Fundamental Journal of Mathematics and Applications, 7(1) (2024), 53-58
Research Paper / Open Access

FUJMA

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1424382

A Note On Kantorovich Type Operators Which Preserve Affine Functions

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Article Information

Keywords: Modulus of continuity; Rate of convergence; Voronovskaya theorem

AMS 2020 Classification: 41A25; 41A36

Abstract

The authors present an integral widening of operators which preserve affine functions. Influenced by the operators which preserve affine functions, we define the integral extension of these operators. We give quantitative type theorem using weighted modulus of continuity. Withal quantitative Voronovskaya theorem is aquired by classical modulus of continuity. When the moments of the operator are known, convergence results with the moments obtained for the Kantorovich form of the same operator is given.

1. Introduction

In mathematical analysis, studies on approximation by linear and positive operators retained its importance for many years. Recently many researchers have studied some generalizations of these operators, especially the Kantorovich form of Bernstein, Baskakov and Szàsz operators. Also they have studied some operators which preserve test functions, exponentials and affine functions (see [1]-[8]).

The Kantorovich version of Bernstein operators [9] defined by replacing the sample values $f\left(\frac{k}{n}\right)$ with the mean values of f in $\left[\frac{k}{n},\frac{k+1}{n}\right]$, namely for $x\in[0,1]$, $n\in\mathbb{N}$ and $f\in L_1[0,1]$, $P_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$, $k=0,1,\dots,n$

$$K_n(f)(x)=(n+1)\sum_{k=0}^{n}P_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)dt. \tag{1.1}$$

Note that K_n is just reproduced 1. These operators provide us to switch a Lebesgue integrable function by means of its mean values on the sets $\left[\frac{k}{n},\frac{k+1}{n}\right]$.

General in use, such a $(L_n)_{n\geq 1}$ sequence of linear and positive operators are specified. In 2016, Agratini studied Kantorovich type operators which preserve affine functions ([2]). Inspire of these general operators which preserve affine functions, we study these operators on weighted spaces.

Let's describe the layout of this work. In first part, nodes and moments are given. The second part belongs to some approximation findings for the operators.

The purpose of this article is to show that if we know the moments of the operators, we find convergence results with the moments obtained for the Kantorovich type generalization of the same operator.

⋙ **Received:** 23-01-2024 ⋙ **Revised:** 27-02-2024 ⋙ **Accepted:** 11-03-2024 ⋙ **Online:** 29-03-2024 ⋙ **Published:** 31-03-2024

2. Properties of the operators

Througout the paper, we consider an interval $\mathbb{R}^+ = [0,\infty)$. In [9], we can see the Kantorovich form of the Bernstein operators as

$$K_n(f)(x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ x\in[0,1],$$

where $f \in L_1[0,1]$. Let $C(\mathbb{R}^+)$ denotes the space of real-valued continuous functions on $\mathbb{R}^+$, now we give L_n operator which can be written as

$$L_n(f;x) = \sum_{k\in J_n} \lambda_{n,k}(x) f(x_{n,k}),\ x\in\mathbb{R}^+ \tag{2.1}$$

where $\lambda_{n,k} \in C(\mathbb{R}^+)$ and $\lambda_{n,k}\geq 0$ and $(n,k)\in\mathbb{N}\times J_n$. Also $(x_{n,k})_{k\in J_n}$ be set on the interval $\mathbb{R}^+$ where $J_n\subseteq\mathbb{N}$ is a set of indices. Now we consider nodes for each $n\in\mathbb{N}$,

$$x_{n,k+1} - x_{n,k} = u_n,\ k\in J_n$$

where $\lim\limits_{n\to\infty} u_n = 0$.
We take into about L_n operators given by (2.1) which preserve affine functions,

$$\sum_{k\in J_n}\lambda_{n,k}(x) = 1 \text{ and } \sum_{k\in J_n}\lambda_{n,k}(x)x_{n,k} = x,\ x\in\mathbb{R}^+.$$

Now let $u_n^* = \sup\limits_{n\in\mathbb{N}} u_n$. If $\mathbb{R}^+ = [0,\infty)$, then we set $A^* = [\frac{u^*}{2},\infty)$.

2.1. Auxiliary Results

We give some results which will be necessary for proofs of theorems. At first, we find some moments and central moments of

$$\widetilde{K_n}(f;x) = \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\int_{x_{n,k}}^{x_{n,k+1}} f(t)dt,\ x\in\mathbb{R}^+ \tag{2.2}$$

operators.

