{p-1} + A^{p-2}B + \ldots + B^{p-1}$ are two commuting positive operators, and so $A^p - B^p$ is also positive. Thus, $B^p \leq A^p$, as desired.

3. General properties

We start this section with the following elementary observation.

Theorem 3.1. Let $T \in \mathfrak{B}(\mathcal{H})$ and $p \in \mathbb{C}[z]$. The following conditions are equivalent:

(i) T is p–accretive;
(ii) $p(T)$ is accretive.
(iii) Re $\langle p(T)x, x\rangle \geq 0$, for all $x \in \mathcal{H}$;
(iv) T^* is $\overline{p}$–accretive.

Proof. $(i) \Leftrightarrow (ii)$: Obvious.

$(i) \Leftrightarrow (iii)$: We have that

$$\begin{aligned} p(T) + \overline{p}(T^*) \geq 0 &\Longleftrightarrow \langle (p(T) + \overline{p}(T^*))x, x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}, \\ &\Longleftrightarrow \langle p(T)x, x\rangle + \langle \overline{p}(T^*)x, x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}, \\ &\Longleftrightarrow \langle p(T)x, x\rangle + \langle x, p(T)x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}, \\ &\Longleftrightarrow \langle p(T)x, x\rangle + \overline{\langle p(T)x, x\rangle} \geq 0, \quad \text{for all } x \in \mathcal{H}, \\ &\Longleftrightarrow \text{Re } \langle p(T)x, x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

$(i) \Leftrightarrow (iv)$: This follows directly from the definition.

□

Theorem 3.2. Let $p \in \mathbb{C}[z]$ and $T \in \mathfrak{B}(\mathcal{H})$ be p–accretive.

(i) If zeroes of p do not belong to $\sigma(T)$, then $p(T)^{-1}$ is accretive.
(ii) If S is unitarily equivalent to T, then S is p–accretive.
(iii) If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces T, then $P_{\mathcal{M}}T|_{\mathcal{M}}$ is p–accretive.

Proof. (i) Assume that the zeroes of p do not belong to $\sigma(T)$. Then, by the Spectral Mapping Theorem, we have that $p(T)$ is invertible. Since $p(T)$ is accretive, for all $x \in \mathcal{H}$, we have

$$0 \leq \text{Re } \left\langle p(T)p(T)^{-1}x, p(T)^{-1}x\right\rangle = \text{Re } \left\langle x, p(T)^{-1}x\right\rangle = \text{Re } \left\langle p(T)^{-1}x, x\right\rangle.$$

Thus, $p(T)^{-1}$ is accretive.

(ii) By assumption, S is unitarily equivalent to T, and so there exists a unitary operator $U \in \mathfrak{B}(\mathcal{H})$ such that $S = U^*TU$. Then $S^* = U^*T^*U$, and it is easy to see that $p(S) = U^*p(T)U$ and $\overline{p}(S^*) = U^*\overline{p}(T^*)U$. From (1.1) now follows that

$$p(S) + \overline{p}(S^*) = U^*p(T)U + U^*\overline{p}(T^*)U = U^*(p(T) + \overline{p}(T^*))U \geq 0,$$

and so S is p–accretive.

(iii) If $\mathcal{M}$ is a closed reducing subspace for T, then T can be represented as

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix}.$$

From here,

$$T^* = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix}, \quad p(T) = \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \text{ and } \overline{p}(T) = \begin{bmatrix} \overline{p}(T_1^*) & 0 \\ 0 & \overline{p}(T_2^*) \end{bmatrix}.$$

Using the fact that T is p–accretive, for any $x \in \mathcal{M}$, we have

$$0 \leq \text{Re} \left\langle \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle = \text{Re} \left\langle p(T_1)x, x \right\rangle.$$

Thus, $P_{\mathcal{M}} T|_{\mathcal{M}} = T_1$ is p–accretive. □

Theorem 3.3. Let $p \in \mathbb{C}[z]$ and $T \in \mathfrak{B}(\mathcal{H})$. If $T = T_1 \oplus T_2$, then T is p–accretive if and only if T_1 and T_2 are p–accretive.

