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Inequalities For Strongly s-Convex Functions Via Atangana-Baleanu Fractional Integral Operators

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Abstract: It is more convenient to use fractional derivatives and integrals to express and represent rapid changes than to use integer derivatives and integrals. For this reason, fractional analysis has been found worthy of study in many fields. In recent years, fractional derivatives and integrals have been discussed together with inequality theory and the studies have attracted attention. In this article, we discuss new Hermite-Hadamard type approximations for strongly $s-$ convex functions with the help of Atangana-Baleanu fractional integral operators. Additionally, new upper bounds have been obtained using various auxiliary inequalities with the help of twice differentiable strongly convex functions.

Atangana-Baleanu Kesirli İntegral Operatörler Yardımıyla Güçlü s-Konveks Fonksiyonlar İçin Eşitsizlikler

Anahtar Kelimeler
Hermite-Hadamard eşitsizliği,
Güçlü s-konveks fonksiyon,
Atangana-Baleanu kesirli integral

Öz: Hızlı değişimleri ifade etmek ve temsil etmek için kesirli türev ve integraller kullanmak tamsayı mertebeden türev ve integralleri kullanmaktan daha uygun olmaktadır. Bu nedenle kesirli analiz birçok alanda çalışılmaya değer bulunmuştur. Son yıllarda da kesirli türev ve integraller eşitsizlik teorisiyle birlikte ele alınmış ve yapılan çalışmalar dikkat çekmiştir. Biz de bu makalede, Atangana-Baleanu kesirli integral operatörler yardımıyla güçlü $s-$ konveks fonksiyonlar için Hermite-Hadamard tipli yeni tahminleri tartışıyoruz. Ayrıca, iki kez türevlenebilen güçlü konveks fonksiyonlar yardımıyla, çeşitli yardımcı eşitsizlikler kullanılarak yeni üst sınırlar elde edilmiştir.

1. INTRODUCTION

The discovery of mathematical inequalities in the twentieth century, leading to numerous new results and research problems, has contributed to the expansion of mathematics into different areas. Many studies, including various classical and new inequalities as well as many new applications and proof methods, have become the main resources for researchers [1,2]. Convex functions, which have a long history, began to find a place in mathematics as a result of the works of Hermite and Hadamard in the late 19th century. Convex functions increased their recognition with Jensen's systematic studies in 1905-1906, became an independent field of mathematical analysis and continued to develop rapidly. This rapid development can be attributed to the inclusion of applications of convex functions in many areas of mathematical analysis and the close relationship between the theory of convex functions and inequality theory [3,4]. The fact that inequalities and convex functions are not only significant in mathematics but also in other branches of science has made these topics a focused point for researchers, leading to numerous ongoing studies in these areas.

The Hermite-Hadamard inequality plays an important and magnificent role in the literature and is stated as follows [5].

If $Q:\Im\subseteq\Re\to\Re$ is convex in $\Im$ for $\xi_1,\xi_2\in\Im$ and $\xi_1<\xi_2$, then

$$Q\left(\frac{\xi_1+\xi_2}{2}\right)\le\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}Q(x)dx\le\frac{Q(\xi_1)+Q(\xi_2)}{2} \tag{1}$$

This inequality specifies the lower and upper bounds of the integral mean value of a convex function. In the last two decades, many studies have been published by mathematicians regarding its generalizations and extensions.

Fractional calculus is a branch of mathematical analysis that encompasses applications and research involving arbitrary order derivatives and integrals [6,7]. In this respect, the concepts of fractional derivative and fractional integral differ from classical derivative and classical integrals and are also more comprehensive than these concepts.

The main objective of this study is to obtain new Hermite-Hadamard inequalities for Atangana-Baleanu fractional integrals. Twice differentiable strongly $s-$ convex functions are employed to achieve the results, and new inequalities are demonstrated by parameter substitutions. It can be easily observed that the results of this study are a generalization of the existing similar results in the literature.

Definition 1: [8] A function such that $Q:\Im\to\Re$, $\Im\subset\Re$ is called convex function if

$$Q\left(\tau\xi_1+(1-\tau)\xi_2\right)\le\tau Q(\xi_1)+(1-\tau)Q(\xi_2)\qquad(2)$$

for all $\xi_1,\xi_2\in\Im$ and $\tau\in[0,1]$.

In 1966, Polyak introduced the class of strongly convex functions and made a contribution to convex analysis.

Definition 2: [9] A function such that $Q:\Im\to\Re$, $\Im\subset\Re$ is called strongly convex function with modulus $\rho>0$, if

$$\begin{aligned}&Q\left(\tau\xi_1+(1-\tau)\xi_2\right)\\&\le\tau Q(\xi_1)+(1-\tau)Q(\xi_2)-\rho\tau(1-\tau)(\xi_1-\xi_2)^2\end{aligned}\qquad(3)$$

for all $\xi_1,\xi_2\in\Im$ and $\tau\in[0,1]$.

Definition 3: [9] A function such that $Q:\Im\to\Re$, $\Im\subset\Re$ is called strongly $s-$ convex function with modulus $\rho>0$, if

$$\begin{aligned}&Q\left(\tau\xi_1+(1-\tau)\xi_2\right)\\&\le\tau^s Q(\xi_1)+(1-\tau)^s Q(\xi_2)-\rho\tau(1-\tau)(\xi_1-\xi_2)^2\end{aligned}\qquad(4)$$

for all $\xi_1,\xi_2\in\Im$ and $\tau\in[0,1]$.

