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Mapping of L^2-norm of Two Multiplied $2\pi-$Periodic Functions to Their *Fourier* Coefficients (Part II)

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Abstract. This work derives an identity that maps between the 2-norm of two multiplied 2π-periodic functions in L^2 space (i.e., $\left\|f.g\right\|^2_{L^2(-\pi,\pi)}$) to the individual *Fourier* coefficients of f and g. Alternately, it maps between the 2-norm of two multiplied discrete-time *Fourier* transforms (i.e., $\left\|\mathscr{F}\{f\}.\mathscr{F}\{g\}\right\|^2_{L^2(-\pi,\pi)}$) to the discrete-time samples of f and g. The results are equality to *Cauchy–Schwarz* inequality, and extend the results of our previous paper that map between $\left\|f\right\|^4_{L^4(-\pi,\pi)}$ to the *Fourier* coefficients of f, alternately $\left\|\mathscr{F}\{f\}\right\|^4_{L^4(-\pi,\pi)}$ to the discrete-time samples of f.

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1. Introduction

For a 2π-periodic function f in *Lebesgue* space (L^4), our previous paper [20] derived an identity that maps between 4-norm of f ($\left\|f\right\|^4_{L^4(-\pi,\pi)}$) and the *Fourier* coefficients of f, such norm is labelled $L^4(-\pi,\pi)$-norm. In this paper, section 2 derives a similar identity that maps between $L^2(-\pi,\pi)$-norm of two multiplied functions f and g ($\left\|f.g\right\|^2_{L^2(-\pi,\pi)}$) and the individual *Fourier* coefficients of f and g. Alternately, it maps between $L^2(-\pi,\pi)$-norm of two multiplied discrete-time *Fourier* transforms (DTFT) $\mathscr{F}\{f\}$ and $\mathscr{F}\{g\}$ ($\left\|\mathscr{F}\{f\}.\mathscr{F}\{g\}\right\|^2_{L^2(-\pi,\pi)}$) and the individual discrete-time samples of f and g. Furthermore, for the discrete-time discrete-frequency case, it maps 2-norm of two multiplied discrete *Fourier* transform (DFT) sequences to their inverse discrete *Fourier* transform (IDFT) sequences and vice versa, or 4-norm of a DFT sequence to its IDFT sequence and vice versa. Section 3 shows scenarios where the results are useful in solving for the exact value in physical problems such as comparison of energies of signals (e.g., detection and estimation) and computing power spectrum of weighted signals (e.g., digital filters). Then, section 4 draws the overall conclusion.

Notation: A bold letter ($\mathbf{u}[n]$) denotes a sequence of index n, and a non-bold ($u[n]$) is its n^{th} element. A superscript asterisk ($.^*$) denotes conjugate transpose. The convolution operator is denoted ($\circledast$) and *Hadamard* product denoted ($\odot$).

Contribution: The well-known *Cauchy–Schwarz* inequality [1,7] states $\langle\mathbf{u}|\mathbf{v}\rangle \leq \|\mathbf{u}\|_2\|\mathbf{v}\|_2$. Let $\mathbf{u}' = \sqrt{\mathbf{u}}$ and $\mathbf{v}' = \sqrt{\mathbf{v}}$, then $\langle\mathbf{u}|\mathbf{v}\rangle = \langle\mathbf{u}'^2|\mathbf{v}'^2\rangle = \sum(\mathbf{u}'\odot\mathbf{v}')^2 = \left\|\mathbf{u}'\odot\mathbf{v}'\right\|^2_2$, and $\|\mathbf{u}\|_2 = \sqrt{\langle\mathbf{u}|\mathbf{u}\rangle} = \sqrt{\sum|\mathbf{u}|^2} = \sqrt{\sum|\mathbf{u}'|^4} = \left\|\mathbf{u}'\right\|^2_4$. The inequality is re-formulated to $\left\|\mathbf{u}'\odot\mathbf{v}'\right\|^2_2 \leq \left\|\mathbf{u}'\right\|^2_4\left\|\mathbf{v}'\right\|^2_4$, and its integral form [4,9] to $\left\|f(t)g(t)\right\|^2_2 \leq \left\|f(t)\right\|^2_4\left\|g(t)\right\|^2_4$. If $f(t), g(t) \in L^2(-\pi,\pi)$, the identities introduced in this paper provide equality to these two forms of *Cauchy–Schwarz* inequality.