Proof. The "*if*" part follows from part *(iii)* of the previous theorem.

Now assume that T_1 and T_2 are p–accretive and let $[x\ y]^\top \in \mathcal{H} \oplus \mathcal{H}$ be arbitrary. Then,

$$\begin{aligned} \text{Re} \left\langle \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle &= \text{Re} \left\langle \begin{bmatrix} p(T_1)x \\ p(T_2)y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \\ &= \text{Re} \left(\langle p(T_1)x, x \rangle + \langle p(T_2)y, y \rangle \right) \\ &= \text{Re} \langle p(T_1)x, x \rangle + \text{Re} \langle p(T_2)y, y \rangle \\ &\geq 0. \end{aligned}$$

Thus, T is p–accretive. □

Theorem 3.4. Let $T \in \mathfrak{B}(\mathcal{H})$. If T is k–accretive for all $1 \leq k \leq n$, then T is p–accretive for any polynomial p of a degree n with nonnegative coefficients.

Proof. Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ be an n–th degree polynomial with nonnegative coefficients. Then,

$$\begin{aligned} \text{Re} \langle p(T)x, x \rangle &= \text{Re} \left\langle \sum_{k=0}^{n} a_k T^k x, x \right\rangle \\ &= \text{Re} \sum_{k=0}^{n} a_k \left\langle T^k x, x \right\rangle \\ &= \sum_{k=0}^{n} a_k \text{Re} \left\langle T^k x, x \right\rangle \\ &= a_0 \|x\|^2 + \sum_{k=1}^{n} a_k \text{Re} \left\langle T^k x, x \right\rangle \\ &\geq 0. \end{aligned}$$

Theorem 3.1 now yields the wanted result. □

Theorem 3.5. Let $T \in \mathfrak{B}(\mathcal{H})$ and $q, r \in \mathbb{C}[z]$. Consider $F = q(T) + \overline{r}(T^*)$ and $G = q(T) - \overline{r}(T^*)$ and let $p(z) = q(z)r(z)$, $z \in \mathbb{C}$. The following conditions are equivalent:

(i) T is p–accretive;

(ii) $GG^* \leq FF^*$.

Proof. By direct computation, we have

$$\begin{aligned} FF^* - GG^* &= (q(T) + \overline{r}(T^*))(\overline{q}(T^*) + r(T)) \\ &\quad - (q(T) - \overline{r}(T^*))(\overline{q}(T^*) - r(T)) \\ &= q(T)\overline{q}(T^*) + q(T)r(T) + \overline{r}(T^*)\overline{q}(T^*) + \overline{r}(T^*)r(T) \\ &\quad - (q(T)\overline{q}(T^*) - q(T)r(T) - \overline{r}(T^*)\overline{q}(T^*) + \overline{r}(T^*)r(T)) \\ &= 2(q(T)r(T) + \overline{r}(T^*)\overline{q}(T^*)) \\ &= 2(p(T) + \overline{p}(T^*)). \end{aligned}$$

Therefore,

$$T \text{ is } p\text{–accretive} \iff p(T) + \overline{p}(T^*) \geq 0 \iff FF^* - GG^* \geq 0,$$

from where the conclusion follows. □

4. The structure of p–accretive operators

The main goal of this section is to give some representations and the structure of polynomially accretive operators. The starting point in our discussion will be the following representation theorem of 2–normal operators proved by Radjavi and Rosenthal in [24]. We present it here in a slightly different form.

Theorem 4.1. [24] Let $T \in \mathfrak{B}(\mathcal{H})$. Operator T is 2–normal if and only if

$$T = \begin{bmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{bmatrix}, \tag{4.1}$$

where A, B are normal, $C \geq 0$, C is one–to–one and $BC = CB$. Moreover, B can be chosen so that $\sigma(B)$ lies in the closed upper half–plane and the Hermitian part of B is non-negative.

