Volume Special Issue, Issue 1, Page 37-42, 2024 https://doi.org/10.46810/tdfd.1424728 Research Article

BİNGÖL UNIVERSITY

Türk Doğa ve Fen Dergisi
Turkish Journal of Nature and Science
www.dergipark.gov.tr/tdfd

Comments on Parallel Curves in 3-Dimensional Lie Group G

Ali ÇAKMAK[1*], Pelin Ezgi EPİK[2]

[1] Ağrı İbrahim Çeçen University, Faculty of Sciences and Arts, Department of Mathematics, Ağrı, Türkiye
[2] Bitlis Eren University, Faculty of Sciences and Arts, Department of Mathematics, Bitlis, Türkiye
Ali ÇAKMAK ORCID No: 0000-0002-2783-9311
Pelin Ezgi EPİK ORCID No: 0009-0005-4621-0120

Corresponding author: acakmak@agri.edu.tr

(Received: 23.01.2024, Accepted: 21.03.2024, Online Publication: 01.10.2024)

Keywords
Lie group,
Parallel curve,
Some special curves

Abstract: In this study, firstly, basic concepts in 3-dimensional Euclidean space and basic information about curves are given and some special curves are examined. Then, basic information about Lie algebra and Lie group basic concepts and curves are given and special curves such as helix, involute-evolute, Bertrand, Mannheim, Smarandache, are defined. Secondly, inspired by these special curves examined in the Lie group, the definitions of the parallel curve in the vector direction, the parallel curve in the direction of the vector B and the parallel curve in the direction of the linear combination of the vectors B and N of a curve according to the Frenet frame are given and characterized. Some theorems and results are obtained by finding the Frenet apparatus of these characterized curves. Finally, the findings are examined in more specific circumstances and new results are found.

3-Boyutlu Lie Grup G de Paralel Eğriler Üzerine Yorumlar

Anahtar Kelimeler
Lie grup,
Paralel eğri,
Bazı özel eğriler

Öz: Bu çalışmada ilk olarak, 3-boyutlu Öklid uzayındaki temel kavramlar ve eğriler ile ilgili temel bilgiler verilmiş ve bazı özel eğriler incelenmiştir. Sonrasında Lie cebiri ve Lie gruplarındaki temel kavramlar ve eğriler ile ilgili temel bilgiler verilmiş helis, involüt-evolüt, Bertrand, Mannheim, Smarandache gibi özel eğrilere ait temel tanım ve teoremler verilmiştir. İkinci olarak, Lie gruplarında incelenen bu özel eğrilerden esinlenerek Lie gruplarında Frenet çatısına göre bir eğrinin N ve B vektörlerinin lineer birleşimi olan vektör yönündeki paralel eğri ve B vektörü yönündeki paralel eğri tanımları verilip karakterize edilmiştir. Karakterize edilen bu eğrilere ait Frenet elemanları bulunarak bazı teoremler ve sonuçlar elde edilmiştir. Son olarak, elde edilen bulgular daha özel hallerde incelenmiş ve yeni sonuçlar bulunmuştur.

1. INTRODUCTION

One of the predominant topics in differential geometry is the theory of curves. In studies related to the theory of curves, researchers often explore various curves such as general helices, slant helices, Salkowski curves, Bertrand curves, and more. While examining these curves, the relations between the curvatures and the Frenet apparatus are used. Consider the curve $\alpha: I \subseteq R \to E^3$ with arc-length parameterized s in three-dimensional Euclidean space. Let $\{T, N, B\}$ be the Frenet vectors of the curve α. The Frenet formulas are given by:

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s), \qquad (1) \\ B'(s) &= -\tau(s)N(s), \end{aligned}$$

where $\kappa > 0$, $\tau \neq 0$ are the curvature and torsion of the curve α, and can be calculated by following:

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3},$$

$$\tau = \frac{< \alpha' \wedge \alpha'', \alpha''' >}{\|\alpha' \wedge \alpha''\|^2},$$

where α', α'', α''' are the first, second, and third-order derivations of the curve α, respectively [1].