Fractional analysis has become a new field, laying the foundation for solving certain differential equation problems. Studies involving Riemann-Liouville fractional derivatives and integrals have begun, and it remains a popular field with ongoing research today [10,11,12,13]. Depending on the type of problem, there are multiple definitions to obtain the best solution. Grünwald-Letnikov, Weyl, Marchaud, Hadamard, Erdèlyi-Kober, Riezs, Chen, Caputo, Osler, Khalil, Fabrizio, Atangana, Baleanu have made significant contributions to this process. In this study, we will focus on Atangana-Baleanu fractional integrals.

In 2015, Caputo and Fabrizio introduced a new definition for the fractional derivative, which Caputo defined in 1967, in order to eliminate the singularity of the kernel function by using the exponential function. In 2017, Abdeljawad and Baleanu presented the Caputo-Fabrizio fractional integral associated with this fractional derivative as following.

Definition 4: [14] A function such that $Q\in H^1(0,\xi_2)$ is called left and right side of Caputo-Fabrizio fractional integral, if

$$\left({}^{CF}_{\xi_1}I^{\psi}\right)Q(x)=\frac{1-\psi}{B(\psi)}Q(x)+\frac{\psi}{B(\psi)}\int_{\xi_1}^{x}Q(y)dy\qquad(5)$$

and

$$\left({}^{CF}I^{\psi}_{\xi_2}\right)Q(x)=\frac{1-\psi}{B(\psi)}Q(x)+\frac{\psi}{B(\psi)}\int_{x}^{\xi_2}Q(y)dy\qquad(6)$$

where $\xi_2>\xi_1$, $\psi\in[0,1]$ and $B(\psi)$ is normalization function.

In 2016, Atangana and Baleanu presented a new derivative operator and associated integral operator containing the Mittag-Leffler function in its kernel. The Mittag-Leffler function offers a more reasonable option than a power law in explaining the physical phenomena around us. This has made the Atangana-Baleanu fractional operator more powerful and flexible. Therefore, many researchers have shown great interest in using this operator [15,16,17,18,19].

Definition 5: [20,21] A function such that $Q\in H^1(0,\xi_2)$ is called left and right side of Atangana-Baleanu fractional integral, if

$$\begin{aligned}&\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)\\&=\frac{1-\psi}{B(\psi)}Q(x)+\frac{\psi}{B(\psi)\Gamma(\psi)}\int_{\xi_1}^{x}Q(\tau)(x-\tau)^{\psi-1}d\tau\end{aligned}\qquad(7)$$

and

$$\begin{aligned}&\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\&=\frac{1-\psi}{B(\psi)}Q(x)+\frac{\psi}{B(\psi)\Gamma(\psi)}\int_{x}^{\xi_2}Q(\tau)(\tau-x)^{\psi-1}d\tau\end{aligned}\qquad(8)$$

where $\xi_2>\xi_1$, $\psi\in[0,1]$ and $B(\psi)$ is normalization function.

The next section focuses on establishing new Hermite-Hadamard type inequalities for Atangana-Baleanu fractional integrals involving twice differentiable strongly $s-$convex functions. To obtain these inequalities Hölder, Power-Mean, Young and Hölder-İşcan inequalities were used.

2. MAIN RESULTS

To prove our main results, we consider the following Lemma given by Set *et al.* in [19].

Lemma 1:[19] $Q:[\xi_1,\xi_2]\to\Re$ be differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$. Then the following identity is valid for Atangana-Baleanu fractional integral operators

$$\begin{aligned}
&\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\
&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\
&-\frac{2(1-\psi)Q(x)}{B(\psi)}\\
&=\frac{(x-\xi_1)^{\psi+1}}{(\psi+1)B(\psi)\Gamma(\psi)}Q'(\xi_1)\\
&+\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\int_0^1(1-\tau)^{\psi+1}Q''(\tau x+(1-\tau)\xi_1)d\tau\\
&-\frac{(\xi_2-x)^{\psi+1}}{(\psi+1)B(\psi)\Gamma(\psi)}Q'(\xi_2)\\
&+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\int_0^1\tau^{\psi+1}Q''(\tau\xi_2+(1-\tau)x)d\tau
\end{aligned}\tag{9}$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$ and Γ is Gamma function.

Theorem 1: Suppose that $Q:[\xi_1,\xi_2]\to\Re$ be a twice differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $L[\xi_1,\xi_2]$. If $|Q''|$ is strongly $s-$convex on $[\xi_1,\xi_2]$, for some $s\in(0,1]$ with modulus $\rho>0$ then following inequalities hold

$$\begin{aligned}
&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\
&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\
&-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|\\
&\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\
&\times\left[|Q''(x)|\beta(2+\psi,1+s)+\frac{|Q''(\xi_1)|}{\psi+s+2}-\frac{\rho(x-\xi_1)^2}{\psi^2+7\psi+12}\right]\\
&+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\
&\times\left[|Q''(x)|\beta(2+\psi,1+s)+\frac{|Q''(\xi_2)|}{\psi+s+2}-\frac{\rho(\xi_2-x)^2}{\psi^2+7\psi+12}\right]
\end{aligned}\tag{10}$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$, Γ is Gamma function and β is Beta function.