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2. Mapping of $L^2(-\pi,\pi)$-norm of Two Multiplied Functions to Their *Fourier* Coefficients

The mapping is given firstly in the case of continuous-time discrete-frequency, then in the case of discrete-time continuous-frequency, and finally for the discrete-time discrete-frequency finite sequences.

2.1. Continuous-time $2\pi-$periodic Functions.

A continuous-in-time, L^1-integrable, complex-valued function $f(t)$ is re-constructed from its frequency components $\hat{f}(\xi)$ using the inverse *Fourier* transform [29],

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi t}\, d\xi, \qquad \forall\ t \in \mathbb{R}.$$

If $f(t)$ is periodic in the interval $[-\pi,\pi]$, then it is synthesized from discrete harmonics, and the integral is reduced to a summation resulting the famous *Fourier* series,

$$f_{2\pi}(t) = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{int}, \qquad t \in [-\pi,\pi],$$

where the *Fourier* coefficients $\hat{c}_n$ are given by $\hat{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{2\pi}(t) e^{-int}\, dt$.

Theorem 2.1. *Let $f(t)$ and $g(t)$ be two complex-valued functions in time t ($f,g:\mathbb{R}\to\mathbb{C}$), and $f(t), g(t)$ be periodic and have finite-length $N = 2M+1$ discrete* Fourier *coefficients as $f(t) = \sum_{n=-M}^{M} \hat{u}_n e^{int}$ and $g(t) = \sum_{n=-M}^{M} \hat{v}_n e^{int}$, where $\hat{u}_n$ and $\hat{v}_n$ are the complex* Fourier *coefficients, and $\hat{u}_n, \hat{v}_n \in \mathbb{C}$, then*

$$\left\|(f.g)(t)\right\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} \left|f(t).g(t)\right|^2\, dt = 2\pi \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}^*_m \Bigg[\sum_{\substack{l=(n-m)-M \\ n\geq m}}^{M} \hat{v}^*_l \hat{v}_{l-(n-m)} + \sum_{\substack{l=(m-n)-M \\ n<m}}^{M} \hat{v}^*_{l-(m-n)} \hat{v}_l \Bigg].$$

For infinite-length Fourier *coefficients (i.e., $M=\infty$),*

$$\left\|(f.g)(t)\right\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} \left|f(t).g(t)\right|^2\, dt = 2\pi \sum_{n\in\mathbb{Z}} \hat{u}_n \sum_{m\in\mathbb{Z}} \hat{u}^*_m \Bigg[\sum_{\substack{l\in\mathbb{Z} \\ n\geq m}} \hat{v}^*_l \hat{v}_{l-(n-m)} + \sum_{\substack{l\in\mathbb{Z} \\ n<m}} \hat{v}^*_{l-(m-n)} \hat{v}_l \Bigg].$$

Proof. It follows same steps of proof of [20, Theorem 2.1] –given also previously in [21]–, and is given again below;

$$\int_{-\pi}^{\pi} \left|f(t).g(t)\right|^2 dt = \int_{-\pi}^{\pi} \left| \sum_{n=-M}^{M} \hat{u}_n e^{int} \sum_{l=-M}^{M} \hat{v}_l e^{ilt} \right|^2 \mathrm{d}t = \int_{-\pi}^{\pi} \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}^*_m \sum_{l=-M}^{M} \hat{v}_l \sum_{k=-M}^{M} \hat{v}^*_k e^{i(n-m+l-k)t}\, \mathrm{d}t,$$

for $n+l = m+k \implies e^{i(n-m+l-k)t} = 1$, and for $n+l \neq m+k \implies e^{i(n-m+l-k)t} = e^{\pm iat}$, where $a = 1,2,...,4M$.

$$= \int_{-\pi}^{\pi} \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}^*_m \sum_{l=-M}^{M} \hat{v}_l \Big(\sum_{\substack{k=-M \\ n+l=m+k}}^{M} \hat{v}^*_k + \sum_{\substack{k=-M \\ n+l\neq m+k}}^{M} \hat{v}^*_k e^{\pm iat} \Big) \mathrm{d}t = 2\pi \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}^*_m \sum_{l=-M}^{M} \hat{v}_l \sum_{\substack{k=-M \\ k=l+n-m}}^{M} \hat{v}^*_k,$$