Offset or parallel curves are defined as curves with points at a constant distance along the normal direction from a given curve in plane [2]. In the literature, the definition of a parallel curve in 3-dimensional Euclidean space appears as different definitions. The first of these is as follows:

Let a curve $\gamma = \gamma(s^*)$ with unit-speed be given, where $\tau \neq 0$. Let the β curve be the parallel curve at a distance

r from the curve γ. Let the Frenet frames of the curves γ and β be given as $\{T, N, B\}$, $\{T_\beta, N_\beta, B_\beta\}$ respectively. So, we can write the following equations:

$$\langle \beta(s) - \gamma, \beta(s) - \gamma \rangle = r^2, \tag{2}$$

$$\langle T_\beta(s), \beta(s) - \gamma \rangle = 0, \tag{3}$$

$$\langle (T_\beta{}'(s), \beta(s) - \gamma) \rangle + \langle (T_\beta(s), T_\beta(s)) \rangle = 0. \tag{4}$$

The Frenet formulas can be obtained by using the equations (1) at the point β depending on the parameter s for a curve $\beta(s)$ with unit speed where $\tau > 0$. Thus, from the equations (2), (3) and (4) we have

$$\beta(s) - \gamma = m_2 N_\beta + m_3 B_\beta, \tag{5}$$

where m_2, m_3 are the appropriate coefficients and $\beta' = T_\beta$ [3]. Another definition is as follows:

The parallel curve to a unit-speed curve $\alpha(s)$ is given by

$$\bar{\alpha} = \alpha(s) + rB(s). \tag{6}$$

Here, $r \neq 0$ is a real constant, $s = s(\bar{s})$ is the arc-length of $\alpha(s)$ and $\bar{s}$ is the arc-length of the parallel curve $\bar{\alpha}$. B is the binormal vector of the curve $\alpha(s)$ [4].

In mathematics, a Lie group is a group that is also a differentiable manifold, characterized by the smoothness of group operations. Lie groups take their name from the Norwegian mathematician Sophus Lie, who laid the foundations of the theory of continuous transformation groups. Essentially, a Lie group is a continuous group, with its elements defined by several real parameters. Many studies have been done on the differential geometry of curves in Lie Groups [5, 6, 7, 8, 9, 10, 11].

In the present study, we discuss using the definitions of parallel curves in the literature in 3-dimensional Lie groups. It also contains theorems and results that give the relations between parallel curve pairs and special curve pairs.

2. MATERIAL AND METHOD

2.1. Fundamental Concepts in Lie Groups

Definition 2.1.1. Let G be a group and M^* is a differentiable manifold,

L_1: Every element of G coincicides with the points of M^*,

L_2: $M^* \times M^* \to M^*, (p, q) \to pq^{-1}$ is differentiable.

In such a way that the axioms are satisfied, the fundamental manifold of the Lie group is obtained as M^*, the fundamental group of the Lie group as G, and the pair (M^*, G) is obtained as the Lie group [12].

Definition 2.1.2. Let G be a Lie group and $\langle , \rangle$ be an invariant metric on G. If the Lie algebra of the Lie group G is given by g and the unit element of the Lie group G is given by e, the Lie algebra is g and the Lie algebra structure $T_g(e)$ is isomorphic. Let $\langle , \rangle$ be an invariant metric on G, and let G be the Levi-Civita connection of the Lie group $\bar{V}$. Here, $\forall K, L, M \in g$ is given by:

$\langle K, [L, M] \rangle = \langle [K, L], M \rangle$

and

$$\bar{V}_K L = \frac{1}{2}[K, L].$$

Assume that $\alpha: I \subset R \to G$ is a curve parameterized by arc-length, and $\{V_1, V_2, \dots, V_n\}$ is an orthonormal base of g. In this case, two vector fields along the curve can be written as $W = \sum_{i=1}^n w_i X_i$ and $Z = \sum_{i=1}^n z_i X_i$, where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Lie product of two vector fields W and Z:

$[W, Z] = \sum_{i=1}^n w_i z_j [X_i, X_j]$,

and the covariant derivative of the vector field W along the curve α is

$$D_{\alpha'} W = \dot{W} + \tfrac{1}{2}[T, W]. \tag{7}$$

Here $T = \alpha', \dot{W} = \sum_{i=1}^n \dot{w}_i X_i = \sum_{i=1}^n \frac{dw_i}{dt} X_i$. If the left invariant vector field restricted to α is W, $\dot{W} = 0$ [9].