Proof: Taking modulus on both sides of Lemma 1, we have

$$\begin{aligned}
&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\
&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\
&-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|\\
&\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1(1-\tau)^{\psi+1}|Q''(\tau x+(1-\tau)\xi_1)|d\tau\\
&+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1\tau^{\psi+1}|Q''(\tau\xi_2+(1-\tau)x)|d\tau
\end{aligned}$$

By using strongly $s-$convexity of $|Q''|$, it yields

$$\begin{aligned}
&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\
&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\
&-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|\\
&\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\
&\times\left(\int_0^1(1-\tau)^{\psi+1}\right)\\
&\times\left(\tau^s|Q''(x)|+(1-t)^s|Q''(\xi_1)|-\rho\tau(1-\tau)(x-\xi_1)^2\right)d\tau\\
&+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\
&\times\left(\int_0^1\tau^{\psi+1}\right)\\
&\times\left(\tau^s|Q''(\xi_2)|+(1-t)^s|Q''(x)|-\rho\tau(1-\tau)(\xi_2-x)^2\right)d\tau.
\end{aligned}$$

By calculating the above integrals, the proof is completed.

Corollary 1: When we choose $s=1$, Theorem 1 yields the following result for Atangana-Baleanu fractional operator

$$\begin{aligned}&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\&-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|\\&\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\&\times\left[\frac{|Q''(x)|}{(\psi+2)(\psi+3)}+\frac{|Q''(\xi_1)|}{\psi+3}-\frac{\rho(x-\xi_1)^2}{\psi^2+7\psi+12}\right]\\&+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\&\times\left[\frac{|Q''(x)|}{(\psi+2)(\psi+3)}+\frac{|Q''(\xi_2)|}{\psi+3}-\frac{\rho(\xi_2-x)^2}{\psi^2+7\psi+12}\right].\end{aligned}\tag{11}$$

Corollary 2: In Theorem 1, if we choose $x=\frac{\xi_1+\xi_2}{2}$, we have

$$\begin{aligned}&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)\\&-\frac{(\xi_2-\xi_1)^{\psi}}{2^{\psi}B(\psi)\Gamma(\psi)}\left(Q(\xi_1)+Q(\xi_2)\right)\\&-\frac{(\xi_2-\xi_1)^{\psi+1}}{2^{\psi+1}(\psi+1)B(\psi)\Gamma(\psi)}\left(Q'(\xi_1)-Q'(\xi_2)\right)\\&-\frac{2(1-\psi)Q\left(\frac{\xi_1+\xi_2}{2}\right)}{B(\psi)}\Bigg|\\&\le\frac{(\xi_2-\xi_1)^{\psi+2}}{2^{\psi+2}(\psi+1)B(\psi)\Gamma(\psi)}\\&\times\Bigg[2\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|\beta(2+\psi,1+s)\\&+\frac{|Q''(\xi_1)|+|Q''(\xi_2)|}{\psi+s+2}-\frac{\rho\left(\frac{\xi_2-\xi_1}{2}\right)^2}{2\left(\psi^2+7\psi+12\right)}\Bigg]\end{aligned}\tag{12}$$

Theorem 2: Suppose that $Q:[\xi_1,\xi_2]\to\Re$ be a twice differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $L[\xi_1,\xi_2]$. If $|Q''|^{\eta}$ is strongly $s-$convex on $[\xi_1,\xi_2]$, for some $s\in(0,1]$ with modulus $\rho>0$ then following inequalities hold,

$$\begin{aligned}&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\&-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|\\&\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\&\times\Bigg[\left(\frac{1}{\psi\mu+\mu+1}\right)^{\frac{1}{\mu}}\\&\times\left(\frac{1}{s+1}\left(|Q''(x)|^{\eta}+|Q''(\xi_1)|^{\eta}\right)-\frac{\rho(x-\xi_1)^2}{6}\right)^{\frac{1}{\eta}}\Bigg]\\&+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\\&\times\Bigg[\left(\frac{1}{\psi\mu+\mu+1}\right)^{\frac{1}{\mu}}\\&\times\left(\frac{1}{s+1}\left(|Q''(x)|^{\eta}+|Q''(\xi_2)|^{\eta}\right)-\frac{\rho(\xi_2-x)^2}{6}\right)^{\frac{1}{\eta}}\Bigg]\end{aligned}\tag{13}$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$, $\mu^{-1}+\eta^{-1}=1$, $\eta>1$, Γ is Gamma function and β is Beta function.

Proof: From Lemma 1 and applying Hölder inequality, we have

$$\begin{aligned}&\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\\&-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}\\&-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|\end{aligned}$$

$$\leq \frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$\times\int_0^1 (1-\tau)^{\psi+1}\left|Q''(\tau x+(1-\tau)\xi_1)\right|d\tau$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$\times\int_0^1 \tau^{\psi+1}\left|Q''(\tau\xi_2+(1-\tau)x)\right|d\tau$$
$$\leq \frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$\times\left[\left(\int_0^1 (1-\tau)^{(\psi+1)\mu}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1\left|Q''(\tau x+(1-\tau)\xi_1)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}\right]$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$\times\left[\left(\int_0^1 \tau^{(\psi+1)\mu}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1\left|Q''(\tau\xi_2+(1-\tau)x)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}\right].$$

Using strongly $s-$ convexity, we conclude

$$\int_0^1\left|Q''(\tau x+(1-\tau)\xi_1)\right|^{\eta}d\tau$$
$$\leq \frac{\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}}{s+1}-\frac{\rho(x-\xi_1)^2}{6}$$
$$\int_0^1\left|Q''(\tau\xi_2+(1-\tau)x)\right|^{\eta}d\tau$$
$$\leq \frac{\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}}{s+1}-\frac{\rho(\xi_2-x)^2}{6}.$$

By a simple computation, we get desired result.