In the case when polynomial p has only even powers, the characterization of polynomially accretive operators is rather simple.

Theorem 4.2. Let $T \in \mathfrak{B}(\mathcal{H})$ be 2–normal operator with the matrix representation given by (4.1) and let $p \in \mathbb{C}[z]$ be a polynomial with even powers only. Then T is p–accretive if and only if A and B are p–accretive.

Proof. First note that, since B and C commute, we have that

$$T^2 = \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & B^2 \end{bmatrix}.$$

Since polynomial p has even powers only, we have that $p(z) = q(z^2)$ for some polynomial $q \in \mathbb{C}[z]$. Therefore,

$$p(T) = q(T^2) = \begin{bmatrix} q(A^2) & 0 & 0 \\ 0 & q(B^2) & 0 \\ 0 & 0 & q(B^2) \end{bmatrix} = \begin{bmatrix} p(A) & 0 & 0 \\ 0 & p(B) & 0 \\ 0 & 0 & p(B) \end{bmatrix}.$$

The conclusion now follows by combining Theorems 3.1 and 3.3. □

Lemma 4.3. Let $T \in \mathfrak{B}(\mathcal{H})$ be 2–normal operator with the matrix representation given by (4.1) and let $p \in \mathbb{C}[z]$ be a polynomial of a degree at least 3 and with exactly one odd power $k \geq 3$. If $\mathrm{Re}\,(p(B))$ and $\mathrm{Re}\,(p(-B))$ have closed ranges, then T is p–accretive if and only if the following conditions hold:

(*i*) A is p–accretive;

(*ii*) $|B| \leq \mu(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}$, for some $\mu > 0$;

(iii) $|B|(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \nu(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}}$ for some $\nu > 0$.

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $n \geq 3$. Using representation (4.1), we have that

$$p(T) = \begin{bmatrix} p(A) & 0 & 0 \\ 0 & p(B) & D \\ 0 & 0 & p(-B) \end{bmatrix},$$

for some D. Since $k \geq 3$ is the only odd integer such that $a_k \neq 0$, we have that $D = a_k B^{k-1} C$. Therefore,

$$p(T) + \overline{p}(T^*) = \begin{bmatrix} p(A) + \overline{p}(A^*) & 0 & 0 \\ 0 & p(B) + \overline{p}(B^*) & a_k B^{k-1} C \\ 0 & \overline{a_k}(B^{k-1}C)^* & p(-B) + \overline{p}(-B^*) \end{bmatrix},$$

i.e.,

$$\mathrm{Re}\,(p(T)) = \begin{bmatrix} \mathrm{Re}\,(p(A)) & 0 & 0 \\ 0 & \mathrm{Re}\,(p(B)) & \frac{a_k}{2} B^{k-1} C \\ 0 & (\frac{a_k}{2} B^{k-1} C)^* & \mathrm{Re}\,(p(-B)) \end{bmatrix}.$$

Thus, we have that T is p–accretive if and only if the following two conditions hold:

(i') $\mathrm{Re}\,(p(A)) \geq 0$;

(ii') $\begin{bmatrix} \mathrm{Re}\,(p(B)) & \frac{a_k}{2} B^{k-1} C \\ (\frac{a_k}{2} B^{k-1} C)^* & \mathrm{Re}\,(p(-B)) \end{bmatrix} \geq 0.$

Obviously, conditions (i) and (i') are equivalent. By Theorem 2.2, condition (ii') is equivalent with the conjunction of the following three conditions:

(i'') $\mathrm{Re}\,(p(B)) \geq 0$;

(ii'') $\mathcal{R}(B^{k-1}C) \subseteq \mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})$;

(iii'') $\mathrm{Re}\,(p(-B)) \geq \frac{|a_k|^2}{4} F^* F$, where $F = \left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} B^{k-1} C$.