Proposition 2.1.1. Assume that $\alpha(s)$ is a curve in the Lie group G, with arc-length parameter s and (T, N, B, κ, τ) is the Frenet apparatus of $\alpha(s)$. In this case, we write

$$\begin{cases} [T, N] = \langle [T, N], B \rangle B = 2\tau_G B \\ [T, B] = \langle [T, B], N \rangle N = -2\tau_G N. \end{cases} \tag{8}$$

In this case, considering that $\alpha(s)$ is a curve in G and s is the arc-length parameter of $\alpha(s)$, from the equations (7) and (8), Frenet formulae are found as follows:

$$\begin{pmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dB}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau - \tau_G \\ 0 & -(\tau - \tau_G) & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Here (T, N, B, κ, τ) is Frenet apparatus of $\alpha(s)$ in G, $\tau_G = \frac{1}{2}\langle [T, N], B \rangle$ [8].

Definition 2.1.3. Let $\alpha: I \subseteq \mathbb{R} \to G$ be a curve with arc-length parameters in the 3-dimensional Lie group G and Frenet apparatus are $\{T, N, B, \kappa, \tau\}$. The function h expressed by the equation

$$h = \frac{\tau - \tau_G}{\kappa} \tag{9}$$

is the harmonic curvature of the curve α [8].

2.2. Special Curves in 3-Dimensional Lie Groups

Definition 2.2.1. Let $\alpha: I \subseteq \mathbb{R} \to G$ be considered as a curve with arc-length parameterized in the three-dimensional Lie group and the unit left invariant vector field $X \in G$. If X and the curve α make a constant angle at all points of the curve α, that is the unit tangent vector field T at point $\alpha(s)$ on curve α is a left invariant vector field X and $\vartheta \neq \frac{\pi}{2}$ if it makes a constant angle, namely

$\langle T(s), X \rangle = \cos\vartheta, s \in I.$

The curve α is the general helix in the Lie group. Here, $X \in g$ and unit, T is tangent vector field of the curve α, the angle $\vartheta \neq \frac{\pi}{2}$ is the fixed angle between X and T [6].

Theorem 2.2.1. Let $\alpha: I \subseteq \mathbb{R} \to G$ be considered as a curve with arc-length parameterized in G. The curve α is general helix if and only if $\tau = c\kappa + \tau_G, c = constant$ such that $\{T, N, B, \kappa, \tau\}$ is Frenet apparatus of α [6].

Definition 2.2.2. Let G be a three-dimensional Lie group with a bi-invariant metric and the curves α, β be two curves in G. If the principal normal vector field of the curve α and the binormal vector field of the curve β are linearly dependent at corresponding points of α and β, α is called a Mannheim curve, the curve β is called the Mannheim curve corresponding to α, and the pair $\{\alpha,\beta\}$ is referred to as a Mannheim curve pair [13].

Definition 2.2.3. Let's consider the unit speed curves $\gamma: I\subset\mathbb{R}\to G$ and $\beta:\bar{I}\subset\mathbb{R}\to G$ in G with the left-invariant metric. Let s and $\bar{s}$ be the arc-length parameters and $\{T,N,B,k_0,\varkappa_0,\alpha\}$, $\{\bar{T},\bar{N},\bar{B},\bar{k}_0,\bar{\varkappa}_0,\bar{\alpha}\}$ be Frenet apparatus of γ and β respectively. If the tangent vectors at corresponding points of the γ and β curves are perpendicular to each other, that is $\langle\bar{T},T\rangle=0$, then the curve β is called the involute of γ and the curve γ is called the evolute of β [14].

Definition 2.2.4. Let's consider the Lie group G defined by a bi-invariant metric. If, at corresponding points of α and β, the principal normal vector field of α and the principal normal vector field of β are linearly dependent, then α is called a Bertrand curve. In this case, β is the Bertrand curve corresponding to α, and the pair $\{\alpha,\beta\}$ is referred to as a Bertrand curve pair [13].

Definition 2.2.5. Let $\alpha: I\subseteq\mathbb{R}\to G$ be a unit speed curve in three-dimensional Lie group and $\{T,N,B,\kappa,\tau\}$ be the Frenet apparatus of the curve α. In this case, the TN -Smarandache curve is $\psi(s_\psi)=\frac{1}{\sqrt{2}}\big(T(s)+N(s)\big)$ [15].

3. RESULTS

In this section, starting with the consideration of parallel curve definitions in 3-dimensional Euclidean space, we will first discuss the definition of a parallel curve in G. Specifically, we will delve into the concept of a parallel curve in the direction of the vectors N and B in Lie group, and then a parallel curve in the direction of the binormal vector B of a curve will be given.

Afterwards, Frenet elements $\{T,N,B,\kappa,\tau\}$ will be obtained for the parallel curves of all two cases. In addition, some theorems and results characterizing these curves will be obtained.