where the exponential terms vanish after integration as the definite integral of $e^{\pm iat}$ is zero for the integral limits $t = \pm\pi$. For the condition on k, not all combinations of $(l+n-m)$ satisfy $k \in [-M, M]$, so the unused combinations are excluded,

$$\begin{aligned} -M \leq &\quad k &&\leq M, \\ -M \leq &\quad l+n-m &&\leq M, \\ (m-n)-M \leq &\quad l &&\leq M+(m-n). \end{aligned} \tag{2.1}$$

$$\text{However,} \qquad -M \leq \quad l \quad \leq M. \tag{2.2}$$

Then, the valid range of l is (2.1) $\cap$ (2.2) $\equiv \begin{cases} -M \leq l \leq M+(m-n), & \text{if } n \geq m, \\ (m-n)-M \leq l \leq M, & \text{if } n < m. \end{cases}$

$$= 2\pi \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}^*_m \Bigg[\sum_{\substack{l=-M \\ n\geq m}}^{M+(m-n)} \hat{v}_l \hat{v}^*_{l+(n-m)} + \sum_{\substack{l=(m-n)-M \\ n<m}}^{M} \hat{v}_l \hat{v}^*_{l+(n-m)} \Bigg]$$

$$= 2\pi \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}^*_m \Bigg[\sum_{\substack{l=(n-m)-M \\ n\geq m}}^{M} \hat{v}^*_l \hat{v}_{l-(n-m)} + \sum_{\substack{l=(m-n)-M \\ n<m}}^{M} \hat{v}^*_{l-(m-n)} \hat{v}_l \Bigg].$$

Where the counter l was shifted down by $(m-n)$ in the case $(n \geq m)$. Moreover, for $\hat{v}_n \in \mathbb{R}$, it simplifies to

$$= 2\pi \sum_{n=-M}^{M} \hat{u}_n \sum_{m=-M}^{M} \hat{u}_m^* \sum_{l=|n-m|-M}^{M} \hat{v}_l \hat{v}_{l-|n-m|}. \qquad \square$$

2.2. **Discrete-time Functions.** A function $f(t)$ sampled at equally-spaced time instants has a continuous frequency response $\hat{f}(\omega)$ periodic in the interval $[-\pi,\pi]$. If the real or complex discrete samples of $f(t)$ are represented as a sequence $\mathbf{x}[n]$, then $\hat{f}(\omega)$ is synthesized using the discrete-time *Fourier* transform (DTFT) [29],

$$\hat{f}_{2\pi}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}, \qquad \omega \in [-\pi,\pi],$$

where the discrete samples $\mathbf{x}[n]$ are given by $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}_{2\pi}(\omega)e^{-in\omega}\,d\omega$.

Theorem 2.2. *Let $f(t) = \mathbf{u}[n]$ and $g(t) = \mathbf{v}[n]$ be two discrete functions sampled at equally spaced points, and have a finite-length $N = 2M+1$, where $\mathbf{u}[n], \mathbf{v}[n] \in \mathbb{C}^N$, and $\hat{f}(\omega) = \sum_{n=-M}^{M} u[n]e^{-i\omega n}$ and $\hat{g}(\omega) = \sum_{n=-M}^{M} v[n]e^{-i\omega n}$ be the discrete-time* Fourier *transform of $f(t)$ and $g(t)$ respectively, then*

$$\left\|(\hat{f}.\hat{g})(\omega)\right\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi}\left|(\hat{f}.\hat{g})(\omega)\right|^2 d\omega = 2\pi\sum_{n=-M}^{M} u[n]\sum_{m=-M}^{M} u^*[m]\Bigg[\sum_{\substack{l=(n-m)-M\\ n\geq m}}^{M} v^*[l]v[l-(n-m)] + \sum_{\substack{l=(m-n)-M\\ n<m}}^{M} v^*[l-(m-n)]v[l]\Bigg].$$

If $\mathbf{v}[.] \in \mathbb{R}^N$, then

$$\left\|(\hat{f}.\hat{g})(\omega)\right\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi}\left|\hat{f}(\omega).\hat{g}(\omega)\right|^2 d\omega = 2\pi\sum_{n=-M}^{M} u[n]\sum_{m=-M}^{M} u^*[m]\sum_{l=|n-m|-M}^{M} v[l]v[l-|n-m|].$$