First, we focus on condition (ii'). Let us show that $(ii'') \implies (ii)$. Note that since C is one–to–one, we have that $\mathcal{R}(C)$ is dense in $\mathcal{H}$. Thus, $B^{k-1}(\mathcal{R}(C)) \subseteq \mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})$ now implies that

$$\mathcal{R}(B^{k-1}) \subseteq \overline{\mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})}.$$

By assumption, a positive operator $\mathrm{Re}\,(p(B))$ has closed range, and thus, $\mathcal{R}(\mathrm{Re}\,(p(B))) = \mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})$. Therefore,

$$\mathcal{R}(B^{k-1}) \subseteq \mathcal{R}(\mathrm{Re}\,(p(B))).$$

By Theorem 2.1, there exists $\mu' > 0$ such that

$$B^{k-1}(B^*)^{k-1} \leq \mu' \left(\mathrm{Re}\,(p(B))\right)^2$$

Using the fact that B is normal, it follows that

$$|B|^{2(k-1)} = (B^* B)^{k-1} \leq \mu' \left(\mathrm{Re}\,(p(B))\right)^2.$$

Since the function $f(x) = x^{\frac{1}{2(k-1)}}$ is operator monotone (Theorem 2.6), we have

$$|B| \leq (\mu')^{\frac{1}{2(k-1)}} (\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}.$$

By taking $\mu = (\mu')^{\frac{1}{2(k-1)}}$, condition (ii) now follows. The reverse implication can be proved in a similar manner by noting that normality of B and Theorem 2.4 imply that $|B|$ and $(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}$ commute, and thus the function $g(x) = x^{2(k-1)}$ preserves monotonicity, by Remark 2.7. Thus, $(ii) \iff (ii'')$.

Let us now show that $(iii) \iff (iii'')$. Assume that (iii'') holds. Since B is normal, C is positive and $BC = CB$, we have that both B and C commute with a positive operator $\mathrm{Re}\,(p(B))$. By Theorem 2.4, they also commute with $\mathrm{Re}\,(p(B)))^{1/2}$, which further implies, by using the Spectral Theorem for normal operators, that they commute with $\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger}$, as well. Thus, by Theorem 2.3, we have that operator $F = \left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} B^{k-1}C$ is normal. Hence, $F^*F = FF^*$, and so

$$\mathrm{Re}\,(p(-B)) \geq \frac{|a_k|^2}{4} FF^*.$$

By Theorem 2.1 and using the closedness of range of $\mathrm{Re}\,(p(-B))$, we conclude that

$$\mathcal{R}(F) \subseteq \mathcal{R}((\mathrm{Re}\,(p(-B))^{1/2})) = \mathcal{R}(\mathrm{Re}\,(p(-B))),$$

i.e.,

$$B^{k-1}\mathcal{R}\left(\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} C\right) \subseteq \mathcal{R}(\mathrm{Re}\,(p(-B))). \tag{4.2}$$

Observe that

$$\begin{aligned}\overline{\mathcal{R}\left(((\mathrm{Re}\,(p(B)))^{1/2})^{\dagger} C\right)} &= \overline{((\mathrm{Re}\,(p(B)))^{1/2})^{\dagger}\, \overline{\mathcal{R}\,(C)}}\\ &= \overline{\mathcal{R}\left(((\mathrm{Re}\,(p(B)))^{1/2})^{\dagger}\right)}\\ &= \overline{\mathcal{R}\left(((\mathrm{Re}\,(p(B)))^{1/2})^{*}\right)}\\ &= \overline{\mathcal{R}\left((\mathrm{Re}\,(p(B)))^{1/2}\right)}\\ &= \mathcal{R}\left(\mathrm{Re}\,(p(B))\right).\end{aligned}$$