3.1. Parallel Curve in the Direction of *NB* in 3-Dimensional Lie Groups

Definition 3.1.1. Let $\alpha: I\subseteq\mathbb{R}\to G$ be considered as a unit speed curve in the three-dimensional Lie group G and $\{T,N,B,\kappa,\tau\}$ as the Frenet apparatus of the curve α. The parallel curve of the curve α in G is

$$\rho(s_\rho)=\alpha(s)+r_1N(s)+r_2B(s). \qquad (10)$$

Here, $r_1\neq 0, r_2\neq 0$ are real constants.

The Frenet frame $\{T_\rho,N_\rho,B_\rho\}$ of the parallel curve $\rho(s_\rho)$ are

$$T_\rho(s_\rho)=\frac{(1-r_1\kappa)T(s)+r_2\kappa hN(s)-r_1\kappa hB(s)}{\sqrt{(1-r_1\kappa)^2+(r_2\kappa h)^2+(r_1\kappa h)^2}}, \qquad (11)$$

for

$$P_\rho=\begin{bmatrix}((1-r_1\kappa)^2+(r_2\kappa h)^2\\ +(r_1\kappa h)^2)(-r_1\kappa'+r_2\kappa^2h)\\ -(1-r_1\kappa)(-r_1\kappa'+r_1^2\kappa\kappa')\\ -(1-r_1\kappa)(r_1^2+r_2^2)(\kappa\kappa'h^2+\kappa^2hh')\end{bmatrix},$$

$$R_\rho=\begin{bmatrix}((1-r_1\kappa)^2+(r_2\kappa h)^2\\ +(r_1\kappa h)^2)\\ (\kappa-r_1\kappa^2+r_2\kappa'h+r_2\kappa h'+r_1\kappa^2h^2)\\ -(r_2\kappa h)(-r_1\kappa'+r_2\kappa\kappa')\\ -(r_2\kappa h)(r_1^2+r_2^2)(\kappa\kappa'h^2+\kappa^2hh')\end{bmatrix},$$

$$S_\rho=\begin{bmatrix}((1-r_1\kappa)^2+(r_2\kappa h)^2\\ +(r_1\kappa h)^2)(-r_2\kappa^2h^2-r_1\kappa'h-r_1\kappa h')\\ +(r_1\kappa h)(-r_1\kappa'+r_1^2\kappa\kappa')\\ +r_1\kappa h(r_1^2+r_2^2)(\kappa\kappa'h^2+\kappa^2hh')\end{bmatrix},$$

$$N_\rho(s_\rho)=\frac{1}{\sqrt{{P_\rho}^2+R_\rho^2+S_\rho^2}}\Big(P_\rho T(s)+R_\rho N(s)+S_\rho B(s)\Big) \qquad (12)$$

and for $p=\sqrt{(1-r_1\kappa)^2+(r_2\kappa h)^2+(r_1\kappa h)^2}$ and $q=\sqrt{P_\rho^2+R_\rho^2+S_\rho^2}$,

$$B_\rho(s_\rho)=\frac{1}{pq}\begin{pmatrix}(r_2\kappa hS_\rho+r_1\kappa hR_\rho)T(s)\\ +(-r_1\kappa hP_\rho-(1-r_1\kappa)S_\rho)N(s)\\ +\Big((1-r_1\kappa)R_\rho-r_2\kappa hP_\rho\Big)B(s)\end{pmatrix}. \qquad (13)$$

The curvature and torsion of the curve $\rho(s_\rho)$ are given by

$$\kappa_\rho=\|\dot{T}\|=\frac{\sqrt{{P_\rho}^2+R_\rho^2+S_\rho^2}}{((1-r_1\kappa)^2+(r_2\kappa h)^2+(r_1\kappa h)^2)^2} \qquad (14)$$

and

$$\tau_\rho=\frac{\begin{matrix}(r_1\kappa^2h-r_1^2\kappa^3h+(r_1^2+r_2^2)\kappa^3h^3)\ell\\ +\big(2r_1r_2\kappa^3h^2+r_1\kappa'h+r_1\kappa h'-r_2\kappa^2h^2-r_1^2\kappa^2h'\big)m\\ +(\kappa-2r_1\kappa^2+r_2\kappa'h+r_2\kappa h'+r_1\kappa^2h^2+r_1^2\kappa^3\\ -r_1r_2\kappa^2h'-r_1^2\kappa^3h^2+r_2^2\kappa^3h^2)n\end{matrix}}{\begin{matrix}(r_1\kappa^2h-r_1^2\kappa^3h+(r_1^2+r_2^2)\kappa^3h^3)^2\\ +(2r_1r_2\kappa^3h^2+r_1\kappa'h+r_1\kappa h'-r_2\kappa^2h^2-r_1^2\kappa^2h')^2\\ +(\kappa-2r_1\kappa^2+r_2\kappa'h+r_2\kappa h'+r_1\kappa^2h^2+r_1^2\kappa^3\\ -r_1r_2\kappa^2h'-r_1^2\kappa^3h^2+r_2^2\kappa^3h^2)^2\end{matrix}}, \qquad (15)$$