For infinite-length discrete-time functions (i.e., $M=\infty$),

$$\left\|(\hat{f}.\hat{g})(\omega)\right\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi}\left|\hat{f}(\omega).\hat{g}(\omega)\right|^2 d\omega = 2\pi\sum_{n\in\mathbb{Z}} u[n]\sum_{m\in\mathbb{Z}} u^*[m]\sum_{l\in\mathbb{Z}} v[l]v[l-|n-m|].$$

Proof. The proof follows the same steps of the proof given in theorem 2.1. $\square$

2.3. **Discrete-time Discrete-frequency Finite Sequences.** The discrete *Fourier* transform (DFT) works on a real or complex finite sequence in time-domain and produce a same length complex finite sequence in frequency domain, alternately, its inverse (IDFT) works on the frequency domain sequence to generate the time domain sequence. The analysis formula or DFT decomposes a sample in frequency domain to the contributions of every sample in time domain, and is given by,

$$\hat{x}[\kappa] = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]\,e^{-i2\pi\kappa n/N}, \quad 0 \leq \kappa \leq N-1.$$

The synthesis formula or IDFT constructs a sample in time domain from the contributions of every sample in frequency domain, and is given by,

$$x[n] = \frac{1}{\sqrt{N}}\sum_{\kappa=0}^{N-1} \hat{x}[\kappa]\,e^{i2\pi\kappa n/N}, \quad 0 \leq n \leq N-1.$$

Theorem 2.3. *Let $\mathbf{u}[n] \overset{\mathscr{F}}{\longleftrightarrow} \hat{\mathbf{u}}[\kappa]$ and $\mathbf{v}[n] \overset{\mathscr{F}}{\longleftrightarrow} \hat{\mathbf{v}}[\kappa]$, where $0 \leq n,\kappa \leq N-1$, be two finite sequences in $\mathbb{C}^N$, and related by discrete* Fourier *transform (DFT) as $\hat{\mathbf{u}}[\kappa], \hat{\mathbf{v}}[\kappa]$ are the DFT's of $\mathbf{u}[n], \mathbf{v}[n]$ respectively, then*

$$\left\|(\mathbf{u}\odot\mathbf{v})[n]\right\|_2^2 = \frac{1}{N}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\Bigg[\sum_{\substack{\ell=\kappa-\mu\\ \kappa\geq\mu}}^{N-1}\hat{v}^*[\ell]\hat{v}[\ell-(\kappa-\mu)] + \sum_{\substack{\ell=\mu-\kappa\\ \kappa<\mu}}^{N-1}\hat{v}^*[\ell-(\mu-\kappa)]\hat{v}[\ell]\Bigg]. \tag{2.3}$$

If $\hat{\mathbf{v}}[.] \in \mathbb{R}^N$, then

$$\left\|(\mathbf{u}\odot\mathbf{v})[n]\right\|_2^2 = \frac{1}{N}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\sum_{\ell=|\kappa-\mu|}^{N-1}\hat{v}[\ell]\hat{v}[\ell-|\kappa-\mu|]. \tag{2.4}$$

Proof.

$$\left\|(\mathbf{u}\odot\mathbf{v})[n]\right\|_2^2=\sum_{n=0}^{N-1}\left|(u\odot v)[n]\right|^2\overset{(I)}{=}\sum_{\kappa=0}^{N-1}\left|\widehat{(u\odot v)}\,[\kappa]\right|^2\overset{(II)}{=}\sum_{\ell=0}^{N-1}\left|\frac{1}{\sqrt{N}}(\hat{u}\circledast\hat{v})_N[\ell]\right|^2\overset{(III)}{=}\frac{1}{N}\sum_{\ell=0}^{2N-2}\left|\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\cdot\hat{v}[\ell-\kappa]\right|^2,$$

where (I) was derived using *Parseval*'s identity (see appendix A), (II) was derived using the *convolution theorem* (see appendix B), and (III) was derived using relation (C.2) proposition C.1 (see appendix C) where the circular convolution of length (N) was replaced by the linear[1] convolution of length $(N+N-1)$, where $0\le\ell\le 2N-2$,

$$=\frac{1}{N}\sum_{\ell=0}^{2N-2}\left(\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\cdot\hat{v}[\ell-\kappa]\right)\left(\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\cdot\hat{v}^*[\ell-\mu]\right)=\frac{1}{N}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\sum_{\ell=0}^{2N-2}\hat{v}^*[\ell-\mu]\hat{v}[\ell-\kappa].$$