Lemma 2.1. *Let L_n defined by (2.1), $n\in\mathbb{N}$, $x\in A^*$ and $e_r(t)=t^r$ for*

$r = 1,\ 2,\ 3,\ 4$. Then we have

(i) $\widetilde{K_n}(e_0)(x) = 1$,
(ii) $\widetilde{K_n}(e_1)(x) = x + \frac{u_n}{2}$,
(iii) $\widetilde{K_n}(e_2)(x) = L_n(e_2)(x) + u_n x - \frac{u_n^2}{3}$,
(iv) $\widetilde{K_n}(e_3)(x) = L_n(e_3)(x) + \frac{3}{2}u_n L_n(e_2)(x) + u_n^2 x - \frac{u_n^3}{4}$,
(v) $\widetilde{K_n}(e_4)(x) = L_n(e_4)(x) + 2u_n L_n(e_3)(x) + 2u_n^2 L_n(e_2)(x) + u_n^3 x - \frac{u_n^4}{5}$.

Proof. (i) It is clear from the definition of the operator $\widetilde{K_n}$.
(ii)

$$\begin{aligned}\widetilde{K_n}(e_1)(x) &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\int_{x_{n,k}}^{x_{n,k+1}} t\,dt\\ &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\frac{1}{2}\left(x_{n,k+1}^2 - x_{n,k}^2\right)\\ &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\frac{1}{2}\left(u_n^2 + 2u_n x_{n,k}\right)\\ &= \frac{u_n}{2} + x.\end{aligned}$$

(iii)

$$\begin{aligned}\widetilde{K_n}(e_2)(x) &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\frac{1}{3}\left(x_{n,k+1}^3-x_{n,k}^3\right)\\ &= \frac{1}{3u_n}\sum_{k\in J_n}\lambda_{n,k}(x)u_n\left[\left(x_{n,k}+u_n\right)^2+x_{n,k}\left(x_{n,k}+u_n\right)+x_{n,k}^2\right],\\ &= \sum_{k\in J_n}\lambda_{n,k}(x)x_{n,k}^2+u_nx+\frac{u_n^2}{3}\\ &= L_n(e_2)(x)+u_nx+\frac{u_n^2}{3}.\end{aligned}$$

(iv)

$$\begin{aligned}\widetilde{K_n}(e_3)(x) &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\frac{1}{4}\left(x_{n,k+1}^4-x_{n,k}^4\right)\\ &= \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\left(x_{n,k+1}-x_{n,k}\right)\left(x_{n,k+1}+x_{n,k}\right)\left(x_{n,k+1}^2+x_{n,k}^2\right)\\ &= \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\left[u_n(2x_{n,k}+u_n)(\left(u_n+x_{n,k}\right)^2+x_{n,k}^2)\right]\\ &= \sum_{k\in J_n}\lambda_{n,k}(x)x_{n,k}^3+\frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)6u_n^2x_{n,k}^2+\frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)4u_n^3x_{n,k}+\frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)u_n^4\\ &= L_n(e_3)(x)+\frac{3}{2}u_nL_n(e_2)(x)+u_n^2x-\frac{u_n^3}{4}.\end{aligned}$$

(v) At that time, (v) can be calculated similarly.

□

Lemma 2.2. *Let $\varphi_x^n(t)=(t-x)^n, n=0,1,2,...$ For the operator $\widetilde{K_n}$ given by (2.2) if we set $\zeta_{n,2}(x)=\widetilde{K_n}(\varphi_x^2(t);x)$ and $\zeta_{n,4}(x)=\widetilde{K_n}(\varphi_x^4(t);x)$, then we have*

$$\zeta_{n,2}(x)=L_n(e_{2,};x)+\frac{u_n^2}{3}-x^2,$$

$$\zeta_{n,4}(x) = L_n(e_4)(x)+(2u_n-4x)L_n(e_3)(x)+(2u_n^2+-6xu_n+6x^2)L_n(e_2)(x)+4x^3u_nL_n(e_1)(x)-3x^4L_n(e_0)(x)+u_n^4-2x^2u_n^2.$$

Proof. By using Lemma 1.1, we obtain

$$\begin{aligned}\zeta_{n,2}(x) &= \widetilde{K_n}(\varphi_x^2(t);x)=L_n(e_{2,};x)+\frac{u_n^2}{3}+u_nx-2x(x+\frac{u_n}{2})+x^2\\ &= L_n(e_{2,};x)+\frac{u_n^2}{3}-x^2.\end{aligned}$$

Now let's calculate $\widetilde{K_n}(\varphi_x^4(t);x)$.