where

$$\ell=\begin{pmatrix}-r_1\kappa''-3r_2\kappa\kappa'h-2r_2\kappa^2h'\\ +r_1\kappa^3-r_1\kappa^3h^2-\kappa^2\end{pmatrix},$$

$$m=\begin{pmatrix}\kappa'-3r_1\kappa\kappa'+r_2\kappa''h+2r_2\kappa'h'+r_2\kappa h''\\ +3r_1\kappa^2hh'+3r_1\kappa\kappa'h^2+r_2\kappa^3h-r_1\kappa^3h^3\end{pmatrix},$$

$$n=\begin{pmatrix}\kappa^2h-r_1\kappa^3h+3r_2\kappa\kappa'h^2+3r_2\kappa^2hh'\\ +r_1\kappa^3h^3-2r_1\kappa'h'-r_1\kappa h''\end{pmatrix}.$$

Theorem 3.1.1. Let the curves $\alpha,\rho: I\subseteq\mathbb{R}\to G$ be considered as Frenet vector fields $\{T,N,B\}$ and $\{T_\rho,N_\rho,B_\rho\}$ respectively. The parallel curve pair (α,ρ) is involute-evolute curve pair if and only if $\kappa=\frac{1}{r_1}$.

Proof. Let the parallel curve pair (α,ρ) be involute-evolute curve pair. In this case, $\langle T,T_\rho\rangle=0$. From the equation (11), we get

$$\langle T,T_\rho\rangle=\frac{(1-r_1\kappa)}{\sqrt{(1-r_1\kappa)^2+(r_2\kappa h)^2+(r_1\kappa h)^2}}T(s)=0$$

which is achieved by $\kappa=\frac{1}{r_1}$

On the contrary, let's $\kappa=\frac{1}{r_1}$. If both sides of the equation (11) are made scalar product by T and $\kappa=\frac{1}{r_1}$ is written, we have $\langle T,T_\rho\rangle=0$.

Hence, the proof is completed.

Theorem 3.1.2. Let the curve pair (α,ρ) be a parallel curve pair with the Frenet vectors $\{T,N,B\}$ and $\{T_\rho,N_\rho,B_\rho\}$ respectively. If (α,ρ) is a Bertrand curve pair, $-r_1\kappa hP_\rho-(1-r_1\kappa)S_\rho=0$.

Proof. Let the curves α and ρ be a Bertrand curve pair, where the principal normal vector field of the curve α is N and the principal normal vector field of the curve ρ is N_ρ. In this case, N and N_ρ are linearly dependent. Now, if we multiply both sides of equation (13) by N_ρ and consider that N and N_ρ are linearly dependent, we obtain

$$\frac{\left(-r_1\kappa hP_\rho-(1-r_1\kappa)S_\rho\right)}{pq}=0.$$

If the necessary adjustments are made in the last equation, we have

$$\left(-r_1\kappa hP_\rho-(1-r_1\kappa)S_\rho\right)=$$

$$\left(-r_1\kappa h\begin{pmatrix}\begin{pmatrix}(1-r_1\kappa)^2+(r_2\kappa h)^2\\+(r_1\kappa h)^2\end{pmatrix}(-r_1\kappa'+r_2\kappa^2h)\\-(1-r_1\kappa)(-r_1\kappa'+r_1^2\kappa\kappa')\\-(1-r_1\kappa)(r_1^2+r_2^2)(\kappa\kappa'h^2+\kappa^2hh')\end{pmatrix}\right)$$

$$-\left(\begin{pmatrix}(1-r_1\kappa)\\\begin{pmatrix}((1-r_1\kappa)^2+(r_2\kappa h)^2\\+(r_1\kappa h)^2)(-r_2\kappa^2h^2-r_1\kappa'h-r_1\kappa h')\\+(r_1\kappa h)(-r_1\kappa'+r_1^2\kappa\kappa')\\+r_1\kappa h(r_1^2+r_2^2)(\kappa\kappa'h^2+\kappa^2hh')\end{pmatrix}\end{pmatrix}\right)=0.$$