Simplify by limiting the range of ℓ to exclude unused combinations. Since the index of $\hat{v}[.]\in[0,N-1]$, then

$$\begin{aligned}0\le\ &\ \ell-\kappa &&\le N-1,\\ \kappa\le\ &\ \ \ell &&\le N+\kappa-1. \end{aligned}\tag{2.5}$$

$$\text{Moreover,}\quad\begin{aligned}0\le\ &\ \ell-\mu &&\le N-1,\\ \mu\le\ &\ \ \ell &&\le N+\mu-1. \end{aligned}\tag{2.6}$$

The usable range of ℓ is (2.5) $\cap$ (2.6) $\equiv\begin{cases}\kappa\le\ell\le N+\mu-1, & \text{if } \kappa\ge\mu,\\ \mu\le\ell\le N+\kappa-1, & \text{if } \kappa<\mu.\end{cases}$

$$=\frac{1}{N}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\left[\sum_{\substack{\ell=\kappa\\ \kappa\ge\mu}}^{N+\mu-1}\hat{v}^*[\ell-\mu]\hat{v}[\ell-\kappa]+\sum_{\substack{\ell=\mu\\ \kappa<\mu}}^{N+\kappa-1}\hat{v}^*[\ell-\mu]\hat{v}[\ell-\kappa]\right],$$

shift down the sums in the two terms inside the brackets by μ and κ, respectively,

$$=\frac{1}{N}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\left[\sum_{\substack{\ell=\kappa-\mu\\ \kappa\ge\mu}}^{N-1}\hat{v}^*[\ell]\hat{v}[\ell-(\kappa-\mu)]+\sum_{\substack{\ell=\mu-\kappa\\ \kappa<\mu}}^{N-1}\hat{v}^*[\ell-(\mu-\kappa)]\hat{v}[\ell]\right].$$

If $\hat{\mathbf{v}}[.]\in\mathbb{R}^N$, then

$$=\frac{1}{N}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]\sum_{\mu=0}^{N-1}\hat{u}^*[\mu]\sum_{\ell=|\kappa-\mu|}^{N-1}\hat{v}[\ell]\hat{v}[\ell-|\kappa-\mu|].$$ □

Corollary 2.4. *The proof of* $\left\|(\mathbf{u}\odot\mathbf{v})[n]\right\|_2^2$ *in Theorem 2.3 applies to* $\|\mathbf{u}[n]\|_4^4$ *by setting* $\mathbf{u}[n]=\mathbf{v}[n]$.

Theorem 2.5. *Let* $\mathbf{u}[n]\overset{\mathscr{F}}{\longleftrightarrow}\hat{\mathbf{u}}[\kappa]$ *and* $\mathbf{v}[n]\overset{\mathscr{F}}{\longleftrightarrow}\hat{\mathbf{v}}[\kappa]$, *where* $0\le n,\kappa\le N-1$, *be two finite sequences in* $\mathbb{C}^N$, *and related by discrete* Fourier *transform as* $\mathbf{u}[n],\mathbf{v}[n]$ *are the IDFT's of* $\hat{\mathbf{u}}[\kappa],\hat{\mathbf{v}}[\kappa]$ *respectively, then*

$$\left\|(\hat{\mathbf{u}}\odot\hat{\mathbf{v}})[n]\right\|_2^2=\frac{1}{N}\sum_{n=0}^{N-1}u[n]\sum_{m=0}^{N-1}u^*[m]\left[\sum_{\substack{l=n-m\\ n\ge m}}^{N-1}v^*[l]v[l-(n-m)]+\sum_{\substack{l=m-n\\ n<m}}^{N-1}v^*[l-(m-n)]v[l]\right].\tag{2.7}$$

If $\mathbf{v}[.]\in\mathbb{R}^N$, *then*

$$\left\|(\hat{\mathbf{u}}\odot\hat{\mathbf{v}})[n]\right\|_2^2=\frac{1}{N}\sum_{n=0}^{N-1}u[n]\sum_{m=0}^{N-1}u^*[m]\sum_{l=|n-m|}^{N-1}v[l]v[l-|n-m|].\tag{2.8}$$

Proof. The proof follows the same steps of the proof of theorem 2.3. □

Corollary 2.6. *The proof of* $\left\|(\hat{\mathbf{u}}\odot\hat{\mathbf{v}})[n]\right\|_2^2$ *in theorem 2.5 applies to* $\left\|\hat{\mathbf{u}}[n]\right\|_4^4$ *by setting* $\hat{\mathbf{u}}[n]=\hat{\mathbf{v}}[n]$.