Corollary 3: Under the assumption of Theorem 2 with $s=1$, we obtain

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\right.$$
$$-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$
$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)} \qquad (14)$$
$$\left.-\frac{2(1-\psi)Q(x)}{B(\psi)}\right|$$

$$\leq \frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$\times\left[\left(\frac{1}{\psi\mu+\mu+1}\right)^{\frac{1}{\mu}}\left(\frac{\left(\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}\right)}{2}-\frac{\rho(x-\xi_1)^2}{6}\right)^{\frac{1}{\eta}}\right]$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$\times\left[\left(\frac{1}{\psi\mu+\mu+1}\right)^{\frac{1}{\mu}}\left(\frac{\left(\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}\right)}{2}-\frac{\rho(\xi_2-x)^2}{6}\right)^{\frac{1}{\eta}}\right].$$

Corollary 4: In Theorem 2, by setting $x=\frac{\xi_1+\xi_2}{2}$, we get

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)\right. \qquad (15)$$
$$-\frac{(\xi_2-\xi_1)^{\psi}}{2^{\psi}B(\psi)\Gamma(\psi)}\left(Q(\xi_1)+Q(\xi_2)\right)$$
$$-\frac{(\xi_2-\xi_1)^{\psi+1}}{2^{\psi+1}(\psi+1)B(\psi)\Gamma(\psi)}\left(Q'(\xi_1)-Q'(\xi_2)\right)$$
$$\left.-\frac{2(1-\psi)Q\left(\frac{\xi_1+\xi_2}{2}\right)}{B(\psi)}\right|$$

$$\leq \frac{(\xi_2-\xi_1)^{\psi+2}}{2^{\psi+2}(\psi+1)B(\psi)\Gamma(\psi)}\times\left(\frac{1}{\psi\mu+\mu+1}\right)^{\frac{1}{\mu}}$$
$$\times\left[\left(\frac{1}{s+1}\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}\right)-\frac{\rho(\xi_2-\xi_1)^2}{24}\right)^{\frac{1}{\eta}}\right.$$
$$\left.+\left(\frac{1}{s+1}\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}\right)-\frac{\rho(\xi_2-\xi_1)^2}{24}\right)^{\frac{1}{\eta}}\right].$$

Theorem 3: Suppose that $Q:[\xi_1,\xi_2]\to\Re$ be a twice differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $L[\xi_1,\xi_2]$. If $\left|Q''\right|^{\eta}$ is strongly $s-$ convex on $[\xi_1,\xi_2]$, for some $s\in(0,1]$ with modulus $\rho>0$ then following inequalities hold,

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\right.$$
$$-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)} \qquad (16)$$

$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$

$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{\eta-1}{\eta(\psi+2)-\mu(\psi+1)-1}\right)^{1-\frac{1}{\eta}}$$

$$\times\left(|Q''(x)|^\eta\beta(1+s,1+\mu+\psi\mu)+\frac{|Q''(\xi_1)|^\eta}{1+s+\mu+\psi\mu}-\frac{\rho(x-\xi_1)^2}{(2+\mu+\psi\mu)(3+\mu+\psi\mu)}\right)^{\frac{1}{\eta}}$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{\eta-1}{\eta(\psi+2)-\mu(\psi+1)-1}\right)^{1-\frac{1}{\eta}}$$

$$\times\left(|Q''(x)|^\eta\beta(1+s,1+\mu+\psi\mu)+\frac{|Q''(\xi_2)|^\eta}{1+s+\mu+\psi\mu}-\frac{\rho(\xi_2-x)^2}{(2+\mu+\psi\mu)(3+\mu+\psi\mu)}\right)^{\frac{1}{\eta}}$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$, $\mu^{-1}+\eta^{-1}=1$, $\eta\geq\mu>1$, Γ is Gamma function and β is Beta function.

Proof: From Lemma 1 and applying Hölder inequality, we have

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$

$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$

$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1(1-\tau)^{\psi+1}\left|Q''(\tau x+(1-\tau)\xi_1)\right|d\tau$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1\tau^{\psi+1}\left|Q''(\tau\xi_2+(1-\tau)x)\right|d\tau$$

$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\int_0^1(1-\tau)^{(\psi+1)\left(\frac{\eta-\mu}{\eta-1}\right)}d\tau\right)^{1-\frac{1}{\eta}}\times\left(\int_0^1(1-\tau)^{(\psi+1)\mu}\left|Q''(\tau x+(1-\tau)\xi_1)\right|^\eta d\tau\right)^{\frac{1}{\eta}}\right]$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\int_0^1\tau^{(\psi+1)\left(\frac{\eta-\mu}{\eta-1}\right)}d\tau\right)^{1-\frac{1}{\eta}}\times\left(\int_0^1\tau^{(\psi+1)\mu}\left|Q''(\tau\xi_2+(1-\tau)x)\right|^\eta d\tau\right)^{\frac{1}{\eta}}\right].$$

By using strongly $s-$convexity of $|Q''|^\eta$ and by simple computation, the proof is completed.