Specifically, when $\kappa=\frac{1}{2r_1}$ and $h=constant$, the following equation is satisfied:

$$\left(-r_1\kappa hP_\rho-(1-r_1\kappa)S_\rho\right)=r_1(\kappa'h+\kappa h'-r_1\kappa^2h')$$
$$+r_2(\kappa^2h^2-2r_1\kappa^3h^2)=0.$$

Corollary 3.1.1. It can be easily seen from Theorem 3.1.2 that if the curvature of the curve $\alpha(s)$ is κ and its torsion is τ and $\kappa=\frac{1}{2r_1}$ and $h=constant$, the curve $\alpha(s)$ is also a helix curve.

3.2. Parallel Curve in the Direction of *B* in 3-Dimensional Lie Groups

Definition 3.2.1. Let $\alpha: I\subseteq\mathbb{R}\to G$ be a unit-speed curve in three-dimensional Lie group G and $\{T,N,B,\kappa,\tau\}$ be the Frenet apparatus of the curve α. The parallel curve of the curve α in the direction of B is defined by

$$\wp(s_\wp)=\alpha(s)+rB(s) \tag{16}$$

where $r\neq 0$ is a real constant.

The Frenet frame $\{T_\wp,N_\wp,B_\wp\}$ of the parallel curve $\wp(s_\wp)$ are

$$T_\wp(s_\wp)=\frac{(T(s)-r\kappa hN(s))}{\sqrt{(1+(r\kappa h)^2}}, \tag{17}$$

for

$$P_\wp=[r\kappa^2h(1+(r\kappa h)^2)-r\kappa'h-r\kappa h')],$$
$$R_\wp=[\kappa(1+(r\kappa h)^2)-r\kappa'h-r\kappa h')], \tag{18}$$
$$S_\wp=[-r\kappa^2h^2(1+(r\kappa h)^2)],$$

$$N_\wp(s_\wp)=\frac{\sqrt{(1+(r\kappa h)^2)}}{\sqrt{P_\wp^2+R_\wp^2+S_\wp^2}}\begin{pmatrix}P_\wp T(s)\\+R_\wp N(s)+S_\wp B(s)\end{pmatrix} \tag{19}$$

and

$$B_\wp(s_\wp)=\frac{1}{p}\begin{pmatrix}(-r\kappa hS_\wp)T(s)\\+(-S_\wp)N(s)+(R_\wp+r\kappa hP_\wp)B(s)\end{pmatrix}, \tag{20}$$

where $p=\sqrt{P_\wp^2+R_\wp^2+S_\wp^2}$. The curvature and torsion of $\rho(s_\wp)$ are given by

$$\kappa_\wp=\|\dot{T}\|=\frac{\sqrt{P_\wp{}^2+R_\wp^2+S_\wp^2}}{(1+(r\kappa h)^2)^2} \tag{21}$$

and

$$\tau_\wp=\frac{(r^2\kappa^3h^3)\ell+(r\kappa^2h^2)m+(\kappa-r\kappa'h-r\kappa h'+r^2\kappa^3h^2)n}{(r^2\kappa^3h^3)^2+(r\kappa^2h^2)^2+(\kappa-r\kappa'h-r\kappa h'+r^2\kappa^3h^2)^2}, \tag{22}$$

where

$\ell=(-\kappa^2+3r\kappa\kappa'h+2r\kappa^2h'),$
$m=(r\kappa^3h+\kappa'-r\kappa''h-2r\kappa'h'-r\kappa h''+r\kappa^3h^3),$
$n=(\kappa^2h-3r\kappa\kappa'h^2-3r\kappa^2hh').$

Corollary 3.2.1. Let the curve pair $(\alpha,\wp)$ be a parallel curve pair with the Frenet vectors $\{T,N,B\}$ and $\{T_\wp,N_\wp,B_\wp\}$ respectively, where $\wp$ is the parallel curve to α in the direction of B. The parallel curve pair $(\alpha,\wp)$ is not a Bertrand curve pair under no circumstances.

Proof. Suppose that $(\alpha,\wp)$ is a Bertrand curve pair. In this case, N and $N_\wp$ are linearly dependent. Considering the equation (20), we multiply both sides of equation (20) by $N_\wp$. Since $S_\wp=r\kappa^2h^2(1+(r\kappa h)^2)\neq 0$, $(\alpha,\wp)$ is not a Bertrand curve pair.