[1]Because the resultant terms of the convolution are all summed -even after squared by outer product-, using linear discrete convolution equals using circular/cyclic convolution (see (C.1) and (C.2) in proposition C.1). The former is used here for its convenience with the following calculations.

Remark 2.7. Another famous representation of DFT that is used in engineering - and in MATLAB® - is

$$\hat{x}[\kappa] = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi\kappa n/N}, \quad 0 \le \kappa \le N-1, \qquad\qquad x[n] = \frac{1}{N}\sum_{\kappa=0}^{N-1} \hat{x}[\kappa]\, e^{i2\pi\kappa n/N}, \quad 0 \le n \le N-1.$$

When this representation is used, theorem 2.3 is modified by multiplying the right hand side (RHS) in equations (2.3) and (2.4) by $1/N^2$, and theorem 2.5 is modified by multiplying the RHS in equations (2.7) and (2.8) by N^2.

3. Applications

Given two deterministic signals $\mathbf{x}[t]$ and $\mathbf{y}[t]$ observed in time and sampled as discrete-time sequences. Therefore, their *power spectral density* (PSD) functions [24, pp.3] [2, pp.174] are periodic within $[-\pi, \pi]$ and defined as,

$$S_x(f) = \left|\hat{x}(f)\right|^2 \quad \text{and} \quad S_y(f) = \left|\hat{y}(f)\right|^2, \qquad \text{where} \qquad \hat{x}(f) = \sum_{t\in\mathbb{Z}} x[t]e^{-i2\pi f t} \quad \text{and} \quad \hat{y}(f) = \sum_{t\in\mathbb{Z}} y[t]e^{-i2\pi f t}.$$

Define a cost function $C(\theta, \hat{\theta})$ as a function of the error ϵ, and assume the criterion of choice to calculate ϵ is the squared error (i.e., $\epsilon \triangleq (\theta - \hat{\theta})^2$). The estimator $\hat{\theta}$ that minimizes the cost $C(\theta, \hat{\theta})$ is referred to as the minimum mean-square error estimate (MMSE) [2, pp.357], and the corresponding cost is $C_{MMSE} = \int_{-\infty}^{\infty} \epsilon \, d\theta = \int_{-\infty}^{\infty} (\theta - \hat{\theta})^2 \, d\theta$. Now, assume it is required to calculate the cost between $S_x(f)$ and $S_y(f)$, then

$$C_{MMSE} = \int_{-\pi}^{\pi} \left[S_x(f) - S_y(f)\right]^2 df,$$

which is a quartic (4^{th}-order) function of $\hat{x}(f)$ and $\hat{y}(f)$, that according to [24, pp.17] has no simple closed-form solution. However, the introduced theorems in this work together with [20] provide a closed-form solution when $\hat{x}(f)$ and $\hat{y}(f)$ are $2\pi-$periodic in frequency (i.e., $\mathbf{x}[t]$ and $\mathbf{y}[t]$ are discrete in time). For instance, assume sequence $\mathbf{y}[t]$ has PSD $S_y(f)$ that needs to be designed as close as possible to $S_x(f)$ in applications such as waveform/filter design [5, 11, 26, 31] and beamforming [12, 13, 21–23, 27, 33, 34], or to compare how close $S_y(f)$ is to $S_x(f)$ in applications such as detection and estimation [8, 14, 17, 18, 25, 28, 32] and machine learning [3, 6, 10, 15, 16, 19, 30]. Then, the cost is calculated as,

$$\begin{aligned} C_{MMSE} &= \int_{-\pi}^{\pi} \left[S_x^2(f) - 2S_x(f)S_y(f) + S_y^2(f)\right] df = \int_{-\pi}^{\pi} \left[\left|\hat{x}(f)\right|^4 - 2\left|\hat{x}(f)\hat{y}(f)\right|^2 + \left|\hat{y}(f)\right|^4\right] df \\ &= \left\|\hat{x}(f)\right\|^4_{L^4(-\pi,\pi)} - 2\left\|\hat{x}(f)\hat{y}(f)\right\|^2_{L^2(-\pi,\pi)} + \left\|\hat{y}(f)\right\|^4_{L^4(-\pi,\pi)}. \end{aligned} \tag{3.1}$$

The first and the last terms in (3.1) were solved in [20, Theorem 2.2] and the middle term was solved in theorem 2.2. The solution is a function in the discrete–time sequences $\mathbf{x}[t]$ and $\mathbf{y}[t]$. Similar scenario if the PSDs are replaced by auto-correlation functions [2, pp.153] then a closed-form solution is possible using [20, Theorem 2.1] and theorem 2.1 and the solution is a function in the discrete *Fourier* coefficients $\hat{\mathbf{x}}[f]$ and $\hat{\mathbf{y}}[f]$. However, the computational complexity of the RHS in these mentioned theorems are of order $O(N^3)$, which makes its usage in real-time applications critical.