$$\begin{aligned}\zeta_{n,4}(x) &= \widetilde{K_n}(\varphi_x^4(t);x)=\widetilde{K_n}(e_4,x)-4\widetilde{K_n}(e_3,x)x+6\widetilde{K_n}(e_2,x)x^2-4\widetilde{K_n}(e_1,x)x^3+x^4\widetilde{K_n}(e_{0,},x)\\ &= L_n(e_{4,};x)+2u_nL_n(e_{3,};x)+2u_n^2L_n(e_{2,};x)+u_n^3x+u_n^4-4x(L_n(e_{3,};x)+\frac{3}{2}u_nL_n(e_{2,};x)+u_n^2x+\frac{u_n^3}{4})\\ &\quad +6x^2(L_n(e_{2,};x)+u_nx+\frac{u_n^2}{3})-4x^3(\frac{u_n}{2}+L_n(e_{1,};x))+x^4L_n(e_{0,};x))\\ &= L_n(e_4)(x)+(2u_n-4x)L_n(e_3)(x)+(2u_n^2+-6xu_n+6x^2)L_n(e_2)(x)+4x^3u_nL_n(e_1)(x)-3x^4L_n(e_0)(x)+u_n^4-2x^2u_n^2,\end{aligned}$$

so the desired result is achieved. □

3. Rate Of Convergence

In this part, setting $f\in\mathbb{R}^+$, approximation result is given for $\widetilde{K_n}$ operator. In [10] and [11], proof of Korovkin theorems are given.

Let $\mu(x)=1+x^2$ be a weight function and K_f be a positive constant depending of f, we define

$$B_\mu\left(\mathbb{R}^+\right)=\left\{f:\mathbb{R}^+\to\mathbb{R}:|f(x)|\leq K_f\ \mu(x)\right\}$$

and

$$C_\mu\left(\mathbb{R}^+\right)=C\left(\mathbb{R}^+\right)\cap B_\mu\left(\mathbb{R}^+\right).$$

Considering the space of functions

$$C_\mu^k\left(\mathbb{R}^+\right)=\left\{f\in C_\mu\left(\mathbb{R}^+\right):\lim_{x\to\infty}\frac{f(x)}{\mu(x)}=K_f<\infty\right\}.$$

Obviously $C_\mu^k\left(\mathbb{R}^+\right)\subset C_\mu\left(\mathbb{R}^+\right)\subset B_\mu\left(\mathbb{R}^+\right)$. Here the norm is defined as

$$\|f\|_\mu=\sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{\mu(x)}.$$

If $f\in C_\mu^k\left(\mathbb{R}^+\right)$, then $\|L_n(f)\|_\mu\leq\|f\|_\mu$ .These results and Korovkin type theorems can be seen in [12, 10, 11].
Let $C^k\left(\mathbb{R}^+\right)$ be the subspace of all the functions $f\in C\left(\mathbb{R}^+\right)$ such that $\lim\limits_{x\to\infty}\frac{|f(x)|}{1+x^2}=k$, where k is a positive constant. For $f\in C^k\left(\mathbb{R}^+\right)$, weighted modulus of continuity is defined by

$$\Omega(f;\delta)=\sup_{|t-x|\leq\delta,\ x\in\mathbb{R}^+}\frac{|f(t)-f(x)|}{(1+x^2)(1+(t-x)^2)}. \tag{3.1}$$

Utilizing 3.1, we give quantitative type theorem.

Theorem 3.1. *If $f\in C_\mu^k\left(\mathbb{R}^+\right)$, then we have*

$$\left|\widetilde{K_n}(f;x)-f(x)\right|\leq 32(1+x^2)\Omega(f;\delta).$$

Proof. From the property of (3.1), we can write

$$\Omega(f;\lambda\delta)\leq 2(1+\lambda)(1+\delta^2)\Omega(f;\delta)$$

for positive λ (see in [13]). By the definition of $\Omega(f;\delta)$ for $f\in C_\mu^k\left(\mathbb{R}^+\right)$ and $x,\,t\in\mathbb{R}^+$ and $\delta>0$, the following inequality is satisfied:

$$|f(t)-f(x)|\leq 16\left(1+x^2\right)\Omega(f;\delta)\left(1+\frac{|t-x|^4}{\delta^4}\right) \tag{3.2}$$

and by using Lemma 1 and (3.2), we have

$$\left|\widetilde{K_n}(f;x)-f(x)\right|\leq\ f(x)\left|1-\widetilde{K_n}(1;x)\right|+\widetilde{K_n}(|f(t)-f(x)|\,;x).$$

Now applying (3.1) to $\widetilde{K_n}$,

$$\begin{aligned}\left|\widetilde{K_n}(f;x)-f(x)\right| &\leq \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\int_{x_{n,k}}^{x_{n,k+1}}|f(t)-f(x)|\,dt\\ &\leq 16\left(1+x^2\right)\Omega(f;\delta)\left(1+\frac{\zeta_{n,4}(x)}{\delta^4}\right),\end{aligned}$$

choosing $\delta=\sqrt[4]{\zeta_{n,4}(x)}$, it follows

$$\left|\widetilde{K_n}(f;x)-f(x)\right|\leq 32\left(1+x^2\right)\Omega\left(f;\sqrt[4]{\zeta_{n,4}(x)}\right),$$

so we obtain desired result. □

Let us denote by $\omega(f;\delta)$, the classical modulus of continuity defined as

$$\omega(f;\delta)=\sup_{|x-t|\leq\delta,x,t\in\mathbb{R}^+}|f(x)-f(t)|. \tag{3.3}$$