Corollary 3.2.2. Let the curve pair $(\alpha,\wp)$ be a parallel curve pair with the Frenet vectors $\{T,N,B\}$ and $\{T_\wp,N_\wp,B_\wp\}$ respectively, where $\wp$ is the parallel curve to α in the direction of B. The parallel curve pair $(\alpha,\wp)$ is not an involute-evolute curve pair under no circumstances.

Proof. Suppose that $(\alpha,\wp)$ is an involute-evolute curve pair. In this case, $T\perp T_\wp$. Considering the equation (17), we multiply both sides of equation (17) by T. Since $\langle T,T_\wp\rangle=\frac{1}{\sqrt{1+(r\kappa h)^2}}$, $T\not\perp T_\wp$. Hence, $(\alpha,\wp)$ is not an involute-evolute curve pair.

Theorem 3.2.1. Let the curve pair $(\alpha,\wp)$ be a parallel curve pair with the Frenet frames $\{T,N,B\}$ and $\{T_\wp,N_\wp,B_\wp\}$ respectively. If the parallel curve pair $(\alpha,\wp)$ is a Mannheim curve pair, $R_\wp=0$.

Proof. Suppose that the parallel curve pair $(\alpha,\wp)$ is a Mannheim curve pair. In this case, N and $B_\wp$ are linearly dependent. Considering the equation (19), if we multiply both sides of equation (19) by $B_\wp$, we obtain $\frac{\sqrt{(1+(r\kappa h)^2)}}{\sqrt{P_\wp^2+R_\wp^2+S_\wp^2}}R_\wp=0$. Since $\sqrt{(1+(r\kappa h)^2)}\neq 0$, we have

$R_\wp=(1+(r\kappa h)^2)\kappa-r\kappa'h-r\kappa h'=0.$

Hence, proof is completed.

In the special case, let $\tau-\tau_G=-\frac{1}{r}$. In this case, we have

$h = \frac{\tau - \tau_G}{\kappa} \Rightarrow h = -\frac{1}{r\kappa}$. (23)

Considering the equation (23), we get the equations (18) the following as:

$$P_\wp = -2\kappa, R_\wp = 2\kappa, S_\wp = 2\kappa h = -\frac{2}{r}$$

These equations are substituted in the equation (21), we obtain

$$\kappa_\wp = \frac{\sqrt{1+2r^2\kappa^2}}{2r} \quad . \quad (24)$$

The equation (24) gives us the relationship between curvatures of the curve pair $(\alpha, \wp)$. In addition to equation (24), it can be expressed in the following theorem and result.

Theorem 3.2.2. Let the curve pair $(\alpha, \wp)$ be a parallel curve pair with the Frenet frames $\{T, N, B\}$ and $\{T_\wp, N_\wp, B_\wp\}$ respectively. If $\tau - \tau_G = -\frac{1}{r}$, the equation (17) is TN -Smarandache curve.

Proof. Substituting the equation $\tau - \tau_G = -\frac{1}{r}$ into equation (9), we get

$$h = \frac{\tau - \tau_G}{\kappa} \Rightarrow h = -\frac{1}{r\kappa}.$$

If we substitute $h = -\frac{1}{r\kappa}$ in equation (17) and we make the necessary simplifications, we have

$$T_\wp = \frac{T+N}{\sqrt{2}}.$$

Considering Definition 2.2.5. it is seen that the last equation is TN -Smarandache curve.

Corollary 3.2.3. Let consider unit-speed curve α with constant curvature κ and its parallel curve $\wp$ with the curvature $\kappa_\wp$. Since $\tau - \tau_G = -\frac{1}{r}$, the curve α is a circular helix. In this case, $\kappa_\wp = constant$ and the curve $\wp$ is a circular helix in the abelian case.

Proof. Considering equation (23), above the equations l, m, n

$$\ell = -\kappa^2, m = -\left(\frac{r^2\kappa^2+1}{r^2}\right), n = \kappa^2 h \quad (25)$$

is obtained as in (25). By using equations (25) into equation (22) and if we make the necessary adjustments, we have

$$\tau_\wp = \frac{\kappa^3 h - \kappa h(-\kappa^2 - \kappa^2 h^2) + 2\kappa^3 h}{(\kappa h)^2 + (\kappa h)^2 + (2\kappa)^2}$$

Since both κ and h are constants, $\tau_\wp = constant$. Since $\tau_{G_\wp} = 0$ in the abelian case (see [6]), $\frac{\tau_\wp}{\kappa_\wp}$ is constant. So, the curve $\wp$ is obtained as a circular helix.