Combining this with (4.2), and again using the fact that $\mathrm{Re}\,(p(-B))$ has closed range, we have

$$\mathcal{R}(B^{k-1}\mathrm{Re}\,(p(B))) \subseteq \mathcal{R}(\mathrm{Re}\,(p(-B))).$$

Theorem 2.1 now implies that there exists $\nu > 0$ such that

$$|B|^{2(k-1)}\mathrm{Re}\,(p(B))^2 \leq \nu'\mathrm{Re}\,(p(-B))^2.$$

Using the fact that $|B|$ and $\mathrm{Re}\,(p(B))$ commute and Theorem 2.5, monotonicity of $f(x) = x^{\frac{1}{2(k-1)}}$ now implies

$$|B|(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \nu(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}},$$

where $\nu = (\nu')^{\frac{1}{2(k-1)}}$. Therefore, (iii) holds.
Using the similar arguments and comments as in part $(ii) \implies (ii'')$, we can show that $(iii) \implies (iii'')$. This completes the proof. □

Remark 4.4. If T and $p \in \mathbb{C}[z]$ are as in Lemma 4.3, we can see that p–accretivity of T implies that the operators A, B and $-B$ are also p–accretive.

Theorem 4.5. Let $T \in \mathfrak{B}(\mathcal{H})$ and $p \in \mathbb{C}[z]$ be as in Lemma 4.3. If operator T is p–accretive then A is p–accretive and there exists $\lambda > 0$ such that

$$\frac{1}{\lambda}|B|^2 \leq |B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \lambda(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}}. \tag{4.3}$$

Moreover, if B is left–invertible, then the reverse implication holds, as well.

Proof. $(\Longrightarrow:)$ Assume that T is p–accretive. By Lemma 4.3, we have that A is p–accretive and there exists $\mu, \nu > 0$ such that

$$|B| \leq \mu(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \tag{4.4}$$

and

$$|B|(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \le \nu(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}}.$$

Let $\lambda = \max\{\mu, \nu\}$. Then the second inequality in (4.3) is obviously satisfied. Now using the fact that B is normal, (4.4) yields

$$\begin{aligned}|B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} &= |B|^{\frac{1}{2}}\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}|B|^{\frac{1}{2}}\\ &\ge \frac{1}{\mu}|B|^{\frac{1}{2}}|B||B|^{\frac{1}{2}} \ge \frac{1}{\lambda}|B|^2.\end{aligned}$$

Hence, (4.3) holds.

($\Longleftarrow$:) To prove the reverse inequality, it is enough to show that condition (ii) in Lemma 4.3 is satisfied. Observe that $|B|$ is invertible since B is left–invertible, by assumption. Therefore, using the normality of B, the inequality $\frac{1}{\lambda}|B|^2 \le |B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}$ implies that

$$|B|^{-\frac{1}{2}}|B|^2|B|^{-\frac{1}{2}} \le \lambda|B|^{-\frac{1}{2}}\left(|B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}\right)|B|^{-\frac{1}{2}},$$

i.e.,

$$|B| \le \lambda(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}.$$

Thus, T is p–accretive and the proof is completed. □

Theorem 4.6. Let $T \in \mathfrak{B}(\mathcal{H})$ and let $k \in \mathbb{N}$. The following conditions are equivalent:

(i) T is 2–normal and $(2k+1)$–accretive;

(ii) $T = T_1 \oplus T_2$, where T_1 is normal and $(2k+1)$–accretive, and T_2 is nilpotent of index 2.

Proof. $(i) \Longrightarrow (ii)$. Assume that (i) holds. Since T is 2–normal, it is given by (4.1). By analysing the proof of Lemma 4.3, we have that $(2k+1)$–accretivity of T implies that A is $(2k+1)$–accretive, and also, the following conditions hold:

(i'') $B^{2k+1} + (B^*)^{2k+1} \ge 0$;

(ii'') $\mathcal{R}(B^{2k}C) \subseteq \mathcal{R}((B^{2k+1} + (B^*)^{2k+1})^{1/2})$;

(iii'') $(-B)^{2k+1} + (-B^*)^{2k+1} \ge 0$.