4. Conclusion

We introduced an analytical solution to the L^2-norm of two multiplied exponential *Fourier* series. Theorem 2.1 maps the L^2-norm of two multiplied $2\pi-$periodic functions (i.e., finite/infinite exponential *Fourier* series) to their individual *Fourier* coefficients. Theorem 2.2 maps between the L^2-norm of two multiplied DTFT to their time samples. The introduced solutions result the exact values and avoid numerical errors happens if the integral is calculated numerically. Theorems 2.3 and 2.5 extended the results to discrete-time discrete-frequency finite sequences, i.e., DFT/IDFT respectively. However, step (*II*) in proof of Theorem 2.3 made a transition from multiplication to convolution which increased the order of computational complexity from $O(N^2)$ to $O(N^3)$, nevertheless, the accuracy of the result remains unchanged because the functions are basically discrete. The identities are useful in physical problems such as comparison of energies of signals (e.g., detection and estimation) and computing power spectrum of a weighted signal (e.g., digital filtering), as they result in the exact value. Besides, the identities hold equality to *Cauchy–Schwarz* inequality for functions periodic in $[-\pi, \pi]$ -satisfied for discrete sequences-, a problem that appears a lot in functional analysis.

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Code and Data Availability

MATLAB® code implementing the introduced theorems is available on https://github.com/h-sharkas/tjmcs.1424850

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Author Contribution Statement

The author has read and agreed to the published version of the manuscript.

Appendix A. *Parseval*'s Identity

For $\hat{x}[\kappa] = \frac{1}{\sqrt{N}} \sum_n x[n]\, e^{-i2\pi\kappa n/N}$ and $x[n] = \frac{1}{\sqrt{N}} \sum_\kappa \hat{x}[\kappa]\, e^{i2\pi\kappa n/N}$, where $0 \le n, \kappa \le N-1$, *Parseval*'s identity states,

$$\|\mathbf{x}[n]\|_2^2 = \left\|\hat{\mathbf{x}}[\kappa]\right\|_2^2 .$$

Proof.

$$\begin{aligned}\sum_{n=0}^{N-1}|x[n]|^2 &= \sum_{n=0}^{N-1} x[n].x^*[n] = \sum_{n=0}^{N-1}\Big(\frac{1}{\sqrt{N}}\sum_{\kappa=0}^{N-1}\hat{x}[\kappa]e^{i2\pi\kappa n/N}\Big).x^*[n] = \frac{1}{\sqrt{N}}\sum_{\kappa=0}^{N-1}\hat{x}[\kappa].\Big(\sum_{n=0}^{N-1}x^*[n]e^{i2\pi\kappa n/N}\Big)\\ &= \frac{1}{\sqrt{N}}\sum_{\kappa=0}^{N-1}\hat{x}[\kappa].\Big(\sum_{n=0}^{N-1}x[n]e^{-i2\pi\kappa n/N}\Big)^* = \frac{1}{\sqrt{N}}\sum_{\kappa=0}^{N-1}\hat{x}[\kappa].\Big(\sqrt{N}\hat{x}[\kappa]\Big)^* = \sum_{\kappa=0}^{N-1}\hat{x}[\kappa].\hat{x}^*[\kappa] = \sum_{\kappa=0}^{N-1}|\hat{x}[\kappa]|^2 .\end{aligned}$$ □

Appendix B. The Convolution Theorem

For $\hat{x}[\kappa] = \frac{1}{\sqrt{N}} \sum_n x[n]\, e^{-i2\pi\kappa n/N}$ and $x[n] = \frac{1}{\sqrt{N}} \sum_\kappa \hat{x}[\kappa]\, e^{i2\pi\kappa n/N}$, where $0 \le n, \kappa \le N-1$; and the N-periodic circular convolution $r_N[m] = (u \circledast v)_N[m] = \sum_n u[n].v_N[(m-n)_{mod\,N}]$, where $0 \le m \le N-1$, the convolution theorem states

$$(\hat{\mathbf{u}} \circledast \hat{\mathbf{v}})_N = \sqrt{N}\,\widehat{(\mathbf{u}\odot\mathbf{v})} .$$