Corollary 5: Under the assumption of Theorem 3 with $s=1$, we obtain

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$

$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg| \tag{17}$$

$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{\eta-1}{\eta(\psi+2)-\mu(\psi+1)-1}\right)^{1-\frac{1}{\eta}}$$

$$\times\left(|Q''(x)|^\eta\beta(2,1+\mu+\psi\mu)+\frac{|Q''(\xi_1)|^\eta}{2+\mu+\psi\mu}-\frac{\rho(x-\xi_1)^2}{(2+\mu+\psi\mu)(3+\mu+\psi\mu)}\right)^{\frac{1}{\eta}}$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{\eta-1}{\eta(\psi+2)-\mu(\psi+1)-1}\right)^{1-\frac{1}{\eta}}$$

$$\times\left(|Q''(x)|^\eta\beta(2,1+\mu+\psi\mu)+\frac{|Q''(\xi_2)|^\eta}{2+\mu+\psi\mu}-\frac{\rho(\xi_2-x)^2}{(2+\mu+\psi\mu)(3+\mu+\psi\mu)}\right)^{\frac{1}{\eta}}.$$

Corollary 6: In Theorem 3, by setting $x=\frac{\xi_1+\xi_2}{2}$, we get

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)-\frac{(\xi_2-\xi_1)^{\psi}}{2^{\psi}B(\psi)\Gamma(\psi)}\left(Q(\xi_1)+Q(\xi_2)\right) \tag{18}$$

$$-\frac{(\xi_2-\xi_1)^{\psi+1}}{2^{\psi+1}(\psi+1)B(\psi)\Gamma(\psi)}\left(Q'(\xi_1)-Q'(\xi_2)\right)-\frac{2(1-\psi)Q\left(\frac{\xi_1+\xi_2}{2}\right)}{B(\psi)}\Bigg|$$

$$\leq \frac{(\xi_2-\xi_1)^{\psi+2}}{2^{\psi+2}(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{\eta-1}{\eta(\psi+2)-\mu(\psi+1)-1}\right)^{1-\frac{1}{\eta}}$$
$$\times\Bigg[\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}\beta(1+s,1+\mu+\psi\mu)+\frac{|Q''(\xi_1)|^{\eta}}{1+s+\mu+\psi\mu}-\frac{\rho(\xi_2-\xi_1)^2}{(2+\mu+\psi\mu)(3+\mu+\psi\mu)}\right)^{\frac{1}{\eta}}$$
$$+\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}\beta(1+s,1+\mu+\psi\mu)+\frac{|Q''(\xi_2)|^{\eta}}{1+s+\mu+\psi\mu}-\frac{\rho(\xi_2-\xi_1)^2}{(2+\mu+\psi\mu)(3+\mu+\psi\mu)}\right)^{\frac{1}{\eta}}\Bigg].$$

Theorem 4: Suppose that $Q:[\xi_1,\xi_2]\to\Re$ be a twice differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $L[\xi_1,\xi_2]$. If $|Q''|^{\eta}$ is strongly $s-$convex on $[\xi_1,\xi_2]$, for some $s\in(0,1]$ with modulus $\rho>0$ then following inequalities hold,

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg| \tag{19}$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{1}{\psi+2}\right)^{\frac{1}{\mu}}\times\left(|Q''(x)|^{\eta}\beta(1+s,2+\psi)+\frac{|Q''(\xi_1)|^{\eta}}{\psi+s+2}-\frac{\rho(x-\xi_1)^2}{\psi^2+7\psi+12}\right)^{\frac{1}{\eta}}$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{1}{\psi+2}\right)^{\frac{1}{\mu}}\times\left(|Q''(x)|^{\eta}\beta(1+s,2+\psi)+\frac{|Q''(\xi_2)|^{\eta}}{\psi+s+2}-\frac{\rho(\xi_2-x)^2}{\psi^2+7\psi+12}\right)^{\frac{1}{\eta}}$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$, $\eta\geq 1$, Γ is Gamma function and β is Beta function.

Proof: From Lemma 1 and using the power mean inequality, we obtain

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1(1-\tau)^{\psi+1}\left|Q''(\tau x+(1-\tau)\xi_1)\right|d\tau+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1\tau^{\psi+1}\left|Q''(\tau\xi_2+(1-\tau)x)\right|d\tau$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\int_0^1(1-\tau)^{(\psi+1)}d\tau\right)^{\frac{1}{\mu}}\times\left(\int_0^1(1-\tau)^{(\psi+1)}\left|Q''(\tau x+(1-\tau)\xi_1)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}\right]$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\int_0^1\tau^{(\psi+1)}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1\tau^{(\psi+1)}\left|Q''(\tau\xi_2+(1-\tau)x)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}\right].$$

Since $|Q''|^{\eta}$ is strongly $s-$ convex, by a simple computation, we have

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\int_0^1(1-\tau)^{(\psi+1)}d\tau\right)^{\frac{1}{\mu}}\times\left(\int_0^1(1-\tau)^{(\psi+1)}\right.\right.$$

$$\times\left(\tau^{s}\left|Q''(x)\right|^{\eta}+(1-\tau)^{s}\left|Q''(\xi_1)\right|^{\eta}-\rho\tau(1-\tau)(x-\xi_1)^{2}\right)d\tau\Bigg)^{\frac{1}{\eta}}\Bigg]$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$\times\left[\left(\int_0^1(\tau)^{(\psi+1)}d\tau\right)^{\frac{1}{\mu}}\times\left(\int_0^1(\tau)^{(\psi+1)}\right.\right.$$

$$\left.\left.\times\left(\tau^{s}\left|Q''(\xi_2)\right|^{\eta}+(1-\tau)^{s}\left|Q''(x)\right|^{\eta}-\rho\tau(1-\tau)(\xi_2-x)^{2}\right)d\tau\right)^{\frac{1}{\eta}}\right]$$

This completes the proof.