Condition (iii'') is equivalent with the fact that

$$-\left(B^{2k+1} + (B^*)^{2k+1}\right) = (-B)^{2k+1} + (-B^*)^{2k+1} \ge 0.$$

This, together with (i''), implies that $B^{2k+1} + (B^*)^{2k+1}$ must be equal to the zero operator. Therefore,

$$\mathcal{R}((B^{2k+1} + (B^*)^{2k+1})^{1/2}) = \{0\}.$$

Condition (ii'') yields that $\mathcal{R}(B^{2k}C) \subseteq \{0\}$, and thus $CB^{2k} = 0$. But C is one–to–one, and so $B^{2k} = 0$. The only nilpotent normal operator is zero operator, and hence, $B = 0$. Let

$$T_1 = A \text{ and } T_2 = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}.$$

Then, $T = T_1 \oplus T_2$, where T_1 is $(2k+1)$–accretive and T_2 is nilpotent of index 2, as required.

$(ii) \Longrightarrow (i)$ Now assume that (ii) holds. Then $T^2 = T_1^2 \oplus 0$ implies that T^2 is normal. Similarly, $T^{2k+1} = T_1^{2k+1} \oplus 0$ yields the $(2k+1)$–accretivity of T. □

Corollary 4.7. Let $T \in \mathfrak{B}(\mathcal{H})$. If T is 2–normal and $(2k+1)$–accretive for some $k \in \mathbb{N}$, then T is n–normal for all $n \ge 2$.

Proof. It follows immediately from the representation of T given in Theorem 4.6. □

Remark 4.8. In general, under the conditions of Corollary 4.7, we cannot conclude that the operator T is normal. To see this, it is enough to take any non–normal operator T such that $T^2 = 0$.

In the following proposition, motivated by [1, Lemma 2.28], we give a necessary condition for the normality of T. First recall that the self–commutator of an operator T, denoted by $[T^*, T]$, is an operator given by $[T^*, T] = T^*T - TT^*$.

Corollary 4.9. Let $T \in \mathfrak{B}(\mathcal{H})$ be such that T is 2–normal and $(2k+1)$–accretive for some $k \in \mathbb{N}$. If $\mathcal{R}([T^*, T]) \subseteq \mathcal{N}(T^l)^\perp$, for some $l \geq 2$, then T is normal.

Proof. Since T is 2–normal and $(2k+1)$–accretive, we have that T is n–normal for all $n \geq 2$, by Corollary 4.7. Specially, T is l–normal and $(l+1)$–normal. Thus,

$$T^l T T^* = T^{l+1} T^* = T^* T^{l+1} = T^* T^l T = T^l T^* T,$$

i.e., $T^l(T^*T - TT^*) = 0$. Thus, $\mathcal{R}([T^*, T]) \subseteq \mathcal{N}(T^l) \cap \mathcal{N}(T^l)^\perp = \{0\}$, from where it follows that $TT^* = T^*T$, i.e., T is normal. □

Corollary 4.10. Let $T \in \mathfrak{B}(\mathcal{H})$. If T is injective, 2–normal and $(2k+1)$–accretive for some $k \in \mathbb{N}$, then T is normal.

The following corollaries are matrix analogues of the previous results, presented in the language of matrix theory.

Corollary 4.11. Let A be a $n \times n$ complex matrix and let $k \in \mathbb{N}$. The following conditions are equivalent:

(i) A^2 is normal and A^{2k+1} is Re–nnd.

(ii) $A = A_1 \oplus A_2$, where A_1 is normal, A_1^{2k+1} is Re-nnd, and A_2 is nilpotent of index 2.