Proof.

$$\begin{aligned}(\hat{u}\circledast\hat{v})_N[m] &= \sum_{\kappa=0}^{N-1}\hat{u}[\kappa].\hat{v}_N[(m-\kappa)_{mod\,N}], \qquad \text{let } \delta = m-\kappa, \text{ and } \mathbf{v}[n] \xleftrightarrow{\mathscr{F}} \hat{\mathbf{v}}[\delta], \text{ where } 0\le n,\delta\le N-1\\ &= \sum_{\kappa=0}^{N-1}\hat{u}[\kappa].\hat{v}[\delta] = \sum_{\kappa=0}^{N-1}\hat{u}[\kappa].\Big(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}v[n]e^{-i2\pi(m-\kappa)n/N}\Big) = \sum_{\kappa=0}^{N-1}\hat{u}[\kappa].\Big(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}v[n]e^{-i2\pi mn/N}e^{i2\pi\kappa n/N}\Big)\\ &= \sum_{n=0}^{N-1}\Big(\frac{1}{\sqrt{N}}\sum_{\kappa=0}^{N-1}\hat{u}[\kappa]e^{i2\pi\kappa n/N}\Big)v[n]e^{-i2\pi mn/N} = \sum_{n=0}^{N-1}u[n].v[n]e^{-i2\pi mn/N} = \sqrt{N}\,\widehat{(u\odot v)}\,[m].\end{aligned}$$ □

Appendix C.

Proposition C.1. *Let* $\mathbf{u}[n]$ *and* $\mathbf{v}[n]$*, where* $0 \le n \le N-1$*, be two finite sequences in* $\mathbb{C}^{1\times N}$*. Let their circular convolution be given by* $\mathbf{r}_N[n] = (\mathbf{u}\circledast\mathbf{v})_N[n]$*, and their linear convolution be given by* $\mathbf{r}[\ell] = (\mathbf{u}\circledast\mathbf{v})[\ell]$*, where* $0\le\ell\le 2N-2$*. Let* $\mathbf{r}_1 = \mathbf{r}[\ell_1]$ *and* $\mathbf{r}_2 = \begin{bmatrix}\mathbf{r}[\ell_2] & 0\end{bmatrix}$*, where* $0\le\ell_1\le N-1$ *and* $N\le\ell_2\le 2N-2$*. Consider the fact that* $\mathbf{r}_N \equiv \mathbf{r}_1+\mathbf{r}_2$*, then*

$$\sum_n \mathbf{r}_N = \sum_n \mathbf{r}_1+\mathbf{r}_2 = \sum_\ell \begin{bmatrix}\mathbf{r}[\ell_1] & \mathbf{r}[\ell_2] & 0\end{bmatrix} = \sum_\ell \mathbf{r}. \tag{C.1}$$

$$\begin{aligned}\sum_n|\mathbf{r}_N|^2 &= \sum_n|\mathbf{r}_1+\mathbf{r}_2|^2 = \sum_n\Big(|\mathbf{r}_1|^2+|\mathbf{r}_2|^2+\mathbf{r}_1^*\mathbf{r}_2+\mathbf{r}_2^*\mathbf{r}_1\Big) = \sum_n\Big(|\mathbf{r}[\ell_1]|^2+|\mathbf{r}[\ell_2]|^2+\mathbf{r}^*[\ell_1]\mathbf{r}[\ell_2]+\mathbf{r}^*[\ell_2]\mathbf{r}[\ell_1]\Big)\\ &= \sum_\ell\begin{bmatrix}\mathbf{r}^*[\ell_1]\\ \mathbf{r}^*[\ell_2]\end{bmatrix}\begin{bmatrix}\mathbf{r}[\ell_1] & \mathbf{r}[\ell_2]\end{bmatrix} = \sum_\ell\Big|\begin{bmatrix}\mathbf{r}[\ell_1] & \mathbf{r}[\ell_2]\end{bmatrix}\Big|^2 = \sum_\ell|\mathbf{r}|^2 .\end{aligned} \tag{C.2}$$

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