Theorem 3.2. *Let $f^{''} \in C(\mathbb{R})$ and $\omega\left(f^{''};\delta\right)$ is the modulus of contiuity of $f^{''}$ such as finite for $\delta > 0$. We have*

$$\left|\frac{1}{\zeta_{n,2}(x)}\left[\left(\widetilde{K_n}f\right)(x)-f(x)\right]-\frac{1}{2}f^{''}(x)\right| \leq \omega\left(f^{''};\frac{\sqrt{\zeta_{n,4}(x)}}{\sqrt{\zeta_{n,2}(x)}}\right).$$

Proof. By using the Taylor expansion at the fixed point x and (3.3) for $\xi \in [x,t]$, we obtain

$$\begin{aligned}
|h(t,x)| &= \left|f(t)-f(x)-\frac{f^{'}(x)}{1!}(t-x)-\frac{f^{''}(x)}{2!}(t-x)^2\right| \\
&= \frac{(t-x)^2}{2!}\left|f^{''}(\xi)-f^{''}(x)\right| \leq \frac{(t-x)^2}{2!}\omega\left(f^{''};|\xi-x|\right) \\
&\leq \frac{(t-x)^2}{2!}\omega\left(f^{''};|t-x|\right) \leq \frac{(t-x)^2}{2!}\left(1+\frac{|t-x|}{\delta}\right)\omega(f^{''};\delta) \\
&= \frac{1}{2}\left((t-x)^2+\frac{|t-x|^3}{\delta}\right)\omega(f^{''};\delta).
\end{aligned}$$

Now applying it to $\widetilde{K_n}$,we have

$$\begin{aligned}
\left|\left(\widetilde{K_n}h(\cdot,x)\right)(x)\right| &= \left|\left(\widetilde{K_n}f\right)(x)-f(x)-f^{'}(x)\ \zeta_{n,1}(x)-\frac{f^{''}(x)}{2}\zeta_{n,2}(x)\right| \\
&= \left|\left(\widetilde{K_n}f\right)(x)-f(x)-\frac{f^{''}(x)}{2}\ \zeta_{n,2}(x)\right| \leq (\widetilde{K_n}\,|h(f;\cdot,x)|)(x) \\
&\leq \frac{1}{2}\cdot\omega(f^{''};\delta)\left(\zeta_{n,2}(x)+\frac{(\widetilde{K_n}\,|e_1-x|^3)(x)}{\delta}\right) \\
&= \frac{\zeta_{n,2}(x)}{2}\cdot\omega(f^{''};\delta)\left(1+\frac{1}{\delta}\cdot\frac{(\widetilde{K_n}\,|e_1-x|^3)(x)}{\zeta_{n,2}(x)}\right).
\end{aligned}$$

If we choose

$$\delta = (\widetilde{K_n}\,|e_1-x|^3)(x)/\zeta_{n,2}(x)$$

and by using

$$(\widetilde{K_n}\,|e_1-x|^3)(x) \leq \sqrt{\zeta_{n,4}(x)}\cdot\sqrt{\zeta_{n,2}(x)},$$

inequality, we can write

$$\left|\left(\widetilde{K_n}f\right)(x)-f(x)-\frac{f^{''}(x)}{2}\sqrt{\zeta_{n,2}(x)}\right| \leq \sqrt{\zeta_{n,2}(x)}\,\omega\left(f^{''};\frac{\sqrt{\zeta_{n,4}(x)}}{\sqrt{\zeta_{n,2}(x)}}\right).$$

Thus we obtain

$$\left|\frac{1}{\sqrt{\zeta_{n,2}(x)}}\left(\widetilde{K_n}f\right)(x)-f(x)-\frac{1}{2}f^{''}(x)\right| \leq \omega\left(f^{''};\frac{\sqrt{\zeta_{n,4}(x)}}{\sqrt{\zeta_{n,2}(x)}}\right).$$

□

4. Conclusion

In this study, we showed that when the moments of an operator are known, some approximation theorems can be given for the Kantorovich type of the same operator using these moments.

Declarations

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions

Author's Contributions: Conceptualization, D.A.A.; methodology D.A.A.; validation, G.U.Y. investigation, D.A.A. and G.U.Y.; resources, D.A.A.; data curation, D.A.A.; writing—original draft preparation, D.A.A.; writing—review and editing, D.A.A. and G.U.Y.; supervision, G.U.Y. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: D. Aydın Arı and G. Uğur Yılmaz, *A note on Kantorovich type operators which preserve affine functions*, Fundam. J. Math. Appl., **7**(1) (2024), 53-58. DOI 10.33401/fujma.1424382