Corollary 7: Under the assumption of Theorem 4 with $s=1$, we obtain

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x) \tag{20}$$
$$-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$
$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{1}{\psi+2}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{\left|Q''(x)\right|^{\eta}}{(\psi+2)(\psi+3)}+\frac{\left|Q''(\xi_1)\right|^{\eta}}{\psi+3}-\frac{\rho(x-\xi_1)^{2}}{\psi^{2}+7\psi+12}\right)^{\frac{1}{\eta}}$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left(\frac{1}{\psi+2}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{\left|Q''(x)\right|^{\eta}}{(\psi+2)(\psi+3)}+\frac{\left|Q''(\xi_2)\right|^{\eta}}{\psi+3}-\frac{\rho(\xi_2-x)^{2}}{\psi^{2}+7\psi+12}\right)^{\frac{1}{\eta}}.$$

Corollary 8: In Theorem 4, if we choose $x=\frac{\xi_1+\xi_2}{2}$, we have

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)$$
$$-\frac{(\xi_2-\xi_1)^{\psi}}{2^{\psi}B(\psi)\Gamma(\psi)}\left(Q(\xi_1)+Q(\xi_2)\right)$$
$$-\frac{(\xi_2-\xi_1)^{\psi+1}}{2^{\psi+1}(\psi+1)B(\psi)\Gamma(\psi)}\left(Q'(\xi_1)-Q'(\xi_2)\right) \tag{21}$$
$$-\frac{2(1-\psi)Q\left(\frac{\xi_1+\xi_2}{2}\right)}{B(\psi)}\Bigg|$$
$$\leq\frac{(\xi_2-\xi_1)^{\psi+2}}{2^{\psi+2}(\psi+1)B(\psi)\Gamma(\psi)}\times\left(\frac{1}{\psi+1}\right)^{\frac{1}{\mu}}$$
$$\times\left[\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}\beta(1+s,2+\psi)\right.\right.$$
$$\left.+\frac{\left|Q''(\xi_1)\right|^{\eta}}{\psi+s+2}-\frac{\rho(\xi_2-\xi_1)^{2}}{4(\xi^{2}+7\xi+12)}\right)^{\frac{1}{\eta}}$$
$$+\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}\beta(1+s,2+\psi)\right.$$
$$\left.\left.+\frac{\left|Q''(\xi_2)\right|^{\eta}}{\psi+s+2}-\frac{\rho(\xi_2-\xi_1)^{2}}{4(\xi^{2}+7\xi+12)}\right)^{\frac{1}{\eta}}\right].$$

Theorem 5: Suppose that $Q:[\xi_1,\xi_2]\rightarrow\Re$ be a twice differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $L[\xi_1,\xi_2]$. If $\left|Q''\right|^{\eta}$ is strongly $s-$convex on $[\xi_1,\xi_2]$, for some $s\in(0,1]$ with modulus $\rho>0$ then following inequalities hold,

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x) \tag{22}$$
$$-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$
$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}$$
$$-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left[\left(\frac{1}{\mu(\psi\mu+\mu+1)}\right)\right.$$
$$\left.+\frac{1}{\eta}\left(\frac{1}{s+1}\left(\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}\right)-\frac{\rho(x-\xi_1)^{2}}{6}\right)\right]$$
$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left[\left(\frac{1}{\mu(\psi\mu+\mu+1)}\right)\right.$$
$$\left.+\frac{1}{\eta}\left(\frac{1}{s+1}\left(\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}\right)-\frac{\rho(\xi_2-x)^{2}}{6}\right)\right]$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$, $\mu^{-1}+\eta^{-1}=1$, $\eta>1$, Γ is Gamma function and β is Beta function.

Proof: By using Lemma 1 and Young inequality, we obtain

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\right|$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1(1-\tau)^{\psi+1}\left|Q''(\tau x+(1-\tau)\xi_1)\right|d\tau+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1\tau^{\psi+1}\left|Q''(\tau\xi_2+(1-\tau)x)\right|d\tau$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\frac{1}{\mu}\left(\int_0^1(1-\tau)^{(\psi+1)\mu}d\tau\right)+\frac{1}{\eta}\left(\int_0^1\left|Q''(\tau x+(1-\tau)\xi_1)\right|^{\eta}d\tau\right)\right]+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\frac{1}{\mu}\left(\int_0^1\tau^{(\psi+1)\mu}d\tau\right)+\frac{1}{\eta}\left(\int_0^1\left|Q''(\tau\xi_2+(1-\tau)x)\right|^{\eta}d\tau\right)\right].$$

By using strongly $s-$convexity of $|Q''|^{\eta}$ and by simple computation, the proof is completed.