Finally, the following theorem can be written by taking the special case of $\kappa = \frac{1}{r}$

Theorem 3.2.3. Let the curve pair $(\alpha, \wp)$ be a parallel curve pair with the Frenet frames $\{T, N, B\}$ and $\{T_\wp, N_\wp, B_\wp\}$ respectively. Let $(\alpha, \wp)$ be Mannheim curve pair, for $\kappa = \frac{1}{r}$. In this case, $h = tan\left(\frac{s}{r} + c\right)$.

Proof. Let $\kappa = \frac{1}{r}$ and the parallel curve pair $(\alpha, \wp)$ be a Mannheim curve pair. In this case N and $B_\wp$ are linearly dependent. Then if both sides of equation (19) are multiplied by $B_\wp$

$$R_\wp = (1 + h^2)\frac{1}{r} - h' = 0 \Rightarrow h' - \frac{(1+h^2)}{r} = 0. \quad (26)$$

From the solution of differential equation (26), we have

$$rh' = h^2 + 1$$

$$arctan(h) = \frac{s}{r} + c \Rightarrow h = tan\left(\frac{s}{r} + c\right).$$

This completes the proof.

4. DISCUSSION AND CONCLUSION

In this study, we investigated parallel curves in the 3-dimensional Lie group, based on the definitions of parallel curves in 3-dimensional Euclidean space. We calculated Frenet apparatus of these curves. We provided the theorems and results that reveal the relationships of the obtained parallel curves and special curves.

Finally, we examined the parallel curve pairs by adding some special cases and found interesting theorems and results.

Acknowledgement

This study was presented as an oral presentation at the "6th International Conference on Life and Engineering Sciences (ICOLES 2023)" conference.

REFERENCES

[1] O' Neill B, Elementary Differential Geometry. Academic Press, New York. 1966.

[2] Willson FN. Theoretical and Practical Graphics. The Macmillan Company. 1898.

[3] Keskin Ö, Yüksel N, Karacan MK, İkiz H. Characterization of the Parallel Curve of the Adjoint Curve in E^3. General Mathematics Notes. 2016; 35:9-18.

[4] Aldossary MT, Gazwani MA. Montion of Parallel Curves and Surfaces in Euclidean 3-Space R3. Global J. Adv. Res. Class. Mod. Geo. 2020; 9:43-56.

[5] Bozkurt Z, Gök I, Okuyucu OZ, Ekmekçi FN. Characterizations of rectifying, normal and osculating curves in three-dimensional compact Lie groups. Life Science Journal. 2013; 10: 819-823.

[6] Çiftçi Ü. A generalization of Lancret's theorem. Journal of Geometry and Physics. 2009; 59:1597-1603.

[7] Kızıltuğ S, Önder M. Associated Curves of Frenet Curves in Three-Dimensional Compact Lie Group. Miskolc Mathematical Notes 2015; 16:953-964.

[8] Okuyucu OZ, Gök I, Yaylı Y, Ekmekçi FN. Slant helices in three-dimensional Lie groups. Applied Mathematics and Computation. 2013, 221:672-683.

[9] Crouch P, Silva LF. The dynamic interpolation problem: On Riemannian manifolds, Lie groups, and symmetric spaces. J. Dyn. Control Syst. 1995; 1:177–202.

[10] Gök I, Okuyucu OZ, Yayli Y, Ekmekci N. Bertrand curves in three-dimensional Lie groups. Miskolc Mathematical Notes. 2016, 17:999–1010.

[11] Çakmak A. New Type Direction Curves in 3-Dimensional Compact Lie Group. Symmetry. 2019, 11:387.
[12] Hacısalihoğlu H. Introduction to High Differential Geometry. Fırat University, Faculty of Science Publications, Elâzığ. 2006.
[13] Okuyucu OZ. Special Curves on Some Manifolds. Ph.D. Thesis, Ankara University, Ankara, 2013.
[14] Kızılay E. Some Generalized Curves in Three Dimensional Compact Lie Groups, M.Sc. Thesis, Bilecik Seyh Edebali University, Bilecik, 2023.
[15] Değirmen C. On Curves in Three Dimensional Compact Lie Groups, M.Sc. Thesis, Bilecik Seyh Edebali University, Bilecik, 2017.