Corollary 4.12. Let A be a $n \times n$ complex matrix. If A^2 is normal and A^{2k+1} is Re–nnd for some $k \in \mathbb{N}$, then A^n is normal for all $n \geq 2$.

Corollary 4.13. Let A be a $n \times n$ complex non–singular matrix. If A^2 is normal and A^{2k+1} is Re–nnd for some $k \in \mathbb{N}$, then A is normal.

At the end, we also consider the connection between n-accretivity and positivity.

Theorem 4.14. Let $T \in \mathfrak{B}(\mathcal{H})$. The following conditions are equivalent:

(i) T is normal and k-accretive for each $k \in \mathbb{N}$;

(ii) $T \geq 0$.

Proof. (i) ⇒ (ii) Let $T = U|T|$ be the polar decomposition of T. Since T is normal, $\mathcal{N}(T) = \mathcal{N}(T^*)$, and we may write

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{pmatrix}.$$

Furthermore, since U is normal, i.e. $U^*U = UU^*$, the initial space $\mathcal{N}(T)^\perp$ coincides with the final space $\overline{\mathcal{R}(T)}$, which directly implies that the restriction of U to the initial space is unitary. Since $\mathcal{N}(U) = \mathcal{N}(T)$, it follows that

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{pmatrix},$$

for some unitary operator $U_1 \in \mathfrak{B}(\mathcal{N}(T)^\perp)$. Thus, without loss of generality, we may assume that T is injective with dense range, and U is unitary.

By [19, Proposition 11.1.46], there is a uniquely determined injective and positive operator $\Theta \in \mathfrak{B}(\mathcal{H})$ with $\sigma(\Theta) \subseteq [0, 2\pi]$ such that

$$U = e^{i\Theta}.$$

Using the series expansion of the exponential function, we have

$$U = \cos\Theta + i\sin\Theta,$$

from where it follows that

$$U^k = \cos(k\Theta) + i\sin(k\Theta), \quad k \in \mathbb{N}.$$

The normality of T implies that $U|T| = |T|U$. Thus, the real part of $T^k = U^k|T|^k$, is given by

$$\mathrm{Re}(T^k) = |T|^k\cos(k\Theta),$$

for each $k \in \mathbb{N}$.

We will show that $\sigma(\Theta) \subseteq \{0, 2\pi\}$. Assume to the contrary, that there exists $\lambda \in (0, 2\pi)$ such that $\lambda \in \sigma(\Theta)$. Then, by the Spectral Mapping Theorem, $\cos(k\lambda) \in \sigma(\cos(k\Theta))$ for each $k \in \mathbb{N}$. However, since $\lambda \in (0, 2\pi)$, there exists $m \in \mathbb{N}$ such that $\mathrm{Re}(\lambda^m) = \cos(m\lambda) < 0$. Thus, $\cos(m\Theta)$ is not a positive operator, and so there exists $x \in \mathcal{H}$ such that $\langle\cos(m\theta)x, x\rangle < 0$. Since $\overline{R(|T|^{\frac{m}{2}})} = \overline{R(T)} = \mathcal{H}$, there exists $x' \in \mathcal{H}$ such that

$$\begin{aligned}\langle\mathrm{Re}(T^m)x', x'\rangle &= \langle|T|^m\cos(m\Theta)x', x'\rangle \\ &= \left\langle\cos(m\Theta)|T|^{\frac{m}{2}}x', |T|^{\frac{m}{2}}x'\right\rangle \\ &< 0,\end{aligned}$$

which is a contradiction with the fact that T is m-accretive. Thus, $\sigma(\Theta) \subseteq \{0, 2\pi\}$, and by the Spectral Mapping Theorem, $\sigma(U) = \{1\}$. Then, $U - I$ is normal, and $\sigma(U - I) = \{0\}$, so it follows that $U = I$. This directly implies that $T = |T| \geq 0$.

$(i) \Rightarrow (ii)$ This is obvious. □

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