Corollary 9: Under the assumption of Theorem 5 with $s=1$, we obtain

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\right| \tag{23}$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\frac{1}{\mu(\psi\mu+\mu+1)}\right)+\frac{\left(|Q''(x)|^{\eta}+|Q''(\xi_1)|^{\eta}\right)}{2\eta}-\frac{\rho(x-\xi_1)^2}{6\eta}\right]+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\frac{1}{\mu(\psi\mu+\mu+1)}\right)+\frac{\left(|Q''(x)|^{\eta}+|Q''(\xi_2)|^{\eta}\right)}{2\eta}-\frac{\rho(\xi_2-x)^2}{6\eta}\right].$$

Corollary 10: In Theorem 5, by setting $x=\frac{\xi_1+\xi_2}{2}$, we get

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)-\frac{(\xi_2-\xi_1)^{\psi}}{2^{\psi}B(\psi)\Gamma(\psi)}\left(Q(\xi_1)+Q(\xi_2)\right)-\frac{(\xi_2-\xi_1)^{\psi+1}}{2^{\psi+1}(\psi+1)B(\psi)\Gamma(\psi)}\left(Q'(\xi_1)-Q'(\xi_2)\right)-\frac{2(1-\psi)Q\left(\frac{\xi_1+\xi_2}{2}\right)}{B(\psi)}\right| \tag{24}$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left[\left(\frac{1}{\mu(\psi\mu+\mu+1)}\right)+\frac{1}{\eta}\left(\frac{1}{s+1}\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+|Q''(\xi_1)|^{\eta}\right)-\frac{\rho(\xi_2-\xi_1)^2}{24}\right)\right]+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\left[\left(\frac{1}{\mu(\psi\mu+\mu+1)}\right)+\frac{1}{\eta}\left(\frac{1}{s+1}\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+|Q''(\xi_2)|^{\eta}\right)-\frac{\rho(\xi_2-\xi_1)^2}{24}\right)\right].$$

Theorem 6: Suppose that $Q:[\xi_1,\xi_2]\to\Re$ be a twice differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $L[\xi_1,\xi_2]$. If $|Q''|^{\eta}$ is strongly $s-$convex on $[\xi_1,\xi_2]$, for some $s\in(0,1]$ with modulus $\rho>0$ then following inequalities hold,

$$\left|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}-\frac{2(1-\psi)Q(x)}{B(\psi)}\right| \tag{25}$$
$$\leq\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\left[\left(\frac{1}{\psi\mu+\mu+2}\right)^{\frac{1}{\mu}}\right.$$

$$\times\left(\frac{\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}}{s^2+3s+2}-\frac{\rho(x-\xi_1)^2}{12}\right)^{\frac{1}{\eta}}$$

$$+\left(\frac{1}{(\psi\mu+\mu+1)(\psi\mu+\mu+2)}\right)^{\frac{1}{\mu}}$$

$$\times\left(\frac{\left|Q''(x)\right|^{\eta}}{s+2}+\frac{\left|Q''(\xi_1)\right|^{\eta}}{s^2+3s+2}-\frac{\rho(x-\xi_1)^2}{12}\right)^{\frac{1}{\eta}}\Bigg]$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$\times\Bigg[\left(\frac{1}{(\psi\mu+\mu+1)(\psi\mu+\mu+2)}\right)^{\frac{1}{\mu}}$$

$$\times\left(\frac{\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}}{s^2+3s+2}-\frac{\rho(\xi_2-x)^2}{12}\right)^{\frac{1}{\eta}}$$

$$+\left(\frac{1}{\psi\mu+\mu+2}\right)^{\frac{1}{\mu}}$$

$$\times\left(\frac{\left|Q''(\xi_2)\right|^{\eta}}{s+2}+\frac{\left|Q''(x)\right|^{\eta}}{s^2+3s+2}-\frac{\rho(\xi_2-x)^2}{12}\right)^{\frac{1}{\eta}}\Bigg]$$

where $\psi\in[0,1]$, $x\in[\xi_1,\xi_2]$, $\eta=\frac{\mu}{\mu-1}$, $\mu>1$, Γ is Gamma function and β is Beta function.

Proof: By using Lemma 1 and Hölder-İşcan inequality, we obtain

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)$$

$$-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$

$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$

$$\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1(1-\tau)^{\psi+1}\left|Q''(\tau x+(1-\tau)\xi_1)\right|d\tau$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}\times\int_0^1\tau^{\psi+1}\left|Q''(\tau\xi_2+(1-\tau)x)\right|d\tau$$

$$\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$\times\Bigg[\left(\int_0^1(1-\tau)^{(\psi+1)\mu+1}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1(1-\tau)\left|Q''(\tau x+(1-\tau)\xi_1)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}$$

$$+\left(\int_0^1\tau(1-\tau)^{(\psi+1)\mu}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1\tau\left|Q''(\tau x+(1-\tau)\xi_1)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}\Bigg]$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$\times\Bigg[\left(\int_0^1(1-\tau)\tau^{(\psi+1)\mu}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1(1-\tau)\left|Q''(\tau\xi_2+(1-\tau)x)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}$$

$$+\left(\int_0^1\tau^{(\psi+1)\mu+1}d\tau\right)^{\frac{1}{\mu}}\left(\int_0^1\tau\left|Q''(\tau\xi_2+(1-\tau)x)\right|^{\eta}d\tau\right)^{\frac{1}{\eta}}\Bigg].$$

By using strongly $s-$ convexity of $\left|Q''\right|^{\eta}$ and by computing the above integrals, we have desired result.

Corollary 11: Under the assumption of Theorem 6 with $s=1$, we obtain

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q(x)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q(x)\tag{26}$$

$$-\frac{(x-\xi_1)^{\psi}Q(\xi_1)+(\xi_2-x)^{\psi}Q(\xi_2)}{B(\psi)\Gamma(\psi)}$$

$$-\frac{(x-\xi_1)^{\psi+1}Q'(\xi_1)-(\xi_2-x)^{\psi+1}Q'(\xi_2)}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$-\frac{2(1-\psi)Q(x)}{B(\psi)}\Bigg|$$

$$\le\frac{(x-\xi_1)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$\times\Bigg[\left(\frac{1}{\psi\mu+\mu+2}\right)^{\frac{1}{\mu}}\left(\frac{\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}}{6}-\frac{\rho(x-\xi_1)^2}{12}\right)^{\frac{1}{\eta}}$$

$$+\left(\frac{1}{(\psi\mu+\mu+1)(\psi\mu+\mu+2)}\right)^{\frac{1}{\mu}}$$

$$\times\left(\frac{\left|Q''(x)\right|^{\eta}}{3}+\frac{\left|Q''(\xi_1)\right|^{\eta}}{6}-\frac{\rho(x-\xi_1)^2}{12}\right)^{\frac{1}{\eta}}\Bigg]$$

$$+\frac{(\xi_2-x)^{\psi+2}}{(\psi+1)B(\psi)\Gamma(\psi)}$$

$$\times\Bigg[\left(\frac{1}{(\psi\mu+\mu+1)(\psi\mu+\mu+2)}\right)^{\frac{1}{\mu}}$$

$$\times\left(\frac{\left|Q''(x)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}}{6}-\frac{\rho(\xi_2-x)^2}{12}\right)^{\frac{1}{\eta}}$$

$$+\left(\frac{1}{\psi\mu+\mu+2}\right)^{\frac{1}{\mu}}\left(\frac{\left|Q''(\xi_2)\right|^{\eta}}{3}+\frac{\left|Q''(x)\right|^{\eta}}{6}-\frac{\rho(\xi_2-x)^2}{12}\right)^{\frac{1}{\eta}}\Bigg].$$

Corollary 12: : In Theorem 6, by setting $x=\frac{\xi_1+\xi_2}{2}$, we get

$$\Bigg|\left({}^{AB}_{\xi_1}I^{\psi}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)+\left({}^{AB}I^{\psi}_{\xi_2}\right)Q\left(\frac{\xi_1+\xi_2}{2}\right)$$
$$-\frac{(\xi_2-\xi_1)^{\psi}}{2^{\psi}B(\psi)\Gamma(\psi)}\left(Q(\xi_1)+Q(\xi_2)\right)$$
$$-\frac{(\xi_2-\xi_1)^{\psi+1}}{2^{\psi+1}(\psi+1)B(\psi)\Gamma(\psi)}\left(Q'(\xi_1)-Q'(\xi_2)\right)$$
$$-\frac{2(1-\psi)Q\left(\frac{\xi_1+\xi_2}{2}\right)}{B(\psi)}\Bigg|$$
$$\leq\frac{(\xi_2-\xi_1)^{\psi+2}}{2^{\psi+2}(\psi+1)B(\psi)\Gamma(\psi)}\Bigg[\left(\frac{1}{\psi\mu+\mu+2}\right)^{\frac{1}{\mu}}$$
$$\times\left(\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+\left|Q''(\xi_1)\right|^{\eta}\right)\frac{1}{s^2+3s+2}-\frac{\rho(\xi_2-\xi_1)^2}{48}\right)^{\frac{1}{\eta}}$$
$$+\left(\frac{1}{(\psi\mu+\mu+1)(\psi\mu+\mu+2)}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{1}{s+2}\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+\frac{\left|Q''(\xi_1)\right|^{\eta}}{s^2+3s+2}-\frac{\rho(\xi_2-\xi_1)^2}{48}\right)^{\frac{1}{\eta}}$$
$$+\left(\frac{1}{(\psi\mu+\mu+1)(\psi\mu+\mu+2)}\right)^{\frac{1}{\mu}}$$
$$\times\left(\left(\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}+\left|Q''(\xi_2)\right|^{\eta}\right)\frac{1}{s^2+3s+2}\right.$$
$$\left.-\frac{\rho(\xi_2-\xi_1)^2}{48}\right)^{\frac{1}{\eta}}$$
$$+\left(\frac{1}{\psi\mu+\mu+2}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{\left|Q''(\xi_1)\right|^{\eta}}{s+2}+\frac{1}{s^2+3s+2}\left|Q''\left(\frac{\xi_1+\xi_2}{2}\right)\right|^{\eta}\right.$$
$$\left.-\frac{\rho(\xi_2-\xi_1)^2}{48}\right)^{\frac{1}{\eta}}\Bigg].$$

3. DISCUSSION AND CONCLUSION

In this study, new integral inequalities for strongly convex functions have been obtained with the help of Atangana-Baleanu fractional integral operators. Additionally, new upper bounds have been obtained by applying different types of auxiliary inequalities. It has also been observed that some of these results are generalizations of inequalities previously obtained in the literature. Researchers interested in this subject can produce new identities and obtain different types of strongly convex function classes, as well as generalize the results with strongly convex function classes by using inequalities in the literature.

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